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SOME REMARKS ABOUT UNIFORMLY LIPSCHITZIAN MAPPINGS AND LIPSCHITZIAN RETRACTIONS

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Abstract. In this paper we present a few facts and open problems about uniformly lipschitzian mappings and lipschitzian retractions.

1. A Few Remarks about Lipschitzian and Uniformly Lipschitzian Mappings

In 1930, J. Schauder [15] published his famous theorem:

Every nonempty, convex and compact subset C of a Banach space X has the fixed point property for continuous mappings.

We give an equivalent form of this theorem in which lipschitzian mappings are involved. For convenience of the reader we recall the definition of this type of mappings.

Let C be a nonempty subset of a Banach space X. A mapping $T : C \to X$ is called *lipschitzian* with constant $k > 0$ if for any $x, y \in C$,

$$\|Tx - Ty\| \leq k\|x - y\|.$$

If $k = 1$, we say that the mapping T is *nonexpansive*.

More information about nonexpansive mappings can be found in [1, 9, 10]. The notions of the lipschitzian mapping and the nonexpansive one can be formulated in an obvious way in metric spaces.

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Now we state the theorem mentioned earlier which is equivalent to Schauder's Fixed Point Theorem.

Every nonempty, convex and compact subset C of a Banach space X has the fixed point property for lipschitzian mappings.

It is worth noting here that a number of other equivalent formulations of Schauder's Fixed Point Theorem, which are not commonly known and quoted in literature, are presented in [7].

Now we restrict our attention to noncompact sets. All balls which appear in our paper are closed. S. Kakutani [11] was probably the first who in 1943 showed that there are continuous mappings of the unit ball in Hilbert space without fixed points. A stronger result is due to V. Klee [12]:

For any nonempty, closed, convex but noncompact subset C of a Banach space X, there exists a continuous mapping $T : C \to C$ which is fixed-point free.

Hence the theorems of Schauder and Klee can be mixed in one property.

Any nonempty, bounded, closed and convex subset C of a Banach space X has the fixed point property for continuous mappings if and only if it is a compact set.

The problem whether in the above theorem continuous mappings can be replaced by lipschitzian ones was open for a long time. It was solved by P. K. Lin and Y. Sternfeld [13]:

For any noncompact, bounded and convex subset C of a Banach space X there is a lipschitzian mapping $T : C \to C$ for which the so called minimal displacement

$$d_T = \inf\{\|x - Tx\| : x \in C\}$$

is positive.

The next theorem about lipschitzian retractions is a consequence of the above result. This result was in fact even published earlier than the last theorem. First B. Nowak [14] proved it for a certain class of spaces and then it was generalized to all Banach spaces by Y. Benyamini and Y. Sternfeld [3].

For any infinite-dimensional Banach space X, there is a lipschitzian retraction of the closed unit ball B onto its sphere S.

This means that there exists a k-lipschitzian mapping $R : B \to S$ such that $Rx = x$ for all $x \in S$. Let us observe that if $R : B \to S$ is a k-lipschitzian retraction and $B(0,r)$, $S(0,r)$ are the closed ball with center 0 and radius r and its sphere, respectively, then

$$R_r(x) = rR\left(\frac{1}{r}x\right) \tag{1.1}$$

for $x \in B(0,r)$ is a retraction of $B(0,r)$ onto $S(0,r)$ with the same Lipschitz constant k. Hence the above theorem shows how different are the cases: finite-dimensional compact balls and infinite-dimensional noncompact balls. An up-to-date bibliography about retractions of balls onto spheres is given in [6].

Directly from the theorem of Y. Benyamini and Y. Sternfeld we obtain the next theorem about lipschitzian retractions:

Let X be an infinite-dimensional Banach space and C a bounded, closed and convex subset with int $C \neq \emptyset$. *Then there is a lipschitzian retraction of C onto its boundary ∂C.*

Proof. Without loss of generality, we can assume that $0 \in C$ and $B(0,r_1) \subset C \subset B(0,r_2)$, where $0 < r_1 \le r_2$. Let p_C be the Minkowski functional of C, $P : C \to B(0,r_1)$ be a radial projection of C onto $B(0,r_1)$ and $R : B(0,r_1) \to S(0,r_1)$ be a k-lipschitzian retraction of the ball $B(0,r_1)$ onto its sphere $S(0,r_1)$. Then the retraction given by the formula

$$R_1(x) = \frac{1}{p_C(R(P(x)))}R(P(x)) \qquad \text{for} \quad x \in C$$

is the claimed lipschitzian retraction of C onto its boundary ∂C with the constant

$$k_1 = 2\frac{r_2}{r_1}\left(\frac{r_2}{r_1}+1\right)k. \qquad \blacksquare$$

Let us observe at this point that lipschitzian retractions mentioned in the last two theorems are in fact uniformly lipschitzian, that is, they satisfy the following definition of uniformly lipschitzian mappings.

Let C be a nonempty, bounded, closed and convex subset of a Banach space X and T be a selfmapping of C. If each iterate T^n, $n = 1, 2, \dots$, has the same Lipschitz constant k, then the mapping T is called *uniformly lipschitzian* (or *k-uniformly lipschitzian*).

Next, if R_1 (R, respectively) is the lipschitzian retraction mentioned in the above theorems, then taking

$$C \ni x \mapsto -\frac{1}{p_C(-R_1(x))}R_1(x) \in \partial C$$

($-R$, respectively) we get the uniformly lipschitzian mapping with a positive minimal displacement (in the case of R it was noticed in [3]). If we compare this remark with Schauder's Fixed Point Theorem, then we come to the following conclusion.

A Banach space X is finite-dimensional if and only if there exists a nonempty, bounded, closed and convex subset C with int $C \neq \emptyset$ *such that for each uniformly lipschitzian mapping $T : C \to C$ we have*

$$d_T = \inf_{x \in C} \|x - Tx\| = 0.$$

The above considerations lead to the following question which is strictly connected with the theorem of P. K. Lin and Y. Sternfeld.

Problem 1.1. *Let C be an arbitrary noncompact, bounded and convex subset of a Banach space X. Does there exist a uniformly lipschitzian mapping $T : C \to C$ for which the minimal distance*

$$d_T = \inf\{\|x - Tx\| : x \in C\}$$

is positive?

The above problem is still open. It is worth noting here that the lipschitzian mapping with a positive minimal distance constructed by P. K. Lin and Y. Sternfeld [13] and its modification given in the book by K. Goebel and W. A. Kirk [9] are both not uniformly lipschitzian.

At the end of this section we recall some facts about uniformly lipschitzian mappings. The notion of uniformly lipschitzian mappings was introduced by K. Goebel and W. A. Kirk to obtain a fixed point theorem for k-uniformly lipschitzian mappings with the constant k greater than 1 – see for the details [8]. They also observed that the class of uniformly lipschitzian selfmappings on C is completely characterized as the class of mappings on C which are nonexpansive relative to some metric equivalent to the norm [9]. Next, if we take a nonexpansive selfmapping T on C and any equivalent norm, then in this equivalent norm the mapping T is uniformly lipschitzian. Recently, J. García-Falset, A. Jiménez-Melado and E. LLoréns-Fuster proved however that not every uniformly lipschitzian mapping can be obtained in this way [4]. In the second section of our paper we show a few other examples of such mappings.

2. Two Constants Connected with Lipschitzian Retractions

As we mentioned at the end of the previous section, every uniformly lipschitzian mapping is nonexpansive with respect to some metric equivalent to the norm, but not for every uniformly lipschitzian mapping T can we find an equivalent norm in which T is nonexpansive. Returning to lipschitzian retractions of balls onto their spheres, we notice that in this case there is no equivalent norm in which they are nonexpansive. Indeed, this is a consequence of the following fact.

Let X be an infinite-dimensional Banach space and let $R : B \to S$ be a lipschitzian retraction of the ball B onto its sphere S. For every equivalent norm the retraction R is k-lipschitzian in the new norm and the constant k is always greater than or equal to 3.

Proof. We show that $k \geq 3$. Let us take an arbitrary norm $\|\cdot\|_1$ equivalent to the original norm $\|\cdot\|$ and let R be k-lipschitzian in the norm $\|\cdot\|_1$. We define $T : B \to S$ by setting

$$Tx = -Rx$$

for $x \in B$. The mapping T is also k-lipschitzian in $\|\cdot\|_1$. For each $x \neq 0$, let $\alpha_x > 0$ be such that

$$\|x\| = \alpha_x \|x\|_1.$$

(Let us observe here that if $x \neq 0$ and if $y = \gamma x$ with $\gamma \neq 0$, then $\alpha_x = \alpha_y$.) It is clear that given $\epsilon > 0$, there exists a unique $x_\epsilon \in B$ satisfying

$$x_\epsilon = \frac{1}{k+\epsilon} T x_\epsilon.$$

Then we have

$$x_\epsilon \neq 0$$

and

$$\|Rx_\epsilon\| = \|Tx_\epsilon\| = \alpha_{x_\epsilon} \|Tx_\epsilon\|_1 = \alpha_{x_\epsilon} \|Rx_\epsilon\|_1,$$

or, in other words,

$$\alpha_{x_\epsilon} = \alpha_{Rx_\epsilon} = \alpha_{Tx_\epsilon},$$

because x_ϵ, Rx_ϵ and Tx_ϵ are collinear and all different from zero. Next we observe that for every $x \in B$,

$$T^2 x = Rx.$$

It then follows that

$$\begin{aligned} 2 &= 2\|Rx_\epsilon\| = 2\alpha_{x_\epsilon}\|Rx_\epsilon\|_1 = \alpha_{x_\epsilon}\|Rx_\epsilon - (-Rx_\epsilon)\|_1 \\ &= \alpha_{x_\epsilon}\|T^2 x_\epsilon - Tx_\epsilon\|_1 \leq k\alpha_{x_\epsilon}\|Tx_\epsilon - x_\epsilon\|_1 \\ &= k\left(1 - \frac{1}{k+\epsilon}\right)\alpha_{x_\epsilon}\|Tx_\epsilon\|_1 = k\left(1 - \frac{1}{k+\epsilon}\right)\|Tx_\epsilon\| \\ &= k\left(1 - \frac{1}{k+\epsilon}\right) = k - \frac{k}{k+\epsilon}. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we get

$$k \geq 3,$$

which completes the proof. ■

Directly from, the above theorem we get the following conollany.

Let X be an infinite-dimensional Banach space and let $R : B \to S$ be a lipschitzian retraction of the ball B onto its sphere S. There is no equivalent norm in which R is nonexpansive.

Let us assume that X is an infinite-dimensional Banach space. Let $k_0(X)$ denote the infimum of the set of all numbers $k > 1$ for which there exists a k-lipschitzian retraction R of the unit ball B onto its unit sphere S [5], and let $\widetilde{k}_0(X)$ denote the infimum of the set of all numbers $k > 1$ for which there exists a retraction R of the unit ball B onto its unit sphere S which is k-lipschitzian in some equivalent norm. It is obvious that

$$3 \leq \widetilde{k}_0(X) \leq k_0(X).$$

Hence we have another open problem.

Problem 2.1. *Does the following equality*

$$k_0(X) = \widetilde{k}_0(X)$$

hold?

3. Final Remarks

In the preceding section lipschitzian retractions of balls onto their spheres were discussed . So let us pass here to a more general situation. In place of the unit ball we consider bounded, closed and convex subsets C with int $C \neq \emptyset$. Without loss of generality we may assume that $0 \in C$ is the interior point of the set C. It then follows that there exist balls $B(0, r_1)$ and $B(0, r_2)$ such that $B(0, r_1) \subset C \subset B(0, r_2)$. The set C can be nonsymmetric and therefore cannot be a ball in any norm. Let us take a norm $\|\cdot\|_1$ equivalent to the original $\|\cdot\|$ and (as in the proof of the previous theorem) let $\alpha_x > 0$ satisfy

$$\|x\| = \alpha_x \|x\|_1$$

for each $x \neq 0$. Let R be a lipschitzian retraction of C onto its boundary ∂C. According to the theorem in Section 1 such a retraction exists. Suppose that it is k-lipschitzian in $\|\cdot\|_1$ and define $T : C \to \partial C$ by setting

$$Tx = -\frac{1}{p_C(-Rx)} Rx$$

for $x \in C$, where p_C is a Minkowski functional of C. The mapping T is also lipschitzian in the norm $\|\cdot\|_1$, say, K-lipschitzian. Then given $\epsilon > 0$, there exists a unique $x_\epsilon \in C$ such that

$$x_\epsilon = \frac{1}{K+\epsilon}Tx_\epsilon = -\frac{1}{(K+\epsilon)p_C(-Rx_\epsilon)}Rx_\epsilon.$$

Consequently, $x_\epsilon \neq 0$. Since x_ϵ, Tx_ϵ and Rx_ϵ are collinear and all different from zero, we have

$$\|Tx_\epsilon\| = \alpha_{x_\epsilon}\|Tx_\epsilon\|_1, \quad \|Rx_\epsilon\| = \alpha_{x_\epsilon}\|Rx_\epsilon\|_1$$

(see the proof of the theorem in Section 2). Next we observe that for every $x \in C$, we have $T^2x = Rx$ and

$$RTx = T^3x = TRx = -\frac{1}{p_C(-Rx)}Rx = Tx.$$

It follows from the above that

$$\begin{aligned}
2r_1 &\leq \left\| -\frac{1}{p_C(-Rx_\epsilon)}Rx_\epsilon - Rx_\epsilon \right\| \\
&= \alpha_{x_\epsilon}\left\| -\frac{1}{p_C(-Rx_\epsilon)}Rx_\epsilon - Rx_\epsilon \right\|_1 \\
&= \alpha_{x_\epsilon}\|Tx_\epsilon - T^2x_\epsilon\|_1 = \alpha_{x_\epsilon}\|RTx_\epsilon - Rx_\epsilon\|_1 \\
&\leq k\alpha_{x_\epsilon}\|Tx_\epsilon - x_\epsilon\|_1 = k\left(1 - \frac{1}{K+\epsilon}\right)\alpha_{x_\epsilon}\|Tx_\epsilon\|_1 \\
&= k\left(1 - \frac{1}{K+\epsilon}\right)\|Tx_\epsilon\| \\
&\leq k\left(1 - \frac{1}{K+\epsilon}\right)r_2 < kr_2.
\end{aligned}$$

Therefore we have $k > 2\frac{r_1}{r_2}$. Summing up the above considerations we arrive at the following remark.

Suppose that C is a bounded, closed and convex subset of an infinite dimensional Banach space X such that int $C \neq \emptyset$. Suppose also that $0 \in C$ and $B(0, r_1) \subset C \subset B(0, r_2)$ with $0 < r_2/r_1 < 2$. Let R be a lipschitzian retraction of C onto its boundary ∂C. Then there is no equivalent norm in which R is nonexpansive.

Thus we can state another open problem.

Problem 3.1. *Let X be an infinite-dimensional Banach space and C its bounded, closed and convex subset with* int $C \neq \emptyset$. *Let $\widetilde{k}_0(X, C)$ denote the infimum of the set of all numbers $k > 1$ for which there exists a retraction R of the set C onto its boundary ∂C which is k-lipschitzian in some equivalent norm. Is $\widetilde{k}_0(X, C)$ always strictly greater than* 1?

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