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CO-SEMISIMPLE MODULES AND GENERALIZED INJECTIVITY

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Abstract. Let $\mathcal{F}$ be a left Gabriel topology on a ring R and $\mathcal{X}$ be a special class of left R-modules (for example, the class of all quasi-continuous left R-modules in $\sigma[M]$, etc.). Suppose that all left R-modules in $\mathcal{X}$ are $\mathcal{F}$-injective. Then, it is proved in this paper that a left R-module M is $\mathcal{F}$-co-semisimple (that is, every $\mathcal{F}$-cocritical left R-module C in $\sigma[M]$ is dense in its M-injective hull) if and only if every $\mathcal{F}$-torsionfree $\mathcal{F}$-finitely cogenerated left R-module N in $\sigma[M]$ is dense in its some essential extensions which are in $\mathcal{X}$. As a corollary we show that a left R-module M is co-semisimple if and only if every finitely cogenerated left R-module in $\sigma[M]$ is continuous (or quasi-continuous, or direct-injective, etc.)

1. Introduction

Let R be a ring with identity and M a left R-module. A left R-module U is called M-injective if for every submodule N of M and homomorphism $\phi : N \longrightarrow U$, ϕ can be lifted to a homomorphism $\psi : M \longrightarrow U$. A left R-module M is called a co-semisimple module by Fuller [2] (and is called a V-module by Ramamurthi [11] and Tominaga [12]) provided every submodule of M is an intersection of maximal submodules. Fuller [2, Proposition 3.1] and Hirano [4, Proposition 3.1] or Dung, Huynh, Smith and Wisbauer [1] proved that M is co-semisimple if and only if every simple left R-module is M-injective. Wisbauer [14] proved that M is co-semisimple if and only if every finitely cogenerated left R-module in $\sigma[M]$ is M-injective. In this paper, we characterize co-semisimple left R-modules via generalized injectivity of some modules.

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Let $\mathcal{F}$ be a left Gabriel topology on R and M a left R-module. We call M a $\mathcal{F}$-co-semisimple module if every $\mathcal{F}$-cocritical left R-module C in $\sigma[M]$ is dense in its M-injective hull $I(C)$. Let $\mathcal{X}$ be a special class of left R-modules (for example, the class of all quasi-continuous left R-modules in $\sigma[M]$, or, the class of all quasi-injective left R-modules in $\sigma[M]$, etc.). We show that if the left Gabriel topology $\mathcal{F}$ is such that all left R-modules in $\mathcal{X}$ are $\mathcal{F}$-injective, then M is $\mathcal{F}$-co-semisimple if and only if every $\mathcal{F}$-torsionfree $\mathcal{F}$-finitely cogenerated left R-module N in $\sigma[M]$ is dense in its some essential extensions which are in $\mathcal{X}$.

As a corollary we show that a left R-module M is a co-semisimple module if and only if every finitely cogenerated left R-module in $\sigma[M]$ is continuous if and only if every finitely cogenerated left R-module in $\sigma[M]$ is quasi-continuous if and only if every finitely cogenerated left R-module in $\sigma[M]$ is direct-injective.

Page and Yousif in [10] proved that a finitely generated left R-module M is a noetherian co-semisimple module if and only if every semisimple left R-module is M-injective. In this paper we also show that a left R-module M is a locally noetherian co-semisimple module if and only if every semisimple left R-module (in $\sigma[M]$) is M-injective if and only if every semisimple left R-module (in $\sigma[M]$) is the direct sum of a finitely cogenerated module and an M-injective module if and only if every essential extension in $\sigma[M]$ of every semisimple left R-module in $\sigma[M]$ is an $\mathcal{X}$-module, where $\mathcal{X}$ is a specified class of left R-modules.

2. Preliminaries

Let M be a left R-module. We say that a left R-module N is subgenerated by M, or that M is a subgenerator for N, if N is isomorphic to a submodule of an M-generated module. Following [14], we denote by $\sigma[M]$ the full subcategory of $R-Mod$ whose objects are all R-modules subgenerated by M. By [14, 17.9], every module N in $\sigma[M]$ has an injective hull $I(N)$ in $\sigma[M]$, which is also called an M-injective hull of N. It is known that the M-injective hulls of a left R-module in $\sigma[M]$ are unique up to isomorphism. In the following, we always denote by $I(N)$ the M-injective hull of N for any left R-module $N \in \sigma[M]$.

Let $\mathcal{X}$ be a class of left R-modules such that if $M \in \mathcal{X}$ then any left R-module isomorphic to M belongs to $\mathcal{X}$. Any member of $\mathcal{X}$ is called an $\mathcal{X}$-module.

Definition 2.1. Let M be a left R-module and $\mathcal{X}$ a class of left R-modules in $\sigma[M]$. We call $\mathcal{X}$ an I-class in the category $\sigma[M]$ if it contains all M-injective left R-modules in $\sigma[M]$ and for any $N \in \sigma[M]$, if there exists an M-injective left R-module L in $\sigma[M]$ such that $N \leq L$ and $N \oplus L$ is in $\mathcal{X}$, then N is L-injective.

If $M = R$, then any I-class in the category $\sigma[M]$ is called an I-class of left R-modules.

Following [19], we call $\mathcal{X}$ an injectivity class in the category $\sigma[M]$ if it is closed under direct summands, contains all quasi-injective left R-modules in $\sigma[M]$ and $N \oplus I(N) \in \mathcal{X}$ implies that N is M-injective. We claim that every injectivity class in the category $\sigma[M]$ is an I-class. In fact, if $\mathcal{X}$ is an injectivity class, then $\mathcal{X}$ contains all M-injective left R-modules in the category $\sigma[M]$. Suppose that N is in $\sigma[M]$ and L an M-injective left R-module in $\sigma[M]$ such that $N \leq L$ and $N \oplus L \in \mathcal{X}$. Then L is an injective object of the category $\sigma[M]$. Thus there exists a homomorphism $g : I(N) \longrightarrow L$ such that $g|_N = \tau$, the natural inclusion map $N \longrightarrow L$. Now it follows that $g : I(N) \longrightarrow L$ is a monomorphism since N is essential in $I(N)$. Thus we have $L = I(N) \oplus P$ for a left R-module P. Therefore $N \oplus L = N \oplus I(N) \oplus P \in \mathcal{X}$. Since $\mathcal{X}$ is closed under direct summands, we obtain that $N \oplus I(N) \in \mathcal{X}$. Now it follows that N is M-injective.

The following proposition gives some examples of I-classes.

Proposition 2.2. *The class of all quasi-injective* (*respectively, continuous, quasi-continuous, direct-injective, NCI, SQC, E-injective*) *left R-modules in $\sigma[M]$ is an I-class.*

Proof. It follows from [5, 9, 15, 17, 18] and [19]. ∎

Let $\mathcal{F}$ be a left Gabriel topology on R. The quotient category $(R, \mathcal{F})-Mod$, associated with $\mathcal{F}$, is the full subcategory of $R-Mod$ whose objects are the $\mathcal{F}$-closed (i.e., $\mathcal{F}$-torsionfree and $\mathcal{F}$-injective) left R-modules, and it is a Grothendieck category. The inclusion functor $i : (R, \mathcal{F}) - Mod \longrightarrow R - Mod$ has a left adjoint $a : R - Mod \longrightarrow (R, \mathcal{F}) - Mod$ which is exact and assigns to each $M \in R - Mod$ its module of quotients $M_{\mathcal{F}}$.

3. $\mathcal{F}$-co-Semisimple Modules

Definition 3.1. Let $\mathcal{F}$ be a left Gabriel topology on R, M a left R-module and N in $\sigma[M]$.

(1) We say N is $\mathcal{F}$-cocyclic in $\sigma[M]$ if there exists an $\mathcal{F}$-cocritical left R-module $C \in \sigma[M]$ such that

$$0 \longrightarrow N_{\mathcal{F}} \longrightarrow I(C)_{\mathcal{F}}$$

is exact. N is called $\sigma - \mathcal{F}$-cocyclic in $\sigma[M]$ if it is a finite direct sum of $\mathcal{F}$-cocyclic left R-modules in $\sigma[M]$.

(2) N is called $\mathcal{F}$-finitely cogenerated in $\sigma[M]$ if there exist $\mathcal{F}$-cocritical left R-modules $C_1, \ldots\ldots, C_n$ in $\sigma[M]$ such that the sequence

$$0 \longrightarrow N_{\mathcal{F}} \longrightarrow \oplus_{i=1}^{n} I(C_i)_{\mathcal{F}}$$

is exact.

If $M = R$ and $\mathcal{F} = \{R\}$, then $\mathcal{F}$-cocyclic modules in $\sigma[M]$ (resp. $\mathcal{F}$-finitely cogenerated modules in $\sigma[M]$) are precisely cocyclic (resp. finitely cogenerated) modules in the usual sense (see [14] and [7]).

Definition 3.2. Let M be a left R-module and $\mathcal{F}$ a left Gabriel topology on R. We call M an $\mathcal{F}$-co-semisimple module if every $\mathcal{F}$-cocritical left R-module C in $\sigma[M]$ is dense in its M-injective hull $I(C)$.

Note that if $\mathcal{F} = \{R\}$, then the $\mathcal{F}$-co-semisimple left R-modules are precisely the co-semisimple left R-modules. On the other hand, if $\mathcal{F}$ is a perfect Gabriel topology, then the inclusion functor $j : (R, \mathcal{F}) - Mod \longrightarrow R_{\mathcal{F}} - Mod$ is an equivalence; hence, for every left R-module M, M is $\mathcal{F}$-co-semisimple if and only if $M_{\mathcal{F}}$ is co-semisimple. If $\mathcal{F}$ is a left Gabriel topology on R such that for every left R-module N in $\sigma[M]$, $N_{\mathcal{F}}$ is an injective object of $(R, \mathcal{F}) - Mod$, then M is an $\mathcal{F}$-co-semisimple module. In particular, If $\mathcal{F}$ is a left Gabriel topology on R such that $(R, \mathcal{F}) - Mod$ is a spectral category (that is, every object is injective), then clearly every left R-module M is $\mathcal{F}$-co-semisimple. Thus, if $\mathcal{G}$ denotes the left Goldie topology, then every left R-module M is $\mathcal{G}$-co-semisimple.

Theorem 3.3. *Let M be a left R-module and $\mathcal{F}$ a left Gabriel topology on R. If $\mathcal{X}$ is an I-class in the category $\sigma[M]$ such that every $\mathcal{X}$-module is $\mathcal{F}$-injective, then the following conditions are equivalent.*

(1) *M is $\mathcal{F}$-co-semisimple.*

(2) *Every $\mathcal{F}$-torsionfree and $\mathcal{F}$-finitely cogenerated left R-module in $\sigma[M]$ is dense in its some essential extensions which are $\mathcal{X}$-modules.*

(3) *Every $\mathcal{F}$-torsionfree and $\sigma - \mathcal{F}$-cocyclic left R-module in $\sigma[M]$ is dense in its some essential extensions which are $\mathcal{X}$-modules.*

(4) *For every $\mathcal{F}$-torsionfree and $\mathcal{F}$-finitely cogenerated left R-module N in $\sigma[M]$, there exists an $\mathcal{X}$-module L with essential submodule N such that* $\mathrm{Rad}_{\mathcal{F}}(L) = 0$.

(5) *For every $\mathcal{F}$-torsionfree and $\sigma - \mathcal{F}$-cocyclic left R-module N in $\sigma[M]$, there exists an $\mathcal{X}$-module L with essential submodule N such that* $\mathrm{Rad}_{\mathcal{F}}(L) = 0$.

Proof. (1) $\Longrightarrow$ (2). Let left R-module $N \in \sigma[M]$ be $\mathcal{F}$-torsionfree and $\mathcal{F}$-finitely cogenerated in $\sigma[M]$. Then there exist $\mathcal{F}$-cocritical left R-modules $C_1, \ldots\ldots, C_m \in \sigma[M]$ such that

$$0 \longrightarrow N_{\mathcal{F}} \xrightarrow{g} \oplus_{i=1}^{m} I(C_i)_{\mathcal{F}}.$$

Consider the following diagram

$$\begin{array}{ccc} N_{\mathcal{F}} & \xrightarrow{\tau} & I(N)_{\mathcal{F}} \\ g\downarrow & & \\ \oplus_{i=1}^{m} I(C_i)_{\mathcal{F}} & & \end{array}$$

It is clear that $\oplus_{i=1}^{m} I(C_i) \in \sigma[M]$ is M-injective. Thus $\oplus_{i=1}^{m} I(C_i)$ is an $\mathcal{X}$-module since $\mathcal{X}$ is an I-class in the category $\sigma[M]$, and so is $\mathcal{F}$-injective by assumption. It is easy to see that $\oplus_{i=1}^{m} I(C_i)$ is $\mathcal{F}$-torsionfree. Thus $\oplus_{i=1}^{m} I(C_i)$ is $\mathcal{F}$-closed. This implies that $i((\oplus_{i=1}^{m} I(C_i))_{\mathcal{F}}) \simeq \oplus_{i=1}^{m} I(C_i)$. Thus $i((\oplus_{i=1}^{m} I(C_i))_{\mathcal{F}})$ is M-injective. Similar argument gives that $i(I(N)_{\mathcal{F}}) \simeq I(N) \in \sigma[M]$. Thus $i(N_{\mathcal{F}}) \in \sigma[M]$. Now, by [14, 16.3], there exists a homomorphism $f : i(I(N)_{\mathcal{F}}) \longrightarrow i((\oplus_{i=1}^{m} I(C_i))_{\mathcal{F}})$ such that $i(g) = fi(\tau)$. Thus we have $g = a(f)\tau$. Since N is essential in $I(N)$ and N is $\mathcal{F}$-torsionfree, by [3, Lemma 0.1], it follows that $N_{\mathcal{F}}$ is an essential subobject of $I(N)_{\mathcal{F}}$. Thus $a(f) : I(N)_{\mathcal{F}} \longrightarrow \oplus_{i=1}^{m} I(C_i)_{\mathcal{F}}$ is a monomorphism. Since M is $\mathcal{F}$-co-semisimple, we have $(C_i)_{\mathcal{F}} = I(C_i)_{\mathcal{F}}$, $i = 1, \ldots\ldots, m$. Thus $I(N)_{\mathcal{F}}$ is isomorphic to a subobject of semisimple object $\oplus_{i=1}^{m} (C_i)_{\mathcal{F}}$ of $(R, \mathcal{F}) - Mod$. Hence $I(N)_{\mathcal{F}}$ is semisimple. This implies that $N_{\mathcal{F}} = I(N)_{\mathcal{F}}$. Clearly $I(N)$ is an $\mathcal{X}$-module since it is M-injective.

(2) $\Longrightarrow$ (3). Let N be an $\mathcal{F}$-torsionfree and $\sigma - \mathcal{F}$-cocyclic left R-module in $\sigma[M]$. Then $N = \oplus_{i=1}^{n} N_i$, where $N_1, \ldots\ldots, N_n$ are $\mathcal{F}$-cocyclic left R-modules in $\sigma[M]$. Thus there exist $\mathcal{F}$-cocritical left R-modules $C_1, \ldots\ldots, C_n \in \sigma[M]$ such that the sequence

$$0 \longrightarrow N_{\mathcal{F}} \longrightarrow \oplus_{i=1}^{n} I(C_i)_{\mathcal{F}}$$

is exact. Thus N is $\mathcal{F}$-finitely cogenerated in $\sigma[M]$.

(3) $\Longrightarrow$ (1). Let N be an $\mathcal{F}$-cocritical left R-module in $\sigma[M]$. Then N is $\mathcal{F}$-torsionfree and $N_{\mathcal{F}}$ is a simple object of $(R, \mathcal{F}) - Mod$. Since $I(N)$ is M-injective, it is easy to see that $I(N)$ is $\mathcal{F}$-closed. Thus $i(I(N)_{\mathcal{F}}) \simeq I(N) \in$

$\sigma[M]$; and hence $i(N_{\mathcal{F}})$, a submodule of $i(I(N)_{\mathcal{F}})$, is in $\sigma[M]$. Since the sequence

$$0 \longrightarrow (i(N_{\mathcal{F}}))_{\mathcal{F}} \longrightarrow (i(I(N)_{\mathcal{F}}))_{\mathcal{F}}$$

is exact, and $(i(I(N)_{\mathcal{F}}))_{\mathcal{F}} \simeq I(N)_{\mathcal{F}}$, we see that $i(N_{\mathcal{F}})$ is $\mathcal{F}$-cocyclic in $\sigma[M]$. It is easy to see that $i(I(N)_{\mathcal{F}}) \in \sigma[M]$ is $\mathcal{F}$-cocyclic, too. Thus $i(N_{\mathcal{F}}) \oplus i(I(N)_{\mathcal{F}})$ is $\sigma - \mathcal{F}$-cocyclic. Clearly $i(N_{\mathcal{F}}) \oplus i(I(N)_{\mathcal{F}})$ is $\mathcal{F}$-torsionfree. Thus, by condition (3), there exists an $\mathcal{X}$-module L such that $i(N_{\mathcal{F}}) \oplus i(I(N)_{\mathcal{F}})$ is an essential submodule of L and $(i(N_{\mathcal{F}}) \oplus i(I(N)_{\mathcal{F}}))_{\mathcal{F}} = L_{\mathcal{F}}$. Thus we get

$$L_{\mathcal{F}} \simeq N_{\mathcal{F}} \oplus I(N)_{\mathcal{F}}.$$

Clearly L is $\mathcal{F}$-closed since it is $\mathcal{F}$-torsionfree and $\mathcal{F}$-injective. Thus

$$i(N_{\mathcal{F}}) \oplus i(I(N)_{\mathcal{F}}) \simeq i(L_{\mathcal{F}}) \simeq L.$$

This implies that $i(N_{\mathcal{F}}) \oplus i(I(N)_{\mathcal{F}})$ is an $\mathcal{X}$-module. From the definition of I-classes it follows that $i(N_{\mathcal{F}})$ is $i(I(N)_{\mathcal{F}})$-injective. Thus there exists a homomorphism $f : i(I(N)_{\mathcal{F}}) \longrightarrow i(N_{\mathcal{F}})$ such that $f|_{i(N_{\mathcal{F}})} = 1$. This implies that there exists a left R-module H such that $i(I(N)_{\mathcal{F}}) = i(N_{\mathcal{F}}) \oplus H$. Thus $I(N)_{\mathcal{F}} = N_{\mathcal{F}} \oplus a(H)$. But $N_{\mathcal{F}}$ is essential in $I(N)_{\mathcal{F}}$ since N is essential in $I(N)$. Thus $N_{\mathcal{F}} = I(N)_{\mathcal{F}}$ and we are done.

(2) $\Longrightarrow$ (4). Let the left R-module $N \in \sigma[M]$ be $\mathcal{F}$-torsionfree and $\mathcal{F}$-finitely cogenerated in $\sigma[M]$. By (2), there exists an essential extension L of N such that L is an $\mathcal{X}$-module and N is dense in L. By [3, Proposition 1.2], we have

$$(\mathrm{Rad}_{\mathcal{F}}(L))_{\mathcal{F}} = \mathrm{Rad}(L_{\mathcal{F}}) = \mathrm{Rad}(N_{\mathcal{F}}).$$

Since N is $\mathcal{F}$-finitely cogenerated in $\sigma[M]$, there exist $\mathcal{F}$-cocritical left R-modules $C_1, \ldots\ldots, C_n$ in $\sigma[M]$ such that the following sequence is exact:

$$0 \longrightarrow N_{\mathcal{F}} \longrightarrow \oplus_{i=1}^{n} I(C_i)_{\mathcal{F}}.$$

By the results proved above, M is $\mathcal{F}$-co-semisimple when condition (2) holds. Therefore every $\mathcal{F}$-cocritical left R-module in $\sigma[M]$ is dense in its M-injective hull. It follows that $I(C_i)_{\mathcal{F}} = (C_i)_{\mathcal{F}}, i = 1, \ldots\ldots, n$. Thus we have

$$\mathrm{Rad}(\oplus_{i=1}^{n} I(C_i)_{\mathcal{F}}) = \oplus_{i=1}^{n} \mathrm{Rad}(I(C_i)_{\mathcal{F}}) = 0,$$

which implies that $\mathrm{Rad}(N_{\mathcal{F}}) = 0$. Thus $\mathrm{Rad}_{\mathcal{F}}(L) = t(\mathrm{Rad}_{\mathcal{F}}(L))$, where t is a left exact radical corresponding to the Gabriel topology $\mathcal{F}$. This means that $\mathrm{Rad}_{\mathcal{F}}(L)$ is an $\mathcal{F}$-torsion module. On the other hand, $\mathrm{Rad}_{\mathcal{F}}(L)$ is an $\mathcal{F}$-saturated submodule of L, and so is $\mathcal{F}$-torsionfree. Therefore we get $\mathrm{Rad}_{\mathcal{F}}(L) = 0$, as required.

(4) $\Longrightarrow$ (5). It is similar to the implication (2) $\Longrightarrow$ (3).

(5) $\Longrightarrow$ (3). Let N be an $\mathcal{F}$-torsionfree and $\sigma-\mathcal{F}$-cocyclic left R-module in $\sigma[M]$. Then there exists an essential extension L of N such that L is an $\mathcal{X}$-module and $\mathrm{Rad}_{\mathcal{F}}(L)=0$. It is enough to show that N is dense in L. By [3, Proposition 1.3], it follows that L is cogenerated by the class of $\mathcal{F}$-cocritical left R-modules. Since N is $\sigma-\mathcal{F}$-cocyclic in $\sigma[M]$, as the implication (2) $\Longrightarrow$ (3), we see N is $\mathcal{F}$-finitely cogenerated in $\sigma[M]$. Thus there exist $\mathcal{F}$-cocritical left R-modules $C_1,\ldots\ldots,C_n$ such that the sequence

$$0\longrightarrow N_{\mathcal{F}}\longrightarrow \oplus_{i=1}^{n}I(C_i)_{\mathcal{F}}$$

is exact. By analogy with the implication (1) $\Longrightarrow$ (2), we obtain that $I(N)_{\mathcal{F}}$ is isomorphic to a subobject of $\oplus_{i=1}^{n}I(C_i)_{\mathcal{F}}$. By [14, 17.10], it follows that $I(N)\simeq I(L)$ since N is essential in L. Thus $I(N)_{\mathcal{F}}\simeq I(L)_{\mathcal{F}}$, which implies that $L_{\mathcal{F}}$ can be embedded to $\oplus_{i=1}^{n}I(C_i)_{\mathcal{F}}$. This means that L is $\mathcal{F}$-finitely cogenerated in $\sigma[M]$. Thus clearly L is $\mathcal{F}$-finitely cogenerated in $R-Mod$. By [3, Proposition 1.7], every family of $\mathcal{F}$-torsionfree modules which cogenerates L does cogenerate it finitely. Thus there exist finite $\mathcal{F}$-cocritical left R-modules $D_1,\ldots\ldots,D_k$ such that the sequence $0\longrightarrow L\longrightarrow \oplus_{i=1}^{k}D_i$ is exact, which implies that the sequence $0\longrightarrow L_{\mathcal{F}}\longrightarrow \oplus_{i=1}^{k}(D_i)_{\mathcal{F}}$ is exact. This means that $L_{\mathcal{F}}$ is a semisimple object of $(R,\mathcal{F})-Mod$. On the other hand, N is an essential submodule of L and so $N_{\mathcal{F}}$ is essential in $L_{\mathcal{F}}$ by [3, Lemma 0.1]. It is then easy to see that $N_{\mathcal{F}}=L_{\mathcal{F}}$; in other words, N is dense in L and we are done. ■

The following corollary generalizes a corresponding result of [7].

Corollary 3.4. *Let $\mathcal{F}$ be a left Gabriel topology on R. If $\mathcal{X}$ is an I-class of left R-modules such that every $\mathcal{X}$-module is $\mathcal{F}$-injective, then the following conditions are equivalent.*

(1) *R is an $\mathcal{F}$-V-ring.*

(2) *Every $\mathcal{F}$-torsionfree and $\mathcal{F}$-finitely cogenerated left R-module is dense in its some essential extensions which are $\mathcal{X}$-modules.*

(3) *Every $\mathcal{F}$-torsionfree and $\sigma-\mathcal{F}$-cocyclic left R-module is dense in its some essential extensions which are $\mathcal{X}$-modules.*

(4) *For every $\mathcal{F}$-torsionfree and $\mathcal{F}$-finitely cogenerated left R-module M, there exists an $\mathcal{X}$-module L with essential submodule M such that* $\mathrm{Rad}_{\mathcal{F}}(L)=0$.

(5) *For every $\mathcal{F}$-torsionfree and $\sigma-\mathcal{F}$-cocyclic left R-module M, there exists an $\mathcal{X}$-module L with essential submodule M such that* $\mathrm{Rad}_{\mathcal{F}}(L)=0$.

Corollary 3.5. *Let M be a left R-module and $\mathcal{X}$ an I-class in the category $\sigma[M]$. Then the following assertions are equivalent.*

(1) *M is co-semisimple.*
(2) *Every finitely cogenerated left R-module in $\sigma[M]$ is an $\mathcal{X}$-module.*
(3) *Every σ-cocyclic left R-module in $\sigma[M]$ is an $\mathcal{X}$-module.*

For $M = R$, Corollary 3.5 gives characterizations of left V-rings by generalized injectivity.

Page and Yousif in [10] proved that a finitely generated left R-module M is a noetherian co-semisimple module if and only if every semisimple left R-module is M-injective. Recall that a left R-module M is locally noetherian if every finitely generated submodule of M is noetherian. We have

Proposition 3.6. *Let M be a left R-module and $\mathcal{X}$ an I-class in the category $\sigma[M]$. Then the following conditions are equivalent.*

(1) *M is a locally noetherian co-semisimple left R-module.*
(2) *Every semisimple left R-module (in $\sigma[M]$) is M-injective.*
(3) *Every semisimple left R-module (in $\sigma[M]$) is the direct sum of a finitely cogenerated left R-module and an M-injective module.*
(4) *For every semisimple left R-module N in $\sigma[M]$, every essential extension in $\sigma[M]$ of N is an $\mathcal{X}$-module.*
(5) *For every semisimple left R-module N in $\sigma[M]$, every submodule of an essential extension in $\sigma[M]$ of N is an $\mathcal{X}$-module.*

If $\mathcal{X}$ is closed under direct summands, then the following are also equivalent.

(6) *For every semisimple left R-module N in $\sigma[M]$, every essential extension in $\sigma[M]$ of N is the direct sum of a finitely cogenerated module and an $\mathcal{X}$-module.*
(7) *For every semisimple left R-module N in $\sigma[M]$, every submodule of an essential extension in $\sigma[M]$ of N is the direct sum of a finitely cogenerated module and an $\mathcal{X}$-module.*

Proof. The equivalence of (1), (2) and (3) is proved in [8].

(4) $\Longrightarrow$ (2). Let N be a semisimple left R-module in $\sigma[M]$. Then $N \oplus I(N)$ is an essential extension of $N \oplus N$. Thus $N \oplus I(N)$ is an $\mathcal{X}$-module by condition (4), which implies that N is $I(N)$-injective by the definition of I-class. Now it is easy to see that $N = I(N)$ is M-injective.

(2) $\Longrightarrow$ (5). Let N be a semisimple left R-modules in $\sigma[M]$ and L a submodule of an essential extension D in $\sigma[M]$ of N. Then N is M-injective

by (2). Thus N is D-injective by [14, 16.3]. Now it is easy to see that $N = D$ and thus D is semisimple and M-injective. Therefore L, a direct summand of D, is M-injective. Since $\mathcal{X}$ contains all M-injective left R-modules in $\sigma[M]$, it follows that every submodule of an essential extension in $\sigma[M]$ of a semisimple left R-module in $\sigma[M]$ is an $\mathcal{X}$-module.

The implications (5) $\Longrightarrow$ (4), (7) $\Longrightarrow$ (6) and (5) $\Longrightarrow$ (7) are clear.

(6) $\Longrightarrow$ (4). Note that the class of semisimple left R-modules is closed under direct sums. It is easy to see that every direct sum of essential extensions in $\sigma[M]$ of semisimple left R-modules in $\sigma[M]$ is essential extension of a semisimple module. Now, by analogy with the proof of the main result of [6], we see that every essential extension in $\sigma[M]$ of a semisimple left R-module in $\sigma[M]$ is an $\mathcal{X}$-module. ■

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