TAIWANESE JOURNAL OF MATHEMATICS
Vol. 28, No. 2, pp. 249–295, April 2024
DOI: 10.11650/tjm/231206

On Successive Minimal Bases of Division Points of Drinfeld Modules

Maozhou Huang

Abstract. We define successive minimal bases (SMBs) for the space of u^n-division points of a Drinfeld $\mathbb{F}_q[t]$-module over a local field, where u is a finite prime of $\mathbb{F}_q[t]$ and n is a positive integer. These SMBs share similar properties to those of SMBs of the lattices associated to Drinfeld modules. We study the relations between these SMBs and those of the lattices. Finally, we apply the relations to study the explicit wild ramification subgroup action on an SMB of the space of u^n-division points and show a function field analogue of Szpiro's conjecture for rank 2 Drinfeld modules under a certain limited situation.

1. Introduction

1.1. Notations

Let us introduce the notation used throughout this paper. Put $A := \mathbb{F}_q[t]$, where $\mathbb{F}_q[t]$ is the polynomial ring in t over the field $\mathbb{F}_q$ whose order is a power of a rational prime p. Let F be a global function field which is a finite extension of the fraction field of A. Let K be the completion of F at a prime w. We also let w denote the valuation associated to K normalized so that $w(K^\times) = \mathbb{Z}$. Fix K^{sep} (resp. K^{alg}) a separable (resp. algebraic) closure of K. Let $\mathbb{C}_w$ denote the completion of K^{alg}. If w is an infinite prime, we also let $\mathbb{C}_\infty$ denote $\mathbb{C}_w$.

Let ϕ be a rank r Drinfeld A-module over K. For an element a in A, let $\phi[a]$ be the A/a-module of a-division points in K^{sep}. It is a free module of rank r. Fix a positive integer n and a finite prime u of A, i.e., a monic irreducible polynomial $u \in A$. The main research objects in this paper are successive minimal bases of $\phi[u^n]$ defined below. For $a \in A$ and $x \in \phi[u^n]$, write $a \cdot_\phi x := \phi_a(x)$ for the action of a on x.

If w is an infinite prime, let Λ denote the rank r A-lattice in $\mathbb{C}_\infty$ and e_ϕ the exponential function from $\mathbb{C}_\infty$ to $\mathbb{C}_\infty$ associated to ϕ via the uniformization. Here we have considered Λ and the domain of e_ϕ as A-modules via the natural embedding $A \to \mathbb{C}_\infty$.

If w is a finite prime, we assume throughout this paper that ϕ has stable reduction over K and the reduction of ϕ has rank $r' \leq r$ unless otherwise specified. Let ψ denote the

Received May 24, 2023; Accepted December 29, 2023.
Communicated by Liang-Chung Hsia.
2020 *Mathematics Subject Classification.* Primary: 11G09; Secondary: 11S15.
Key words and phrases. Drinfeld modules, successive minimal bases, higher ramification subgroups, conductors, Szpiro conjecture.

rank r' Drinfeld module over K having good reduction, Λ the rank $r-r'$ A-lattice in $\mathbb{C}_w$, and e_ϕ the exponential function from $\mathbb{C}_w$ to $\mathbb{C}_w$ associated to ϕ via the Tate uniformization (see [6, Section 7] or Section 2.1). Here we consider Λ and the domain of e_ϕ as A-modules via ψ, i.e., we have the action of a on ω to be $a \cdot_\psi \omega := \psi_a(\omega)$ for any $a \in A$ and any ω in Λ or $\mathbb{C}_w$.

Let $|-|$ denote one of the following functions.

(F1) If w is an infinite prime, we have the absolute value $|-|$ on K which extends the absolute value $|-| = q^{\deg(-)}$ on $\mathbb{F}_q((\frac{1}{t}))$. This absolute value may be extended to $\mathbb{C}_\infty$.

(F2) Assume that w is a finite prime of F. Following [8, Section 1], define a function $|-|$ on K by

$$\text{for } x \in K, \quad |x| = \begin{cases} (-w(x))^{1/r'} & \text{if } w(x) < 0, \\ -w(x)^{1/r'} & \text{if } w(x) \geq 0, \\ |0| = -\infty & \text{if } x = 0. \end{cases}$$

We may extend this function to $\mathbb{C}_w$. This function is not an absolute value. However, the ultrametic inequality holds. For $x \in \mathbb{C}_w$, we still call $|x|$ the absolute value of x.

1.2. On SMBs of u^n-division points

The main definition is

Definition 1.1. Let $|-|$ denote the function in (F1) or (F2). We call a family of elements $\{\lambda_i\}_{i=1,\ldots,r}$ an *SMB* (*successive minimal basis*) of $\phi[u^n]$ if for each i, the elements $\lambda_1, \ldots, \lambda_i$ in $\phi[u^n]$ satisfy

(1) $\lambda_1, \ldots, \lambda_i$ are A/u^n-linearly independent;

(2) $|\lambda_i|$ is minimal among the absolute values of elements λ in $\phi[u^n]$ such that $\lambda_1, \ldots, \lambda_{i-1}, \lambda$ are A/u^n-linearly independent.

Here we have imitated the definition of SMBs of the lattices Λ (see [17, Section 4] or [10, Section 3]). Let us remark that

Remark 1.2. (1) in the definition implies that $\{\lambda_1, \ldots, \lambda_r\}$ is an A/u^n-basis (or a generating set) of $\phi[u^n]$. The condition (2) above can be replaced with "$w(\lambda_i)$ is the largest among the valuations of elements λ in $\phi[u^n]$ such that $\lambda_1, \ldots, \lambda_{i-1}, \lambda$ are A/u^n-linearly independent". In Definition 2.8, we will extend the definition of SMBs of $\phi[u^n]$ to the case where ϕ does not necessarily have stable reduction over K.

If w is a finite prime, let $u^{-n}\Lambda$ denote the A-module consisting of all roots of $\psi_{u^n}(X)-\omega$ for all $\omega \in \Lambda$. For any infinite or finite prime w, by the uniformization or the Tate uniformization of ϕ, we have an isomorphism of A/u^n-modules

$$\mathcal{E}_\phi\colon u^{-n}\Lambda/\Lambda \to \phi[u^n]$$

induced by e_ϕ. Hence one may expect that there are relations between SMBs of $\phi[u^n]$ and those of Λ.

Let $|-|$ denote the absolute value in (F1) (resp. the function in (F2)) if w is an infinite prime (resp. a finite prime). Put $|u^n|_\infty = q^{\deg(u^n)}$.

Theorem 1.3. (1) *Let w be an infinite prime.*

- (see Theorem 3.3) *Let $\{\omega_i\}_{i=1,\dots,r}$ be an SMB of Λ. Then the images $e_\phi(\omega_i/u^n)$ for $i=1,\dots,r$ form an SMB of $\phi[u^n]$.*
- (see Corollary 3.11(1)) *Let l be a positive integer and $\{\eta_i\}_{i=1,\dots,r}$ an SMB of $\phi[u^l]$. Let $\{\lambda_i\}_{i=1,\dots,r}$ be an SMB of $\phi[u^n]$. Assume that n satisfies $|u^n|_\infty > |\eta_r|/|\eta_1|$. Under this assumption, for each $i=1,\dots,r$, the element λ_i has only one preimage under e_ϕ, denoted $\log_\phi(\lambda_i)$, with absolute value $< |\omega|$ for any $\omega \in \Lambda\setminus\{0\}$. Then the family of elements $\{u^n \log_\phi(\lambda_i)\}_{i=1,\dots,r} \subset \mathbb{C}_\infty$ is an SMB of Λ.*

(2) *Let w be a finite prime.*

- (see Theorem 4.6) *Let $\{\omega_i\}_{i=1,\dots,r'}$ (resp. $\{\omega_i^0\}_{i=r'+1,\dots,r}$) be an SMB of $\psi[u^n]$ (resp. Λ). Let ω_i be a root of $\psi_{u^n}(X)-\omega_i^0$ for $i=r'+1,\dots,r$. Then the images $e_\phi(\omega_i)$ for $i=1,\dots,r$ form an SMB of $\phi[u^n]$.*
- (see Corollary 4.12(1) and (2)) *Let l be a positive integer and $\{\eta_i\}_{i=1,\dots,r}$ an SMB of $\phi[u^l]$. Let $\{\lambda_i\}_{i=1,\dots,r}$ be an SMB of $\phi[u^n]$. Assume that n satisfies $|u^n|_\infty > |\eta_r|/|\eta_{r'+1}|$. Under this assumption, for each $i=1,\dots,r$, the element λ_i has only one preimage, denoted $\log_\phi(\lambda_i)$, with absolute value $< |\omega|$ for any $\omega \in \Lambda\setminus\{0\}$. Then the family of elements $\{\log_\phi(\lambda_i)\}_{i=1,\dots,r'} \subset \mathbb{C}_w$ (resp. $\{u^n \cdot_\psi \log_\phi(\lambda_i)\}_{i=r'+1,\dots,r} \subset \mathbb{C}_w$) is an SMB of $\psi[u^n]$ (resp. of Λ).*

It turns out that the SMBs of $\phi[u^n]$ have the following properties.

Proposition 1.4. *Let $\{\lambda_i\}_{i=1,\dots,r}$ be an SMB of $\phi[u^n]$.*

(1) (see Proposition 2.10) *The sequence $|\lambda_1| \le |\lambda_2| \le \cdots \le |\lambda_r|$ associated to an SMB of $\phi[u^n]$ is an invariant of $\phi[u^n]$, i.e., for any SMB $\{\lambda_i'\}_{i=1,\dots,r}$ of $\phi[u^n]$, we have $|\lambda_i'| = |\lambda_i|$ for all i.*

(2) (see Proposition 3.12 and 4.13) *Assume that u is not divisible by the prime w, i.e., $w(u) \leq 0$. Then we have*

$$\left|\sum_i a_i \cdot_\phi \lambda_i\right| = \max_i \left\{|a_i \cdot_\phi \lambda_i|\right\}$$

for any $a_i \in A \mod u^n$.

(3) (see Proposition 2.13) *There exists an SMB $\{\lambda_i'\}_{i=1,\dots,r}$ of $\phi[u^{n+1}]$ such that $u \cdot_\phi \lambda_i' = \lambda_i$ for all i. The elements $u \cdot_\phi \lambda_i$ for $i = 1, \dots, r$ form an SMB of $\phi[u^{n-1}]$.*

Here the properties (1) and (2) are similar to those of SMBs of lattices (see Propositions 2.4 and 2.5). We remark that (2) essentially follows from similar properties of SMBs of lattices (see Proposition 2.5 or [17, Lemma 4.2]). We hope to know whether the condition "$w(u) \leq 0$" in (2) can be removed.

Let $K(\Lambda)$ (resp. $K(u^{-n}\Lambda)$ and $K(\phi[u^n])$) denote the extension of K generated by all elements in Λ (resp. $u^{-n}\Lambda$ and $\phi[u^n]$). By Theorem 1.3, we are able to show

Proposition 1.5. *Let l be a positive integer and $\{\eta_i\}_{i=1,\dots,r}$ an SMB of $\phi[u^l]$. Let $\{\lambda_i\}_{i=1,\dots,r}$ be an SMB of $\phi[u^n]$.*

(1) (see Corollary 3.11(2)) *If w is an infinite prime and n is large enough so that $|u^n|_\infty > |\eta_r|/|\eta_1|$, then we have $K(\Lambda) = K(\phi[u^n])$.*

(2) (see Corollary 4.12(3)) *If w is a finite prime and n is large enough so that $|u^n|_\infty > |\eta_r|/|\eta_{r'+1}|$, then we have $K(u^{-n}\Lambda) = K(\phi[u^n])$.*

The claim (1) is an effective version of [12, Proposition 2.1].

1.3. An application to rank 2 Drinfeld modules

Let u be a finite prime of A. Let ϕ be a rank 2 Drinfeld A-module over K which does not necessarily have stable reduction when w is finite. Let $\{\lambda_i\}_{i=1,2}$ be an SMB of $\phi[u^n]$. Let $\boldsymbol{j}$ denote the j-invariant of ϕ. Assume

$$\begin{cases} \text{either } \big(w(\boldsymbol{j}) < w(t)q \text{ and } p \nmid w(\boldsymbol{j})\big), \text{ or } w(\boldsymbol{j}) \geq w(t)q & \text{if } w \text{ is infinite,} \\ \text{either } \big(w(\boldsymbol{j}) < 0 \text{ and } p \nmid w(\boldsymbol{j})\big), \text{ or } w(\boldsymbol{j}) \geq 0 & \text{if } w \text{ is finite.} \end{cases} \tag{1.1}$$

For a positive integer n, let $G(n)_1$ denote the wild ramification subgroup, i.e., the first lower ramification subgroup, of $\mathrm{Gal}(K(\phi[u^n])/K)$. In [1, Theorems 3.9 and 3.13, Lemmas 3.14 and 3.15], for u having degree 1 and any n, the action of $G(n)_1$ on $\{\lambda_i\}_{i=1,2}$ has been studied assuming moreover $q \neq 2$ when w is a finite prime. In Sections 5 and 6, we study the action of $G(n)_1$ on $\{\lambda_i\}_{i=1,2}$ *without* requiring $(\deg(u) = 1)$ and ($q \neq 2$ when w is finite).

Theorem 1.6. *Let ϕ be a rank 2 Drinfeld A-module over K which does not necessarily have stable reduction when w is finite. Let u be a finite prime of A with $\deg(u) = d$. Let $\{\lambda_i\}_{i=1,2}$ be an SMB of $\phi[u^n]$.*

(1) (see Theorem 5.9) *Let w be an infinite prime. Assume $w(\boldsymbol{j}) < w(t)q$ and $p \nmid w(\boldsymbol{j})$. Let m be the integer such that $w(\boldsymbol{j}) \in (w(t)q^{m+1}, w(t)q^m)$. Put $d = \deg(u)$. Assume $n \geq m/d$.*

- *Any element in $G(n)_1$ fixes λ_1;*
- *Let $A^{<m}$ denote the subgroup of A consisting of elements with degree $< m$. Then the map*
$$G(n)_1 \to A^{<m} \cdot_\phi \lambda_1, \quad \sigma \mapsto \sigma(\lambda_2) - \lambda_2$$
is an isomorphism of groups.

(2) (see Corollary 6.4) *Let w be a finite prime satisfying $w \nmid u$. Assume $w(\boldsymbol{j}) < 0$ and $p \nmid w(\boldsymbol{j})$.*

- *Any element in $G(n)_1$ fixes λ_1;*
- *There is an isomorphism of groups*
$$G(n)_1 \to A \cdot_\phi \lambda_1, \quad \sigma \mapsto \sigma(\lambda_2) - \lambda_2.$$

Example 5.10 provides an instance where w is an infinite prime, $w(\boldsymbol{j}) < w(t)q$, $p \mid w(\boldsymbol{j})$, and the extension $K(\phi[u^n])/K$ is not wildly ramified. Let us remark that (1) if w is an infinite prime and $w(\boldsymbol{j}) \geq w(t)q$, the extension $K(\phi[u^n])/K$ is at worst tamely ramified such that $G(n)_1$ is a trivial group for any $n \geq 1$; (2) if w is a finite prime and $w(\boldsymbol{j}) \geq 0$, then ϕ has potentially good reduction at w such that the extension $K(\phi[u^n])/K$ is at worst tamely ramified and the group $G(n)_1$ is trivial for any $n \geq 1$.

Let ϕ be a rank 2 Drinfeld A-module over F. With the assumptions on its j-invariant in (1.1), we define and calculate the conductors of ϕ at each prime w of F using the u-adic Tate module with $u \nmid w$. Finally, we show a function field analogue of Szpiro's conjecture in Theorem 6.6, which slightly generalizes [1, Theorem 4.3].

Motivated by [9, Proposition 3.2], we may expect that there are generalizations of the results in Sections 5 and 6 to Drinfeld A-modules ϕ of rank r over K satisfying $\phi_t(X) = tX + a_s X^{q^s} + a_r X^{q^r} \in K[X]$. We have obtained a generalization of Proposition 5.1 for such ϕ (see Remark 5.2). There are difficulties in generalizing Theorem 1.6. We do not further investigate the generalizations in the present paper. Some partial results will appear in the author's doctoral thesis [11]. For instance, the explicit action of the wild ramification subgroup $\mathrm{Gal}(K(\phi[t])/K)_1$ on an SMB $\{\xi_i\}_{i=1,\dots,r}$ of $\phi[t]$ has been worked out in [11, Theorem 3.3.16] under certain limited situations.

1.4. Contents

Except for Section 6.3, we consider Drinfeld A-modules over a localization K of a global function field. In Section 2, we first review the basics of the SMB of lattices. The rest of this section is devoted to the basics of SMBs of $\phi[u^n]$. In Section 3, we mainly show the infinite prime case of Theorem 1.3. For an element ω_i of an SMB of the lattice Λ as in Theorem 1.3(1) and an element a_i in A with a limited degree, we describe the absolute value of $e_\phi(a_i\omega_i)$ in Corollary 3.2(1). This is the key result of this section and its proof is inspired by that of [10, Lemma 3.4]. Section 4 consists of finite prime analogues of the results in Section 3. The analogue of Corollary 3.2(1) is Corollary 4.5(1).

In Section 5 (resp. Section 6), we apply the results in the previous sections to a rank 2 Drinfeld A-module ϕ over K with w being infinite (resp. finite). We first calculate the valuations of elements of SMBs of Λ and $\phi[u^n]$ in Sections 5.1 and 6.1. In Section 5.2, we calculate the conductors of ϕ in Lemma 5.5. Then we study the action of the wild ramification subgroup of the Galois group $\mathrm{Gal}(K(\phi[u^n])/K)$ on an SMB of $\phi[u^n]$ in Theorem 5.9. Section 6.2 consists of finite prime analogues of the results in Section 5.2. In Section 6.3, we obtain a function field analogue of Szpiro's conjecture under certain assumptions.

In Appendix A, when w is an infinite prime, the conductor of a rank r Drinfeld A-module over K is defined. In Appendix B, we calculate the Herbrand ψ-function of the extension of K generated by the roots of a certain polynomial with degree being a power of q.

2. Basics of SMBs

Let $|-|$ denote the absolute value in (F1) (resp. the function in (F2)) if w is an infinite prime (resp. a finite prime) defined in Section 1.1.

2.1. SMBs of lattices

In this subsection, we recall first the basics of SMBs of lattices and then the (Tate) uniformization of Drinfeld A-modules. Consider $\mathbb{C}_\infty$ as an A-module via the embedding $A \to \mathbb{C}_\infty$. If w is a finite prime, consider $\mathbb{C}_w$ as an A-module via a Drinfeld A-module ψ having good reduction of rank r'. The next lemma will be applied implicitly in this paper.

Lemma 2.1. (1) *If w is an infinite prime, we have $|a\omega| = |a| \cdot |\omega|$ for any $a \in A$ and $\omega \in \mathbb{C}_\infty$.*

(2) (see [8, Section 1]) *Let w be a finite prime. Then we have $|a \cdot_\psi \omega| = |a|_\infty \cdot |\omega|$, i.e., $w(a \cdot_\psi \omega) = |a|_\infty^{r'} \cdot w(\omega)$ for any $a \in A$ and any $\omega \in \mathbb{C}_w$ having valuation < 0, where $|a|_\infty = q^{\deg(a)}$.*

Proof. (1) is clear. We show (2). Put $g = r' \cdot \deg(a)$, $a_0 = a$ and $\sum_{i=0}^{g} a_i X^{q^i} = \psi_a(X)$. As the Drinfeld module ψ has good reduction, we have $w(a_i) \geq 0$ and $w(a_g) = 0$. Hence the assumption $w(\omega) < 0$ implies that the valuation $w(a_g \omega^{q^g})$ is the strictly smallest among $w(a_i \omega^{q^i})$ for all i. As $w(a_g) = 0$, we have $w(a_g \omega^{q^g}) = q^g w(\omega)$, i.e., $|a \cdot_\psi \omega| = |a|_\infty \cdot |\omega|$. □

Let L be an A-lattice of rank r in $\mathbb{C}_\infty$ or an A-lattice of rank r in $\mathbb{C}_w$ such that each nonzero element in the lattice has valuation < 0.

Definition 2.2. (see [17, Section 4] or [10, Section 3]) A family of elements $\{\omega_i\}_{i=1,\ldots,r}$ in L is called an SMB of L if for each i, the elements $\omega_1, \ldots, \omega_i$ satisfy

(1) $\omega_1, \ldots, \omega_i$ are A-linearly independent;

(2) $|\omega_i|$ is minimal among the absolute values of elements ω in L such that $\omega_1, \ldots, \omega_{i-1}, \omega$ are A-linearly independent.

Remark 2.3. If elements λ_i for $i = 1, \ldots, r$ of $\phi[u^n]$ are A/u^n-linearly independent (cf. Definition 1.1(1)), then $\{\lambda_i\}_{i=1,\ldots,r}$ is an A/u^n-basis of $\phi[u^n]$. On the other hand, if elements ω_i for $i = 1, \ldots, r$ of L are A-linearly independent, then $\{\omega_i\}_{i=1,\ldots,r}$ is not necessarily an A-basis of L.

Proposition 2.4. *Let $\{\omega_i\}_{i=1,\ldots,r}$ be a family of elements in L.*

(1) *This family is an SMB if and only if for each i, the elements $\omega_1, \ldots, \omega_i$ satisfy*

- *$\omega_1, \ldots, \omega_i$ are A-linearly independent;*
- *we have $|\omega_i| = l_i$, where*

$$l_i = \min\big\{\rho \in \mathbb{R} \mid \textit{the ball in } \mathbb{C}_\infty \textit{ or } \mathbb{C}_w \textit{ around } 0 \textit{ of radius } \rho \textit{ contains at least } i \textit{ elements in } L \textit{ which are } A\textit{-linearly independent}\big\}.$$

(2) *The sequence $|\omega_1| \leq |\omega_2| \leq \cdots \leq |\omega_r|$ for an SMB $\{\omega_i\}_{i=1,\ldots,r}$ is an invariant of L, i.e., for any SMB $\{\omega'_i\}_{i=1,\ldots,r}$ of L, we have $|\omega_i| = |\omega'_i|$ for all i.*

Proposition 2.5. *Let $\{\omega_i\}_{i=1,\ldots,r}$ be a family of elements in L so that $|\omega_1| \leq |\omega_2| \leq \cdots \leq |\omega_r|$. Then this family is an SMB of L if and only if*

(1) *$\omega_1, \ldots, \omega_r$ form an A-basis of L;*

(2) *we have $\big|\sum_i a_i \omega_i\big| = \max_i\{|a_i \omega_i|\}$ for any $a_i \in A$.*

Proof. This has been proved in [17, Lemma 4.2]. □

For the subfield K of $\mathbb{C}_w$, we say that L is $\mathrm{Gal}(K^{\mathrm{sep}}/K)$-invariant if each element in the Galois group maps L into L. The following lemma concerns the extension generated by elements ω in the lattice with $|\omega|$ being minimal.

Lemma 2.6. *Let $\{\omega_i\}_{i=1,\dots,r}$ be an SMB of L such that $|\omega_1| = \cdots = |\omega_s| < |\omega_{s+1}|$ for some positive integer $s < r$. Assume that*

- *the extension M/K generated by ω_i for $i = 1, \dots, s$ is separable;*
- *the lattice L is $\mathrm{Gal}(K^{\mathrm{sep}}/K)$-invariant.*

(1) *The extension M/K is Galois.*

(2) *The extension M/K is at worst tamely ramified.*

Proof. We show (1). Let $\widehat{M}$ denote the Galois closure of M/K so that $\widehat{M}$ is exactly the compositum of ςM for all $\varsigma \in \mathrm{Gal}(\widehat{M}/K)$. We have $\widehat{M} = M$. Indeed, if $\widehat{M}/M$ is nontrivial, there exists some element $\varsigma \in \mathrm{Gal}(\widehat{M}/K)$ such that $\varsigma(\omega_j) \notin M$ for j to be one of $1, \dots, s$. Note that M contains the A-module $\bigoplus_{i=1,\dots,s} A\omega_i$ (here $A\omega_i := \{a \cdot_\psi \omega_i \mid a \in A\}$ if the prime w is finite). As elements in $L \setminus \bigoplus_{i=1,\dots,s} A\omega_i$ have strictly smaller valuations than that of ω_i for $i = 1, \dots, s$ and Galois actions preserve valuations, this implies that $\varsigma(\omega_j) \notin L$. If ς also denotes a preimage of ς under $\mathrm{Gal}(K^{\mathrm{sep}}/K) \to \mathrm{Gal}(\widehat{M}/K)$, then $\varsigma(\omega_j) \notin L$ contradicts that L is $\mathrm{Gal}(K^{\mathrm{sep}}/K)$-invariant.

As for (2), we show that M/K is tamely ramified. Assume the converse so that the wild ramification subgroup $\mathrm{Gal}(M/K)_1$ is nontrivial. Let w_M denote the normalized valuation associated to M. For σ to be a nontrivial element in $\mathrm{Gal}(M/K)_1$, we have

$$1 \leq w_M(\sigma(\omega_i)\omega_i^{-1} - 1)$$

for each i. We also have $\sigma(\omega_j) - \omega_j \neq 0$ for j to be one of $1, \dots, s$. Note that $w_M(\omega_j)$ is the largest among the valuations of all nonzero elements in L. As $\sigma(\omega_j) - \omega_j \in L$ (L is $\mathrm{Gal}(K^{\mathrm{sep}}/K)$-invariant), we have

$$w_M(\sigma(\omega_j)\omega_j^{-1} - 1) = w_M(\sigma(\omega_j) - \omega_j) - w_M(\omega_j) \leq 0.$$

This gives a contradiction. □

Next, we recall the uniformization and the Tate uniformization. Let ϕ be a rank r Drinfeld A-module over K. If w is an infinite prime, then the uniformization associates to the Drinfeld module ϕ a $\mathrm{Gal}(K^{\mathrm{sep}}/K)$-invariant A-lattice Λ and an exponential function e_ϕ on $\mathbb{C}_\infty$ such that for each $a \in A$, the following diagram commutes, and its two rows are short exact sequences

$$\begin{tikzcd} \Lambda \arrow[r, hook] \arrow[d, "a"] & \mathbb{C}_\infty \arrow[r, "e_\phi"] \arrow[d, "a"] & \mathbb{C}_\infty \arrow[d, "\phi_a"] \\ \Lambda \arrow[r, hook] & \mathbb{C}_\infty \arrow[r, "e_\phi"] & \mathbb{C}_\infty. \end{tikzcd}$$

Here the exponential function is explicitly

$$e_\phi \colon \mathbb{C}_\infty \to \mathbb{C}_\infty, \quad \omega \mapsto \omega \prod_{\mu \in \Lambda \setminus \{0\}} (1 - \omega/\mu)$$

and the coefficients of $\phi_a(X)$ map to $\mathbb{C}_\infty$ via the embedding $K \hookrightarrow \mathbb{C}_\infty$. The commutativity of the right square in the diagram means $e_\phi(a\omega) = a \cdot_\phi e_\phi(\omega)$ for any $\omega \in \mathbb{C}_\infty$.

Remark 2.7 (SMBs and isomorphic Drinfeld modules). For any $b \in K^{\mathrm{sep}} \setminus \{0\}$, we have the Drinfeld module $b\phi b^{-1}$ isomorphic to ϕ. The uniformization associates to $b\phi b^{-1}$ the lattice $b\Lambda$. If the family $\{\omega_i\}_{i=1,\dots,r}$ is an SMB of Λ, then $\{b\omega_i\}_{i=1,\dots,r}$ is an SMB of $b\Lambda$.

If w is a finite prime, assume that ϕ has stable reduction over K and the reduction of ϕ has rank $r' < r$. According to [6, Section 7], there are the following data associated to ϕ:

(1) A rank r' Drinfeld A-module ψ over K has good reduction;

(2) A $\mathrm{Gal}(K^{\mathrm{sep}}/K)$-invariant A-lattice Λ has rank $r - r'$ with the A action induced by ψ. Each element of Λ has valuation < 0.

(3) An analytic entire surjective homomorphism

$$e_\phi \colon \mathbb{C}_w \to \mathbb{C}_w, \quad \omega \mapsto \omega \prod_{\mu \in \Lambda \setminus \{0\}} (1 - \omega/\mu)$$

such that for each $a \in A$, the following diagram commutes, and its two rows are short exact sequences

$$\begin{array}{ccccc} \Lambda & \hookrightarrow & \mathbb{C}_w & \xrightarrow{e_\phi} & \mathbb{C}_w \\ \downarrow{\scriptstyle \psi_a} & & \downarrow{\scriptstyle \psi_a} & & \downarrow{\scriptstyle \phi_a} \\ \Lambda & \hookrightarrow & \mathbb{C}_w & \xrightarrow{e_\phi} & \mathbb{C}_w. \end{array}$$

The commutativity of the right square means $e_\phi(a \cdot_\psi \omega) = a \cdot_\phi e_\phi(\omega)$ for any $\omega \in \mathbb{C}_w$.

We call these data the Tate uniformization of ϕ.

2.2. SMBs of the module of u^n-division points

In this subsection, let ϕ be a rank r Drinfeld A-module over K which does not necessarily have stable reduction. Using Remark 1.2, we may extend Definition 1.1.

Definition 2.8 (Extending Definition 1.1)**.** Let n be a positive integer and u a finite prime of A. A family of elements $\{\lambda_i\}_{i=1,\dots,r}$ is an *SMB* of $\phi[u^n]$ if for each i, the elements $\lambda_1, \dots, \lambda_i$ in $\phi[u^n]$ satisfy

(1) $\lambda_1, \ldots, \lambda_i$ are A/u^n-linearly independent;

(2) $w(\lambda_i)$ is the largest among the valuations of elements λ in $\phi[u^n]$ such that $\lambda_1, \ldots, \lambda_{i-1}, \lambda$ are A/u^n-linearly independent.

Remark 2.9. For any $b \in K^{\mathrm{sep}} \setminus \{0\}$, a family $\{\lambda_i\}_{i=1,\ldots,r}$ is an SMB of $\phi[u^n]$ if and only if the family $\{b\lambda_i\}_{i=1,\ldots,r}$ is an SMB of $b\phi b^{-1}[u^n]$. Especially, this holds when w is a finite prime and b is an element in some tamely ramified extension L of K so that $b\phi b^{-1}$ has stable reduction over L.

The rest of this subsection is concerned with two basic properties of SMBs of $\phi[u^n]$.

Proposition 2.10. (cf. Proposition 2.4) *Let $\{\lambda_i\}_{i=1,\ldots,r}$ be a family of elements in $\phi[u^n]$.*

(1) *This family is an SMB if and only if for each i, the elements $\lambda_1, \ldots, \lambda_i$ satisfy*

- *$\lambda_1, \ldots, \lambda_i$ are A/u^n-linearly independent;*
- *we have $w(\lambda_i) = l_i$, where*

$$l_i = \max\big\{\rho \in \mathbb{R} \mid \textit{the ball } \{\lambda \in K^{\mathrm{sep}} \mid w(\lambda) \geq \rho\} \textit{ contains at least } i \textit{ elements in } \phi[u^n] \textit{ which are } A/u^n\textit{-linearly independent}\big\}.$$

(2) *The sequence $w(\lambda_1) \geq w(\lambda_2) \geq \cdots \geq w(\lambda_r)$ for an SMB $\{\lambda_i\}_{i=1,\ldots,r}$ is an invariant of $\phi[u^n]$.*

Assume that ϕ has stable reduction when w is finite. Then the sequence $|\lambda_1| \leq |\lambda_2| \leq \cdots \leq |\lambda_r|$ for an SMB $\{\lambda_i\}_{i=1,\ldots,r}$ is an invariant of $\phi[u^n]$.

Proof of Proposition 2.10. (2) straightforwardly follows from (1). We then show (1). The "$\Leftarrow$" is straightforward. For "$\Rightarrow$", the first dot in (1) is the same as Definition 2.8(1). Clearly, we have $l_i \geq w(\lambda_i)$ for all i and $l_1 = w(\lambda_1)$. Then we proceed by induction. We fix any i, assume $l_j = w(\lambda_j)$ for $j < i$, and show $l_i = w(\lambda_i)$. We assume $l_i > w(\lambda_i)$ and find a contradiction. There exist elements $\eta_1, \ldots, \eta_i \in \phi[u^n]$ such that $\eta_1, \ldots, \eta_i$ are A/u^n-linearly independent and $w(\eta_j) \geq l_i > w(\lambda_i)$ for $j = 1, \ldots, i$.

Put $\overline{\eta}_j := u^{n-1} \cdot_\phi \eta_j$ for $j \leq i$ and $\overline{\lambda}_j := u^{n-1} \cdot_\phi \lambda_j$ for $j < i$. We claim that there is some k such that $\overline{\eta}_k$ and $\overline{\lambda}_1, \ldots, \overline{\lambda}_{i-1}$ are A/u-linearly independent. Assume the inverse. Then we have equations

$$b_l \cdot_\phi \overline{\eta}_l + \sum_{j=1}^{i-1} a_{l,j} \cdot_\phi \overline{\lambda}_j = 0$$

for all $l = 1, \ldots, i$, where $a_{l,j} \in A \mod u$ and $b_l \in A \mod u$ with $b_l \not\equiv 0 \mod u$ for each l. For each l, we obtain

$$\overline{\eta}_l = -\sum_{j=1}^{i-1} a_{l,j}/b_l \cdot_\phi \overline{\lambda}_j,$$

where each $a_{l,j}/b_l \in A \mod u$ satisfies $b_l(a_{l,j}/b_l) \equiv a_{l,j} \mod u$. Hence $\overline{\lambda}_1, \dots, \overline{\lambda}_{i-1}$ generate an i-dimensional A/u-vector space, which is absurd.

Next, we claim that η_k and $\lambda_1, \dots, \lambda_{i-1}$ are A/u^n-linearly independent. Assume the inverse and we have

$$c_k \cdot_\phi \eta_k + \sum_{j=1}^{i-1} a_j \cdot_\phi \lambda_j = 0, \tag{2.1}$$

where each $a_j \in A \mod u^n$ and $c_k \in A \mod u^n$ with $c_k \not\equiv 0 \mod u^n$. We may write $c_k = c'_k u^m$ with $m < n$ and $c'_k \in A$ not divisible by u. Then we have $u^m \mid a_j$ for all $j < i$, for otherwise, by (2.1), we have $\sum_{j=1}^{i-1} a_j u^{n-m} \cdot_\phi \lambda_j = 0$ with $a_j u^{n-m} \not\equiv 0 \mod u^n$ for some j. We may write $a_j = a'_j u^m$ for $a'_j \in A$. Hence we have by (2.1),

$$0 = c_k u^{n-1-m} \cdot_\phi \eta_k + \sum_{j=1}^{i-1} a_j u^{n-1-m} \cdot_\phi \lambda_j = c'_k \cdot_\phi \overline{\eta}_k + \sum_{j=1}^{i-1} a'_j \cdot_\phi \overline{\lambda}_j$$

with $c'_k \in A$ not divisible by u. This contradicts that $\overline{\eta}_k$ and $\overline{\lambda}_1, \dots, \overline{\lambda}_{i-1}$ are A/u-linearly independent. We have obtained A/u^n-linearly independent elements $\lambda_1, \dots, \lambda_{i-1}, \eta_k$ such that $w(\eta_k) \geq l_i > w(\lambda_i)$. This contradicts Definition 2.8(2). □

In the remainder of this subsection, we construct an SMB of $\phi[u^n]$ for any positive integer n.

Lemma 2.11. *Let $\{\lambda_i\}_{i=1,\dots,r}$ be an SMB of $\phi[u^n]$. For each i and $a \in A$ with $a \not\equiv 0 \mod u^n$, the element λ_i has the largest valuation among the roots λ of $\phi_a(X) - a \cdot_\phi \lambda_i$ such that $\lambda \in \phi[u^n]$.*

Proof. Let λ be a root of $\phi_a(X) - a \cdot_\phi \lambda_i$ such that $\lambda \in \phi[u^n]$. Assume $w(\lambda) > w(\lambda_i)$. It suffices to show that $\lambda_1, \dots, \lambda_{i-1}, \lambda$ are A/u^n-linearly independent because this implies that the inequality $w(\lambda) > w(\lambda_i)$ contradicts Definition 2.8(2). Assume that there exist $b_j \in A \mod u^n$ with $b_i \not\equiv 0$ such that $b_i \cdot_\phi \lambda + \sum_{j<i} b_j \cdot_\phi \lambda_j = 0$. Let c be the minimal common multiple of a and b_i such that $c = b'_i b_i = a' a$ for some b'_i and $a' \in A$. Consider the equation $b'_i \cdot_\phi \left(b_i \cdot_\phi \lambda + \sum_{j<i} b_j \cdot_\phi \lambda_j\right) = 0$. Since $b'_i b_i \cdot_\phi \lambda = a' a \cdot_\phi \lambda = a' a \cdot_\phi \lambda_i = c \cdot_\phi \lambda_i$, we have

$$c \cdot_\phi \lambda_i + \sum_{j<i} b'_i b_j \cdot_\phi \lambda_j = 0. \tag{2.2}$$

We have $u^n \nmid c$, for otherwise one of a or b_i is divisible by u^n. Hence the nonzero coefficients in the equation (2.2) contradict that $\lambda_1, \dots, \lambda_i$ are A/u^n-linearly independent. □

Corollary 2.12. *With the notation in the lemma, for each i and $a \in A$ being a power of u, the element λ_i has the largest valuation among the roots of $\phi_a(X) - a \cdot_\phi \lambda_i$.*

Proposition 2.13. *Let $\{\lambda_i\}_{i=1,\dots,r}$ be an SMB of $\phi[u^n]$.*

(1) *For each i, let λ'_i be a root of $\phi_u(X)-\lambda_i$ having the largest valuation. Then $\{\lambda'_i\}_{i=1,\dots,r}$ is an SMB of $\phi[u^{n+1}]$.*

(2) *The family of elements $\{u \cdot_\phi \lambda_i\}_{i=1,\dots,r}$ is an SMB of $\phi[u^{n-1}]$.*

Proof. (1) We check Definition 2.8(1) using induction on i. The base case is clear. Assume $\lambda'_1, \dots, \lambda'_{i-1}$ are A/u^{n+1}-linearly independent. Assume conversely that there are $a_j \in A$ mod u^{n+1} with $a_i \not\equiv 0$ such that $\sum_{j=1}^{i} a_j \cdot_\phi \lambda'_j = 0$. For $j = 1, \dots, i$, since $u \cdot_\phi \lambda'_j = \lambda_j$ and $\lambda_1, \dots, \lambda_i$ are A/u^n-linearly independent, we have $ua_j \equiv 0 \mod u^{n+1}$ and hence $u^n \mid a_j$. There are $b_j \in A$ with $b_i \not\equiv 0 \mod u$ such that $a_j = b_j u^n$ for all j. Hence

$$0 = \sum_{j=1}^{i} a_j \cdot_\phi \lambda'_j = \sum_{j=1}^{i} b_j u^{n-1} \cdot_\phi \lambda_j$$

with $b_i u^{n-1}$ not divisible by u^n, which is absurd.

As for Definition 2.8(2), we show $w(\lambda'_i) \geq w(\lambda)$ for each $\lambda \in \phi[u^{n+1}]$ such that $\lambda'_1, \dots, \lambda'_{i-1}, \lambda$ are A/u^{n+1}-linearly independent. Notice $u \cdot_\phi \lambda \in \phi[u^n]$ and that the elements $\lambda_1, \dots, \lambda_{i-1}, u \cdot_\phi \lambda$ are A/u^n-linearly independent. We have $w(\lambda_i) \geq w(u \cdot_\phi \lambda)$ as $\{\lambda_i\}_{i=1,\dots,r}$ is an SMB of $\phi[u^n]$. Note that $w(\lambda'_i)$ is the largest among the valuations of roots of $\phi_u(X) - \lambda_i$. By comparing the Newton polygons of $\phi_u(X) - \lambda_i$ and $\phi_u(X) - u \cdot_\phi \lambda$, we have $w(\lambda'_i) \geq w(\lambda)$.

(2) It is straightforward to check Definition 2.8(1). Let λ be an element of $\phi[u^{n-1}]$ such that $u \cdot_\phi \lambda_1, \dots, u \cdot_\phi \lambda_{i-1}, \lambda$ are A/u^{n-1}-linearly independent. To show $w(u \cdot_\phi \lambda_i) \geq w(\lambda)$, we assume conversely $w(u \cdot_\phi \lambda_i) < w(\lambda)$. By comparing the Newton polygons of $\phi_u(X) - u \cdot_\phi \lambda_i$ and $\phi_u(X) - \lambda$, there is a root λ' of $\phi_u(X) - \lambda$ such that $w(\lambda') > w(\lambda_i)$. We have $\lambda' \in \phi[u^n]$ as all roots of $\phi_u(X) - \lambda$ belong to $\phi[u^n]$. Similar to the proof of (1), one shows that $\lambda_1, \dots, \lambda_{i-1}, \lambda'$ are A/u^n-linearly independent. Hence the inequality $w(\lambda') > w(\lambda_i)$ contradicts that $\{\lambda_i\}_{i=1,\dots,r}$ is an SMB of $\phi[u^n]$. $\square$

We can find an SMB of $\phi[u]$ in the following way. Put

$$\begin{aligned} &\lambda_{1,1} := \text{an element in } \phi[u] \setminus \{0\} \text{ with the largest valuation,} \\ &\lambda_{i,1} := \text{an element in } \phi[u] \setminus \bigoplus_{j<i} (A/u) \cdot_\phi \lambda_{j,1} \text{ with the largest valuation} \end{aligned} \tag{2.3}$$

for $i = 2, 3, \dots, r$. Since A/u is a field, the elements $\lambda_{i,1}$ for $i = 1, \dots, r$ are A/u-linearly independent and form an SMB of $\phi[u]$. Applying the proposition, we have

Corollary 2.14. *Let $\{\lambda_{i,1}\}_{i=1,\dots,r}$ be an SMB of $\phi[u]$ defined above. Inductively, let $\lambda_{i,j}$ be a root of $\phi_u(X) - \lambda_{i,j-1}$ having the largest valuation for each i and each integer $j \geq 2$. Then for each positive integer n, we have that $\{\lambda_{i,n}\}_{i=1,\dots,r}$ is an SMB of $\phi[u^n]$.*

3. Relations between SMBs, the infinite prime case

Let w denote an infinite prime, $|-|$ the absolute value in (F1) and $\{\omega_i\}_{i=1,\dots,r}$ an SMB of Λ throughout this section. For a positive integer n and a finite prime u of A, we study the relations between SMBs of Λ and those of $\phi[u^n]$.

Lemma 3.1. *Let a be an element in A. For $\omega = \sum_j a_j\omega_j \in \Lambda$ with $a_j \in A$, let i be an index so that $|a_i\omega_i| = |\omega|$, i.e., $|a_i\omega_i| = \max_j\{|a_j\omega_j|\}$. Assume $\deg(a_i) < \deg(a)$. Then we have*

$$\left|e_\phi\left(\frac{\omega}{a}\right)\right| = \left|e_\phi\left(\frac{a_i\omega_i}{a}\right)\right|.$$

Proof. We have

$$e_\phi\left(\frac{\omega}{a}\right) = \frac{\omega}{a}\prod_{\mu\in\Lambda\setminus\{0\}}\left(1-\frac{\omega}{a\mu}\right).$$

Its absolute value is

$$\left|\frac{\omega}{a}\right|\cdot\prod_{\substack{\mu\in\Lambda\setminus\{0\}\\ |a\mu|\le|\omega|}}\left|1-\frac{\omega}{a\mu}\right|.$$

For $\mu\in\Lambda$ satisfying $|a\mu|<|\omega|$, we have by the ultrametric inequality

$$\left|1-\frac{\omega}{a\mu}\right| = \left|\frac{\omega}{a\mu}\right| = \left|\frac{a_i\omega_i}{a\mu}\right| = \left|1-\frac{a_i\omega_i}{a\mu}\right|.$$

Next, for $\mu\in\Lambda$ satisfying $|a\mu| = |\omega| = |a_i\omega_i|$, we show

$$\left|1-\frac{\omega}{a\mu}\right| = \left|1-\frac{a_i\omega_i}{a\mu}\right| = 1.$$

It suffices to show

$$|\omega - a\mu| = |\omega| \quad\text{and}\quad |a_i\omega_i - a\mu| = |a_i\omega_i|.$$

Since $|a_i|<|a|$, we have μ belonging to $\bigoplus_{j<i}A\omega_j$, for otherwise we have $|a\mu|\ge|a\omega_i|>|a_i\omega_i|$ by Proposition 2.5(2). Applying Proposition 2.5(2) to $|\omega-a\mu|$ and $|a_i\omega_i-a\mu|$, we obtain the desired equalities. $\square$

Corollary 3.2. *Let a be an element in A.*

(1) *For any $i=1,\dots,r$ and any $a_i\in A$ satisfying $\deg(a_i)<\deg(a)$, we have*

$$\left|e_\phi\left(\frac{a_i\omega_i}{a}\right)\right| = \left|\frac{a_i\omega_i}{a}\right|\cdot\prod_{\substack{\mu\in\Lambda\setminus\{0\}\\ |a\mu|<|a_i\omega_i|}}|a_i\omega_i|/|a\mu|.$$

(2) *For any positive integers $i,j\le r$, let a_i and a_j be elements in A with degrees strictly smaller than that of a. Assume $|a_j\omega_j|\le|a_i\omega_i|$. Then*

$$\left|e_\phi\left(\frac{a_j\omega_j}{a}\right)\right| \le \left|e_\phi\left(\frac{a_i\omega_i}{a}\right)\right|.$$

(3) *With the notation in the lemma, we have*

$$\left|e_\phi\left(\frac{\omega}{a}\right)\right| = \max_j\left\{\left|a_j \cdot_\phi e_\phi\left(\frac{\omega_j}{a}\right)\right|\right\}.$$

(4) *For any positive integer $i \le r$ and $b \in A$ satisfying* $\deg(b) < \deg(a)$, *we have*

$$|b| \cdot \left|e_\phi\left(\frac{\omega_i}{a}\right)\right| \le \left|b \cdot_\phi e_\phi\left(\frac{\omega_i}{a}\right)\right|.$$

Proof. (1) has been shown in the proof of the lemma. As for (2), by the assumption, we have

$$\{\mu \in \Lambda \mid |a\mu| < |a_j\omega_j|\} \subset \{\mu \in \Lambda \mid |a\mu| < |a_i\omega_i|\}. \tag{3.1}$$

If μ satisfies $|a\mu| < |a_j\omega_j|$, we have $|a_j\omega_j|/|a\mu| \le |a_i\omega_i|/|a\mu|$. Combining this inequality and (3.1), we have the desired inequality by (1). For (3), as $a \cdot_\phi e_\phi(\omega) = e_\phi(a\omega)$ for any $a \in A$ and any $\omega \in \mathbb{C}_\infty$, it remains to show

$$\left|e_\phi\left(\frac{\omega}{a}\right)\right| = \max_j\left\{\left|e_\phi\left(\frac{a_j\omega_j}{a}\right)\right|\right\}.$$

This equality follows from Lemma 3.1 and (2). As for (4), note $|\omega_i| < |b\omega_i|$. One can show (4) similar to the proof of (2). □

Theorem 3.3. *For any finite prime u of A and any positive integer n, the family of elements $\{e_\phi(\omega_i/u^n)\}_{i=1,\dots,r}$ is an SMB of $\phi[u^n]$.*

Proof. Put $\lambda_i = e_\phi(\omega_i/u^n)$ for all i. Note that $\omega_1/u^n, \dots, \omega_r/u^n$ are A/u^n-linearly independent as elements in $u^{-n}\Lambda/\Lambda$. By the A/u^n-module isomorphism $\mathcal{E}_\phi\colon u^{-n}\Lambda/\Lambda \to \phi[u^n]$ induced by e_ϕ, we have that $\lambda_1, \dots, \lambda_r$ are A/u^n-linearly independent.

Fix a positive integer $i \le r$. To check Definition 1.1(2), we show that $|\lambda_i|$ is minimal among the absolute values of elements in $\phi[u^n] \setminus \bigoplus_{j<i}(A/u^n) \cdot_\phi \lambda_j$ (in $\phi[u^n] \setminus \{0\}$ if $i = 1$). Put $\lambda = \sum_j a_j \cdot_\phi \lambda_j$ with $a_j \in A \mod u^n$ such that there is $a_k \not\equiv 0$ for some $k \ge i$. We show $|\lambda_i| \le |\lambda|$. Without loss of generality, we assume that $\deg(a_j) < \deg(u^n)$ for any j. Let l be an index so that $|a_l\omega_l| = \left|\sum_j a_j\omega_j\right|$. By Corollary 3.2(3), we have

$$|\lambda| = |a_l \cdot_\phi \lambda_l|.$$

As $|a_k\omega_k| \le |a_l\omega_l|$, Corollary 3.2(2) implies

$$\left|e_\phi\left(\frac{a_k\omega_k}{u^n}\right)\right| \le \left|e_\phi\left(\frac{a_l\omega_l}{u^n}\right)\right|,$$

hence $|a_k \cdot_\phi \lambda_k| \le |a_l \cdot_\phi \lambda_l|$. As $|\omega_i| \le |\omega_k|$, Corollary 3.2(2) also implies $|\lambda_i| \le |\lambda_k|$. By Corollary 3.2(4), we have $|a_k| \cdot |\lambda_k| \le |a_k \cdot_\phi \lambda_k|$. Combining the equality and inequalities, we have

$$|\lambda_i| \le |\lambda_k| \le |a_k| \cdot |\lambda_k| \le |a_k \cdot_\phi \lambda_k| \le |a_l \cdot_\phi \lambda_l| = |\lambda|.$$ □

Remark 3.4. We have shown in the above proof that $|\lambda_1|$ is minimal among the absolute values of nonzero elements in $\phi[u^n]$. Let $\{\lambda'_i\}_{i=1,\dots,r}$ be an SMB of $\phi[u^n]$. By Theorem 3.8 below, we know that there exists an SMB $\{\omega'_i\}_{i=1,\dots,r}$ of Λ such that $e_\phi(\omega'_i/u^n) = \lambda'_i$ for all i. Hence λ'_1 has the minimal absolute value among elements in $\phi[u^n] \setminus \{0\}$.

Corollary 3.5. *Let $\{\lambda_i\}_{i=1,\dots,r}$ be an SMB of $\phi[u^n]$.*

(1) *If n is large enough so that $|u^n| \geq |\omega_r|/|\omega_1|$, then for $i = 1, \dots, r$, we have $|\lambda_i| \cdot |u^n| = |\omega_i|$.*

(2) *For any positive integer n, we have $|\lambda_r|/|\lambda_1| \geq |\omega_r|/|\omega_1|$.*

(3) *If n is large enough so that $|u^n| > |\omega_r|/|\omega_1|$, then we have $|\lambda_i| < |\omega_1|$ for $i = 1, \dots, r$.*

Proof. We show (1). Fix i to be one of $1, \dots, r$. Corollary 3.2(1) implies

$$\left| e_\phi\left(\frac{\omega_i}{u^n}\right)\right| = \left|\frac{\omega_i}{u^n}\right| \cdot \prod_{\substack{\mu \in \Lambda \setminus \{0\} \\ |u^n\mu| < |\omega_i|}} |\omega_i|/|u^n\mu|. \tag{3.2}$$

For any $\mu \in \Lambda$, we have

$$|u^n\mu| \geq |u^n\omega_1| \geq |\omega_r| \geq |\omega_i|$$

by the hypothesis. Hence (3.2) implies

$$\left| e_\phi\left(\frac{\omega_i}{u^n}\right)\right| = \left|\frac{\omega_i}{u^n}\right|.$$

By Theorem 3.3, the family $\{e_\phi(\omega_i/u^n)\}_{i=1,\dots,r}$ is an SMB of $\phi[u^n]$. Hence we have

$$|\lambda_i| = \left| e_\phi\left(\frac{\omega_i}{u^n}\right)\right| \quad \text{for any } i \tag{3.3}$$

by Proposition 2.10(2). (1) follows. Notice that (3.2) implies

$$\left| e_\phi\left(\frac{\omega_1}{u^n}\right)\right| = \left|\frac{\omega_1}{u^n}\right| \quad \text{and} \quad \left| e_\phi\left(\frac{\omega_i}{u^n}\right)\right| \geq \left|\frac{\omega_i}{u^n}\right| \quad \text{for any } i.$$

(2) follows from (3.3). Since we know $|\lambda_r| = |\omega_r|/|u^n|$ by (1), we have

$$|\lambda_i| \leq |\lambda_r| = |\omega_r|/|u^n| < |\omega_r|/(|\omega_r|/|\omega_1|) = |\omega_1|$$

and (3) follows. □

Remark 3.6. By Corollary 3.5(1) and (2), we have $|\lambda_i| \cdot |u^n| = |\omega_i|$ if n is large enough so that $|u^n| \geq |\lambda_r|/|\lambda_1|$.

Put $B := \{\omega \in \mathbb{C}_\infty \mid |\omega| < |\omega_1|\}$. Since $B \cap \Lambda = \emptyset$, the exponential function e_ϕ is injective on B. For any $\omega \in \mathbb{C}_\infty$, we have

$$|e_\phi(\omega)| = |\omega| \cdot \prod_{\substack{\mu \in \Lambda \setminus \{0\} \\ |\mu| \le |\omega|}} \left| 1 - \frac{\omega}{\mu} \right|. \tag{3.4}$$

Hence $|e_\phi(\omega)| = |\omega|$ for $\omega \in B$. This implies $e_\phi(B) \subset B$. Put $C := e_\phi(B)$. There is an inverse $\log_\phi\colon C \to B$ of e_ϕ defined by a power series with coefficients in K and $e_\phi : B \rightleftarrows C : \log_\phi$ are inverse to each other.

Lemma 3.7. (1) *We have $C = B$.*

(2) *We have the following maps which are inverse to each other*

$$e_\phi : B \cap \mathcal{L} \rightleftarrows B \cap \phi[u^n] : \log_\phi,$$

where

$$\mathcal{L} := \left\{ \sum_i a_i(\omega_i/u^n) \;\middle|\; a_i \in A \textit{ with } \deg(a_i) < \deg(u^n) \right\}$$

is a set of representatives of all elements in $u^{-n}\Lambda/\Lambda$.

(3) *For any $\lambda \in B \cap \phi[u^n]$, we have $|\log_\phi(\lambda)| = |\lambda|$.*

Proof. We show (1) using a property of the image of the open disk B under the power series e_ϕ. Let c_i be elements in $\mathbb{C}_\infty$ so that

$$\sum_{i \ge 1} c_i \omega^i := \omega \prod_{\mu \in \Lambda \setminus \{0\}} (1 - \omega/\mu) = e_\phi(\omega).$$

We first calculate the minimal integer d such that $|c_d||\omega_1^d|$ is maximal among $|c_i||\omega_1^i|$ for all i, i.e., d is the Weierstrass degree of e_ϕ on B. Clearly $c_1 = 1$. As $|\omega_1/\mu| \le 1$ for any $\mu \in \Lambda \setminus \{0\}$, we have the following inequalities

$$|c_i||\omega_1^i| \le \sup_{\mu_j \in \Lambda \setminus \{0\}} \left\{ |\omega_1| \cdot \left| \prod_{j=1}^{i-1} \omega_1/\mu_j \right| \right\} \le |\omega_1| = |c_1||\omega_1|$$

for integers $i \ge 2$. Hence $d = 1$. By [2, Theorem 3.15], we have $C = e_\phi(B) = B$.

As we have bijections $e_\phi\colon B \to B$ and $e_\phi\colon \mathcal{L} \to \phi[u^n]$, (2) follows. As for (3), by (2), we have $\log_\phi(\lambda) \in B \cap \mathcal{L}$ and $e_\phi(\log_\phi(\lambda)) = \lambda$. Hence we have $|\log_\phi(\lambda)| = |\lambda|$ by (3.4). $\square$

Let $\{\lambda_i\}_{i=1,\dots,r}$ denote an SMB of $\phi[u^n]$. Assume that the positive integer n is large enough so that $|u^n| > |\omega_r|/|\omega_1|$. By Corollary 3.5(3) and Lemma 3.7(1), for each i, we have $\lambda_i \in B \cap \phi[u^n] = C \cap \phi[u^n]$ and we put $\omega_i' := \log_\phi(\lambda_i)$.

Theorem 3.8. *The family $\{u^n\omega_i'\}_{i=1,\dots,r}$ is an SMB of Λ.*

We need a lemma in the proof.

Lemma 3.9. *Let $\{\eta_i\}_{i=1,\dots,r}$ be a family of elements in $u^{-n}\Lambda$. It is an SMB of $u^{-n}\Lambda$ if and only if $\{u^n\eta_i\}_{i=1,\dots,r}$ is an SMB of Λ.*

Proof. For any $a_i \in A$, we have

$$\left|\sum_i a_i u^n \eta_i\right| = |u^n| \cdot \left|\sum_i a_i \eta_i\right|.$$

Then the lemma follows from Proposition 2.5. □

Proof of Theorem 3.8. By Lemma 3.9, it suffices to show that the family of elements $\{\omega_i'\}_{i=1,\dots,r}$ is an SMB of $u^{-n}\Lambda$. To check the first dot in Proposition 2.4(1), we show that $\omega_1', \dots, \omega_r'$ are A-linearly independent. Assume that there exist nonzero $a_i \in A$ such that $\sum_i a_i\omega_i' = 0$. We may assume $u^n \nmid a_i$ for some i, for otherwise we divide both sides of the equation $\sum_i a_i\omega_i' = 0$ by some power of u. Note that the map e_ϕ is A/u^n-linear. As some a_i satisfies $a_i \not\equiv 0 \mod u^n$ and $\lambda_1, \dots, \lambda_r$ are A/u^n-linearly independent, we have $e_\phi\big(\sum_i a_i\omega_i'\big) = \sum_i a_i \cdot_\phi \lambda_i \neq 0$. This is absurd.

Next, we check the second dot in Proposition 2.4(1). Let $l_1 \le l_2 \le \cdots \le l_r$ be the invariant of $u^{-n}\Lambda$ as in Proposition 2.4(2). Fix i to be a positive integer $\le r$. It suffices to show $l_i = |\omega_i'|$. We have $l_i \le |\omega_i'|$. Let us assume $l_i < |\omega_i'|$. As $\lambda_i \in B \cap \phi[u^n]$, we have $|\omega_i'| = |\lambda_i|$ by Lemma 3.7(3). Hence $l_i < |\omega_i'| = |\lambda_i| < |\omega_1|$. By Proposition 2.4(1), there is an SMB $\{\eta_j\}_{j=1,\dots,r}$ of $u^{-n}\Lambda$ such that $|\eta_i| = l_i < |\omega_1|$. As $|\eta_i| < |\omega_1|$, we know $|e_\phi(\eta_i)| = |\eta_i|$ from (3.4). We have

$$|e_\phi(\eta_i)| = |\eta_i| = l_i < |\omega_i'| = |\lambda_i|$$

and hence $|e_\phi(\eta_i)| < |\lambda_i|$. On the other hand, note that $\{u^n\eta_j\}_{j=1,\dots,r}$ is an SMB of Λ by Lemma 3.9. By Theorem 3.3, the elements $e_\phi(\eta_j)$ for $j = 1, \dots, r$ form an SMB of $\phi[u^n]$. By Proposition 2.10(2), this contradicts $|e_\phi(\eta_i)| < |\lambda_i|$. □

Finally, we give applications of Theorems 3.3 and 3.8.

Proposition 3.10. *If n is large enough so that $|u^n| > |\omega_r|/|\omega_1|$, then we have*

$$K(\Lambda) = K(\phi[u^n]),$$

where $K(\Lambda)$ (resp. $K(\phi[u^n])$) is the extension of K generated by all elements in Λ (resp. in $\phi[u^n]$).

Proof. (cf. the proof of [12, Proposition 2.1]) Note that e_ϕ is given by a power series with coefficients in K. For any $x \in K^{\mathrm{sep}}$, we have $e_\phi(x) \in K(x)$ since the field $K(x)$ is complete. Since the exponential map e_ϕ induces a bijection $u^{-n}\Lambda/\Lambda \to \phi[u^n]$, for any λ in $\phi[u^n]$, there exists $\omega \in u^{-n}\Lambda$ such that $e_\phi(\omega) = \lambda$. This implies $K(\lambda) \subset K(\omega)$ and $K(\phi[u^n]) \subset K(\Lambda)$.

Note that $\log_\phi$ is given by a power series with coefficients in K. For any $y \in C \cap K^{\mathrm{sep}}$, we similarly have $\log_\phi(y) \in K(y)$. Let $\{\lambda_i\}_{i=1,\dots,r}$ be an SMB of $\phi[u^n]$. As $|u^n| > |\omega_r|/|\omega_1|$, by Theorem 3.8, the elements $u^n\omega_i'$ for $i = 1,\dots,r$ form an SMB of Λ, where $\omega_i' = \log_\phi(\lambda_i)$. Since $K(\omega_i') \subset K(\lambda_i)$ for each i, we have $K(\Lambda) \subset K(\phi[u^n])$. □

Combining Corollary 3.5(2), Theorem 3.8 and Proposition 3.10, we have

Corollary 3.11. *Let l be a positive integer and $\{\eta_i\}_{i=1,\dots,r}$ an SMB of $\phi[u^l]$. Let $\{\lambda_i\}_{i=1,\dots,r}$ be an SMB of $\phi[u^n]$. If n is large enough so that $|u^n| > |\eta_r|/|\eta_1|$, then we have*

(1) *the family $\{u^n \log_\phi(\lambda_i)\}_{i=1,\dots,r}$ is an SMB of Λ;*

(2) *$K(\Lambda) = K(\phi[u^n])$.*

Proposition 3.12. *Let $\{\lambda_i\}_{i=1,\dots,r}$ be an SMB of $\phi[u^n]$. We have*

$$\left|\sum_i a_i \cdot_\phi \lambda_i\right| = \max_i \left\{|a_i \cdot_\phi \lambda_i|\right\}$$

for any $a_i \in A \mod u^n$.

Proof. Without loss of generality, we assume $\deg(a_i) < \deg(u^n)$ for all i. Assume first that n is large enough so that $|u^n| > |\lambda_r|/|\lambda_1|$ (see Corollary 3.5(2)). By Theorem 3.8, the elements $u^n\omega_i'$ for $i = 1,\dots,r$ form an SMB of Λ, where $\omega_i' = \log_\phi(\lambda_i)$. By Corollary 3.2(3), we have

$$\left|e_\phi\left(\sum_i a_i\omega_i'\right)\right| = \max_i \left\{|a_i \cdot_\phi e_\phi(\omega_i')|\right\}.$$

As $e_\phi\big(\sum_i a_i\omega_i'\big) = \sum_i a_i \cdot_\phi \lambda_i$, the claim follows.

For any n, let n' be an integer $\geq n$ so that $|u^{n'}| > |\lambda_r|/|\lambda_1|$. By Proposition 2.13(1), there is an SMB $\{\lambda_i'\}_{i=1,\dots,r}$ of $\phi[u^{n'}]$ such that $u^{n'-n} \cdot_\phi \lambda_i' = \lambda_i$ for all i. Then the desired equation for $\{\lambda_i\}_{i=1,\dots,r}$ follows from that for $\{\lambda_i'\}_{i=1,\dots,r}$. □

Proposition 3.13. (cf. Lemma 2.6) *Let $\{\lambda_i\}_{i=1,\dots,r}$ be an SMB of $\phi[u^n]$ such that $|\lambda_1| = \cdots = |\lambda_s| < |\lambda_{s+1}|$ for some positive integer s.*

(1) *The extension of K generated by $\lambda_1,\dots,\lambda_s$ is at worst tamely ramified.*

(2) *For an element* σ *in the wild ramification subgroup* $\mathrm{Gal}(K(\phi[u^n])/K)_1$*, we have* $\sigma(\lambda_j) = \lambda_j$ *for* $j = 1, \ldots, s$.

Proof. (2) straightforwardly follows from (1). We show (1). Let n' be an integer satisfying $n' \geq n$ and $|u^{n'}| > |\lambda_r|/|\lambda_1|$. By Proposition 2.13(1), we know that there exists an SMB $\{\lambda'_i\}_{i=1,\ldots,r}$ of $\phi[u^{n'}]$ such that $u^{n'-n} \cdot_\phi \lambda'_i = \lambda_i$ for all i. By Corollary 2.12, we have $|\lambda'_1| = \cdots = |\lambda'_s| < |\lambda'_{s+1}|$. By Corollary 3.11(1), we know that there exists an SMB $\{\omega_i\}_{i=1,\ldots,r}$ of Λ such that $e_\phi(\omega_i/u^{n'}) = \lambda'_i$ for all i. Corollary 3.5 implies that $|\omega_1| = \cdots = |\omega_s| < |\omega_{s+1}|$. As $e_\phi(\omega_i/u^n) = \lambda_i$, we have $K(\lambda_i) \subset K(\omega_i)$ for all i. Then the result follows from Lemma 2.6. □

4. Relations between SMBs, the finite prime case

Throughout this section, let w denote a finite prime and assume that ϕ has stable reduction over K. Assume that the reduction of ϕ has rank $r' < r$ unless otherwise specified. Let $\{\omega_i^0\}_{i=r'+1,\ldots,r}$ be an SMB of Λ. Let $|-|$ denote the function in (F2) in Section 1.1 and put $|a|_\infty := q^{\deg(a)}$ for any $a \in A$. For a positive integer n and a finite prime u of A, we study the relations between SMBs of $\psi[u^n]$, those of Λ, and those of $\phi[u^n]$.

First, we are concerned with the valuations of the elements in the A-module $u^{-n}\Lambda$, i.e., the roots of $\psi_{u^n}(X) - \omega$ for all $\omega \in \Lambda$.

Lemma 4.1. *Let* a *be an element in* A.

(1) *Each root of* $\psi_a(X)$ *has valuation* ≥ 0. *Moreover, all nonzero roots of* $\psi_a(X)$ *have valuation* $= 0$ *if and only if* $w(a) = 0$.

(2) *For a nonzero element* $\omega \in \Lambda$*, each root of* $\psi_a(X) - \omega$ *has valuation* < 0.

(3) *An element* $\omega \in a^{-1}\Lambda$ *belongs to* $\psi[a]$ *if and only if it has valuation* ≥ 0.

Proof. Put $g := r' \cdot \deg(a)$, $a_0 := a$, $\sum_{i=0}^{g} a_i X^{q^i} := \psi_a(X)$ and $P_i = (q^i, w(a_i))$ for $i = 0, \ldots, g$. As $w(a_i) \geq 0$ and $w(a_g) = 0$, the segments in the Newton polygon of $\psi_a(X)$ have slopes ≤ 0. If $w(a_0) = 0$, then the Newton polygon of $\psi_a(X)$ consists of exactly one segment P_0P_g which has slope 0. Hence each root of $\psi_a(X)$ has valuation $= 0$. If $w(a_0) > 0$, then the left-most segment in the Newton polygon of $\psi_a(X)$ has negative slope. Hence some nonzero root of $\psi_a(X)$ has valuation > 0.

As for (2), put $Q := (0, w(\omega))$. As $w(\omega) < 0$, $w(a_i) \geq 0$ for all i, and $w(a_g) = 0$, the Newton polygon of $\psi_a(X) - \omega$ consists of exactly one segment QP_g whose slope is $-w(\omega)/q^g > 0$. Hence (2) follows. From (1) and (2), we know (3). □

Fix a root ω_i of $\psi_{u^n}(X)-\omega_i^0$ for $i=r'+1,\dots,r$. The elements $\omega_{r'+1},\dots,\omega_r$ are A-linearly independent. For all $a_i\in A$, we have

$$|u^n|_\infty\cdot\left|\sum_{i=r'+1}^{r} a_i\cdot_\psi\omega_i\right|=\left|\sum_{i=r'+1}^{r} a_iu^n\cdot_\psi\omega_i\right|=\left|\sum_{i=r'+1}^{r} a_i\cdot_\psi\omega_i^0\right|.$$

Hence, by Proposition 2.5, we have

$$\left|\sum_{i=r'+1}^{r} a_i\cdot_\psi\omega_i\right|=\max_{i=r'+1,\dots,r}\left\{|a_i\cdot_\psi\omega_i|\right\}\tag{4.1}$$

for any $a_i\in A$.

In the remainder of this section, let $\{\omega_i\}_{i=1,\dots,r'}$ be an SMB of $\psi[u^n]$ and $\omega_{r'+1},\dots,\omega_r$ be elements in $u^{-n}\Lambda$ defined as above. The family $\{\omega_i\}_{i=1,\dots,r}$ form an A/u^n-basis of $u^{-n}\Lambda/\Lambda$. Next, we study the relations between $\{\omega_i\}_{i=1,\dots,r}$ and SMBs of $\phi[u^n]$.

Lemma 4.2. (1) *For all $a_i\in A$, we have*

$$\left|\sum_i a_i\cdot_\psi\omega_i\right|=\begin{cases}\left|\sum_{i\le r'}a_i\cdot_\psi\omega_i\right|\le 0 & \textit{all } a_i=0 \textit{ for } i>r',\\ \left|\sum_{i>r'}a_i\cdot_\psi\omega_i\right|>0 & \textit{some } a_i\neq 0 \textit{ for } i>r'.\end{cases}$$

(2) *Let a_i be elements in A for $i=1,\dots,r$. Assume either $w(u)=0$, or some a_i is nonzero for $i>r'$. Then we have*

$$\left|\sum_i a_i\cdot_\psi\omega_i\right|=\max_i\left\{|a_i\cdot_\psi\omega_i|\right\}.$$

Proof. (1) Since $\sum_{i\le r'}a_i\cdot_\psi\omega_i\in\psi[u^n]$, we have $\left|\sum_{i\le r'}a_i\cdot_\psi\omega_i\right|\le 0$ by Lemma 4.1(3). Since $u^n\cdot_\psi\omega_i$ for all $i=r'+1,\dots,r$ are elements in Λ, we have $|u^n|_\infty\cdot|\omega_i|>0$ and hence $|a_i|_\infty\cdot|\omega_i|>0$ if a_i is nonzero. Hence, by (4.1) and the ultrametric inequality, we have $\left|\sum_i a_i\cdot_\psi\omega_i\right|=\left|\sum_{i>r'}a_i\cdot_\psi\omega_i\right|>0$ if some a_i for $i>r'$ is nonzero. (1) follows.

(2) If some $a_i\neq 0$ for $i>r'$, the desired equality follows from (1) and (4.1). By Lemma 4.1(1), the assumption $w(u)=0$ implies that the elements in $\psi[u^n]$ have valuation 0. Hence $\left|\sum_{i\le r'}a_i\cdot_\psi\omega_i\right|=0$ and $|a_i\cdot_\psi\omega_i|=0$ for all $i\le r'$. The desired equality similarly follows. □

Recall for any $\omega\in\mathbb{C}_w$, we have

$$e_\phi(\omega)=\omega\prod_{\mu\in\Lambda\setminus\{0\}}\left(1-\frac{\omega}{\mu}\right).$$

Its valuation is

$$w(e_\phi(\omega))=w(\omega)+\sum_{\substack{\mu\in\Lambda\setminus\{0\}\\ w(\mu)\ge w(\omega)}}w\left(1-\frac{\omega}{\mu}\right).\tag{4.2}$$

For certain $\omega=\sum_i a_i\cdot_\psi\omega_i\in u^{-n}\Lambda$, we are to calculate $|e_\phi(\omega)|$.

Lemma 4.3. *If* $\omega = \sum_{i \le r'} a_i \cdot_\psi \omega_i$ *with* $a_i \in A \mod u^n$, *we have*

$$|e_\phi(\omega)| = |\omega|.$$

Proof. By (4.2), it suffices to show $w(1 - \omega/\mu) = 0$ for each $\mu \in \Lambda$. Notice $w(\omega) \ge 0$ by Lemma 4.2(1). Since $w(\mu) < 0$ for any $\mu \in \Lambda$, we have $w(1-\omega/\mu) = 0$ by the ultrametric inequality. ☐

Lemma 4.4. (cf. Lemma 3.1) *For* $\omega = \sum_j a_j \cdot_\psi \omega_j \in u^{-n}\Lambda$, *assume some* a_j *for* $j > r'$ *is nonzero. Let* i *be an integer* $> r'$ *such that* $|\omega| = |a_i \cdot_\psi \omega_i| = \max_j\{|a_j \cdot_\psi \omega_j|\}$ *(by Lemma* 4.2(2)*). Assume* $\deg(a_i) < \deg(u^n)$*. Then we have*

$$|e_\phi(\omega)| = |e_\phi(a_i \cdot_\psi \omega_i)|.$$

Proof. By (4.2), it suffices to show

$$w\left(1 - \frac{\omega}{\mu}\right) = w\left(1 - \frac{a_i \cdot_\psi \omega_i}{\mu}\right)$$

for each $\mu \in \Lambda$ with $w(\mu) \ge w(\omega)$. If $w(\mu) > w(\omega)$, then we have by the ultrametric inequality that

$$w\left(1 - \frac{\omega}{\mu}\right) = w\left(\frac{\omega}{\mu}\right) = w\left(\frac{a_i \cdot_\psi \omega_i}{\mu}\right) = w\left(1 - \frac{a_i \cdot_\psi \omega_i}{\mu}\right).$$

Next, we show

$$w\left(1 - \frac{\omega}{\mu}\right) = w\left(1 - \frac{a_i \cdot_\psi \omega_i}{\mu}\right) = 0$$

if $w(\mu) = w(\omega) = w(a_i \cdot_\psi \omega_i)$. It suffices to show

$$w(\omega - \mu) = w(\omega) \quad \text{and} \quad w(a_i \cdot_\psi \omega_i - \mu) = w(a_i \cdot_\psi \omega_i).$$

As $\deg(a_i) < \deg(u^n)$, we have

$$|\omega| = |a_i \cdot_\psi \omega_i| = |a_i|_\infty \cdot |\omega_i| < |u^n|_\infty \cdot |\omega_i| = |\omega_i^0|$$

and hence $|\mu| = |\omega| < |\omega_i^0|$. This implies $\mu \in \bigoplus_{j=r'+1}^{i-1} A \cdot_\psi \omega_j^0$, for otherwise we have $|\mu| \ge |\omega_i^0|$ by Proposition 2.5(2). Applying Lemma 4.2(2) to $|\omega - \mu|$ and $|a_i \cdot_\psi \omega_i - \mu|$, we obtain the desired equalities. ☐

Corollary 4.5. (cf. Corollary 3.2)

(1) *With the notation in Lemma* 4.4*, we have*

$$w(e_\phi(\omega)) = w(\omega) + \sum_{\substack{\mu \in \Lambda \setminus \{0\} \\ w(\mu) > w(\omega)}} (w(\omega) - w(\mu)).$$

Particularly, for any $i = 1, \ldots, r$ and any $a_i \in A \setminus \{0\}$ satisfying $\deg(a_i) < \deg(u^n)$, we have

$$w(e_\phi(a_i \cdot_\psi \omega_i)) = w(a_i \cdot_\psi \omega_i) + \sum_{\substack{\mu \in \Lambda \setminus \{0\} \\ w(\mu) > w(a_i \cdot_\psi \omega_i)}} (w(a_i \cdot_\psi \omega_i) - w(\mu)).$$

(2) *For any positive integers $i, j \leq r$, let a_i and a_j be elements in A with degree strictly smaller than that of a. Assume $|a_j \cdot_\psi \omega_j| \leq |a_i \cdot_\psi \omega_i|$. Then*

$$|e_\phi(a_j \cdot_\psi \omega_j)| \leq |e_\phi(a_i \cdot_\psi \omega_i)|.$$

(3) *With the notation in Lemma 4.4, we have*

$$|e_\phi(\omega)| = \max_j \big\{ |a_j \cdot_\phi e_\phi(\omega_j)| \big\}.$$

(4) *For any positive integer $i = r' + 1, \ldots, r$ and $b \in A$ satisfying $\deg(b) < \deg(a)$, we have*

$$|b|_\infty \cdot |e_\phi(\omega_i)| \leq |b \cdot_\phi e_\phi(\omega_i)|.$$

Proof. If $i \leq r'$, then we have $w(e_\phi(a_i \cdot_\psi \omega_i)) = w(a_i \cdot_\psi \omega_i)$ by Lemma 4.3. The rest of (1) has been shown in Lemma 4.4. Similar to the proof of Corollary 3.2(2) (resp. (3)), the claim (2) (resp. (3)) follows from (1) (resp. Lemma 4.4 and (2)).

We show (4). Note $b \cdot_\phi e_\phi(\omega_i) = e_\phi(b \cdot_\psi \omega_i)$. By (1), the desired inequality in (4) is equivalent to

(4.3)

$$|b|_\infty^{r'} \cdot \left(w(\omega_i) + \sum_{\substack{\mu \in \Lambda \setminus \{0\} \\ w(\mu) > w(\omega_i)}} (w(\omega_i) - w(\mu)) \right) \geq w(b \cdot_\psi \omega_i) + \sum_{\substack{\mu \in \Lambda \setminus \{0\} \\ w(\mu) > w(b \cdot_\psi \omega_i)}} (w(b \cdot_\psi \omega_i) - w(\mu)).$$

By Lemma 2.1(2), we may write the left in this inequality to be

$$w(b \cdot_\psi \omega_i) + \sum_{\substack{\mu \in \Lambda \setminus \{0\} \\ w(\mu) > w(\omega_i)}} (w(b \cdot_\psi \omega_i) - w(b \cdot_\psi \mu)).$$

Then (4.3) follows from the inclusion

$$\big\{ b \cdot_\psi \mu \in b \cdot_\psi \Lambda \mid w(b \cdot_\psi \mu) > w(b \cdot_\psi \omega_i) \big\} \subset \{ \mu \in \Lambda \mid w(\mu) > w(b \cdot_\psi \omega_i) \}. \qquad \square$$

Theorem 4.6. (cf. Theorem 3.3) *For any finite prime u of A and any positive integer n, let $\{\omega_i\}_{i=1,\ldots,r}$ be the elements in $u^{-n}\Lambda$ defined before Lemma 4.2. Then the family of elements $\{e_\phi(\omega_i)\}_{i=1,\ldots,r}$ is an SMB of $\phi[u^n]$.*

Proof. Put $\lambda_i := e_\phi(\omega_i)$ for all i. Since $\omega_1, \ldots, \omega_r$ form an A/u^n-basis of $u^{-n}\Lambda/\Lambda$, their images under the A/u^n-module isomorphism $\mathcal{E}_\phi \colon u^{-n}\Lambda/\Lambda \to \phi[u^n]$ are A/u^n-linearly independent.

We check Definition 1.1(2). Fix a positive integer $i \le r$. For $\lambda = \sum_j a_j \cdot_\phi \lambda_j$ with $a_j \in A \mod u^n$ such that $\lambda_1, \ldots, \lambda_{i-1}, \lambda$ are A/u^n-linearly independent, we show $|\lambda_i| \le |\lambda|$. Without loss of generality, we assume $\deg(a_j) < \deg(u^n)$ for any j.

Assume first $i \le r'$. If $a_j = 0$ for all $j > r'$, the desired inequality follows from $\{\omega_j\}_{j=1,\ldots,r'}$ being an SMB of $\psi[u^n]$ and Lemma 4.3. If $a_j \neq 0 \mod u^n$ for some $j > r'$, we can apply Corollary 4.5(1), and we have $\left|\sum_j a_j \cdot_\psi \omega_j\right| \le \left|\sum_j a_j \cdot_\phi \lambda_j\right|$. We know $\left|\sum_j a_j \cdot_\psi \omega_j\right| > 0$ from Lemma 4.2(1). By Lemmas 4.2(1) and 4.3, we have $|\lambda_i| = |\omega_i| \le 0$. Hence

$$|\lambda_i| = |\omega_i| \le 0 < \left|\sum_j a_j \cdot_\psi \omega_j\right| \le \left|\sum_j a_j \cdot_\phi \lambda_j\right|.$$

As for the case $i \ge r'+1$, note that there is $a_k \neq 0$ for some $k \ge i$ as $\lambda_1, \ldots, \lambda_{i-1}, \lambda$ are A/u^n-linearly independent. Similar to the proof of Theorem 3.3, one can apply Corollary 4.5(2), (3) and (4) to show the inequality $|\lambda_i| \le |\lambda|$. □

Corollary 4.7. (cf. Corollary 3.5) *Let $\{\lambda_i\}_{i=1,\ldots,r}$ be an SMB of $\phi[u^n]$.*

(1) *If n is large enough so that $|u^n|_\infty \ge |\omega_r^0|/|\omega_{r'+1}^0|$, then for $i = 1, \ldots, r$, we have $|\lambda_i| = |\omega_i|$.*

(2) *For any positive integer n, we have $|\lambda_r|/|\lambda_{r'+1}| \ge |\omega_r^0|/|\omega_{r'+1}^0|$.*

(3) *If n is large enough so that $|u^n|_\infty > |\omega_r^0|/|\omega_{r'+1}^0|$, then we have $|\lambda_i| < |\omega_{r'+1}^0|$ for $i = 1, \ldots, r$.*

Proof. The equation $|\lambda_i| = |\omega_i|$ for $i = 1, \ldots, r'$ follows from Lemma 4.3. Similar to the proof of Corollary 3.5, one can apply Corollary 4.5(1), Theorem 4.6 and Proposition 2.10(2) to show the rest of the lemma. □

Put $B := \{\omega \in \mathbb{C}_w \mid |\omega| < |\omega_{r'+1}^0|\}$. Since $B \cap \Lambda = \emptyset$, the exponential function e_ϕ is injective on B. By (4.2), we have $|e_\phi(\omega)| = |\omega|$ for $\omega \in B$. This implies $e_\phi(B) \subset B$. Put $C := e_\phi(B)$. There is an inverse $\log_\phi \colon C \to B$ of e_ϕ defined by a power series with coefficients in K and $e_\phi : B \rightleftarrows C : \log_\phi$ are inverse to each other.

Lemma 4.8. (cf. Lemma 3.7)

(1) *We have $C \cap \phi[u^n] = B \cap \phi[u^n]$.*

(2) *We have the following maps which are inverse to each other*

$$e_\phi : B \cap \mathcal{L} \rightleftarrows B \cap \phi[u^n] : \log_\phi,$$

where

$$\mathcal{L} := \left\{ \sum_i a_i \cdot_\psi \omega_i \;\middle|\; a_i \in A \text{ with } \deg(a_i) < \deg(u^n) \right\}$$

is a set of representatives of all elements in $u^{-n}\Lambda/\Lambda$.

(3) *For any* $\lambda \in B \cap \phi[u^n]$, *we have* $|\log_\phi(\lambda)| = |\lambda|$.

Following the strategy of the proof of Lemma 3.7(1), we can show Lemma 4.8(1) alternatively.

Proof of Lemma 4.8. (1) We know $C \cap \phi[u^n] \subset B \cap \phi[u^n]$, which implies $\#B \cap \phi[u^n] \geq \#C \cap \phi[u^n]$, where $\#B \cap \phi[u^n]$ denotes the cardinality of the set $B \cap \phi[u^n]$. We show

$$\#C \cap \phi[u^n] \geq \#B \cap \mathcal{L} \geq \#B \cap \phi[u^n] \geq \#C \cap \phi[u^n].$$

As e_ϕ is injective on $\mathcal{L}$, we have $\#B \cap \mathcal{L} \leq \#C \cap \phi[u^n]$ and it remains to show $\#B \cap \mathcal{L} \geq \#B \cap \phi[u^n]$.

Put $B^c := \{\omega \in \mathbb{C}_w \mid |\omega| \geq |\omega^0_{r'+1}|\}$, which is complementary to B in $\mathbb{C}_w$. For any $\omega = \sum_j a_j \cdot_\psi \omega_j \in B^c \cap \mathcal{L}$, there exists $a_j \neq 0$ for some $j > r'$, for otherwise we have $|\omega| \leq 0 < |\omega^0_{r'+1}|$ by Lemma 4.2(1). By Corollary 4.5(1), we have

$$|e_\phi(\omega)| \geq |\omega| \geq |\omega^0_{r'+1}|.$$

Hence $e_\phi(B^c \cap \mathcal{L}) \subset B^c \cap \phi[u^n]$. As e_ϕ is injective on $\mathcal{L}$, we have $\#B^c \cap \mathcal{L} \leq \#B^c \cap \phi[u^n]$. This implies $\#B \cap \mathcal{L} \geq \#B \cap \phi[u^n]$, as desired.

(2) The map $e_\phi \colon B \cap \mathcal{L} \to B \cap \phi[u^n]$ is injective and is also surjective as $\#B \cap \mathcal{L} = \#B \cap \phi[u^n]$. Hence (2) follows.

(3) By (2), we have $\log_\phi(\lambda) \in B \cap \mathcal{L}$ and $e_\phi(\log_\phi(\lambda)) = \lambda$. Hence (3) follows from Lemma 4.3 and Corollary 4.5(1). ☐

Lemma 4.9. *Let* $\{\lambda_i\}_{i=1,\dots,r}$ *be an SMB of* $\phi[u^n]$. *We have* $w(\lambda_i) \geq 0$ *for* $i \leq r'$ *and* $w(\lambda_i) < 0$ *for* $i > r'$.

Proof. For a positive integer j, let $\{\lambda_{i,j}\}_{i=1,\dots,r}$ be an SMB of $\phi[u^j]$ as in Corollary 2.14. By Proposition 2.10(2), we have $w(\lambda_i) = w(\lambda_{i,n})$ for all i. It suffices to show $w(\lambda_{r',n}) \geq 0$ and $w(\lambda_{r'+1,n}) < 0$.

We first show $w(\lambda_{r',1}) \geq 0$ and $w(\lambda_{r'+1,1}) < 0$. Put $d := \deg(u)$, $u_0 := u$, $\sum_{i=0}^{rd} u_i X^{q^i} := \phi_u(X)$ and $P_i := (q^i, w(u_i))$ for $i = 0, \dots, rd$. As ϕ has stable reduction, we have $w(u_i) \geq 0$ for all i, $w(u_{r'd}) = 0$, and $w(u_i) > 0$ for all $i > r'd$. Hence the point $P_{r'd}$ is a vertex of the Newton polygon of $\phi_u(X)$. The segments on the left (resp. right) of $P_{r'd}$ have slopes ≤ 0 (resp. slopes > 0). Hence there are exactly $q^{r'd}$ roots with valuations ≥ 0. Here $0 \in \phi[u]$ is considered to have valuation > 0.

We show $w(\lambda_{r',1}) \geq 0$ and $w(\lambda_{r'+1,1}) < 0$ by induction. By (2.3), we have $w(\lambda_{1,1}) \geq 0$. Fix a positive integer $k \leq r'$ and assume $w(\lambda_{i,1}) \geq 0$ for $i < k$. Then the elements $\lambda_{i,1}$ for $i < k$ generate an A/u-vector subspace of $\phi[u]$ containing $q^{(k-1)d}$ elements. Since ϕ has stable reduction, for any $a \in A$, all coefficients of $\phi_a(X)$ have valuation ≥ 0. By the ultrametric inequality, we have $w(a \cdot_\phi \lambda_{i,1}) \geq 0$ for any $a \in A \mod u$ and $i < k$. Hence all the elements in the vector subspace have valuations ≥ 0. Since $q^{(k-1)d} < q^{r'd}$, there are elements in $\phi[u] \setminus \bigoplus_{i<k}(A/u) \cdot_\phi \lambda_{i,1}$ having valuation ≥ 0. By (2.3), we have $w(\lambda_{k,1}) \geq 0$. For $k = r'+1$, we have the same inductive hypothesis as above. However, since $q^{(k-1)d} = q^{r'd}$, each element in $\phi[u] \setminus \bigoplus_{i<k}(A/u) \cdot_\phi \lambda_{i,1}$ has valuation < 0 and hence $w(\lambda_{r'+1,1}) < 0$.

Next, we show $w(\lambda_{r',n}) \geq 0$ (resp. $w(\lambda_{r'+1,n}) < 0$) by induction. Assume $w(\lambda_{r',j-1}) \geq 0$ (resp. $w(\lambda_{r'+1,j-1}) < 0$). By Corollary 2.14, the element $\lambda_{r',j}$ (resp. $\lambda_{r'+1,j}$) is a root of $\phi_u(X) - \lambda_{r',j-1}$ (resp. $\phi_u(X) - \lambda_{r'+1,j-1}$) having the largest valuation. By the induction hypothesis and the valuations of the coefficients of $\phi_u(X)$, the left-most segment in the Newton polygon of $\phi_u(X) - \lambda_{r',j-1}$ (resp. $\phi_u(X) - \lambda_{r'+1,j-1}$) has slope ≤ 0 (resp. > 0). Hence we have $w(\lambda_{r',j}) \geq 0$ and $w(\lambda_{r'+1,j}) < 0$. □

Let $\{\lambda_i\}_{i=1,\dots,r}$ denote an SMB of $\phi[u^n]$. Assume that the positive integer n is large enough so that $|u^n|_\infty > |\omega^0_r|/|\omega^0_{r'+1}|$. By Corollary 4.7(3) and Lemma 4.8(1), for each i, we have $\lambda_i \in B \cap \phi[u^n] = C \cap \phi[u^n]$ and we put $\omega'_i := \log_\phi(\lambda_i)$.

Theorem 4.10. (cf. Theorem 3.8)

(1) *The family of elements* $\{\omega'_i\}_{i=1,\dots,r'}$ *is an SMB of* $\psi[u^n]$.

(2) *The family of elements* $\{u^n \cdot_\psi \omega'_i\}_{i=r'+1,\dots,r}$ *is an SMB of* Λ.

Proof. (1) To check Definition 1.1(1), we show that the elements ω'_i for $i \leq r'$ belong to $\psi[u^n]$ and are A/u^n-linearly independent. By Lemma 4.8(3) and Lemma 4.9, we have $w(\omega'_i) = w(\lambda_i) \geq 0$ for $i \leq r'$. By Lemma 4.1(3), this implies that $\omega'_i \in \psi[u^n]$ for $i \leq r'$. Note that $\mathcal{E}_\phi \colon u^{-n}\Lambda/\Lambda \to \phi[u^n]$ is an A/u^n-module isomorphism induced by e_ϕ and $e_\phi(\omega'_i) = \lambda_i$. If $\sum_{i\leq r'} a_i \cdot_\psi \omega'_i = 0$ with $a_i \in A \mod u^n$, then we have $\sum_{i\leq r'} a_i \cdot_\phi \lambda_i = 0$. This implies $a_i \equiv 0 \mod u^n$ and hence the desired linear independence.

As $\{\lambda_i\}_{i=1,\dots,r}$ is an SMB of $\phi[u^n]$, we can straightforwardly check Definition 1.1(2) using Lemma 4.3.

(2) Similar to (1), we can apply Lemmas 4.9, 4.8(3), 4.1(3) to show $\omega'_i \notin \psi[u^n]$ such that $u^n \cdot_\psi \omega'_i$ for $i > r'$ belong to Λ. We check the two dots in Proposition 2.4(1). Let us show that $\omega'_{r'+1}, \dots, \omega'_r$ are A-linearly independent first. If there exist $a_i \in A$ such that $\sum_{i>r'} a_i \cdot_\psi \omega'_i = 0$, we can show $a_i \equiv 0 \mod u^n$ for all i similar to (1). Assume $a_i \neq 0$ for some i. Let m be the integer such that $u^m \mid a_i$ for all $i > r'$ and $u^{m+1} \nmid a_i$ for some

i. Then there exist $b_i \in A$ such that $a_i = b_i u^m$ for all $i > r'$ and $b_i \not\equiv 0 \mod u$ for some i. Hence $\sum_{i>r'} b_i \cdot_\psi \omega_i'$ is a root of $\psi_{u^m}(X)$ and we denote this root by ω. On the other hand,

$$u^n \cdot_\psi \omega = \sum_{i>r'} b_i \cdot_\psi \left(u^n \cdot_\psi \omega_i'\right) \in \Lambda.$$

Since $\Lambda \cap \psi[u^m] = 0$, we have $u^n \cdot_\psi \omega = 0$ and hence $\omega \in \psi[u^n]$. By (1), there exist $b_i \in A \mod u^n$ for $i \le r'$ such that $\omega = \sum_{i\le r'} b_i \cdot_\psi \omega_i'$. This equality implies

$$0 = e_\phi\left(\sum_{i>r'} b_i \cdot_\psi \omega_i' - \sum_{i\le r'} b_i \cdot_\psi \omega_i'\right) = \sum_{i>r'} b_i \cdot_\phi \lambda_i - \sum_{i\le r'} b_i \cdot_\phi \lambda_i.$$

As some $b_i \not\equiv 0 \mod u^n$, this is absurd.

Finally, we check the second dot in Proposition 2.4(1). Let $l_{r'+1} \le \cdots \le l_r$ denote the invariant of Λ as in Proposition 2.4(2). Fix i to be a positive integer satisfying $r' < i \le r$. It suffices to show $l_i = |u^n \cdot_\psi \omega_i'|$. We have $l_i \le |u^n \cdot_\psi \omega_i'|$. Let us assume $l_i < |u^n \cdot_\psi \omega_i'|$. Since $\lambda_i \in B \cap \phi[u^n]$, we have $|\omega_i'| = |\lambda_i|$ by Lemma 4.8(3). Hence $l_i/|u^n|_\infty < |\omega_i'| = |\lambda_i| < |\omega^0_{r'+1}|$. By Proposition 2.4, there is an SMB $\{\eta^0_j\}_{j=r'+1,\dots,r}$ of Λ such that $|\eta^0_i| = l_i$. Let η_j be a root of $\psi_{u^n}(X) - \eta^0_j$ for all j (cf. the definition of ω_j before Lemma 4.2). As $|\eta_i| = l_i/|u^n|_\infty < |\omega^0_{r'+1}|$, we have $|e_\phi(\eta_i)| = |\eta_i|$ by (4.2). This implies

$$|e_\phi(\eta_i)| = |\eta_i| = l_i/|u^n|_\infty < |\omega_i'| = |\lambda_i|.$$

By Theorem 4.6, the elements $e_\phi(\omega_j')$ for $j = 1, \dots, r'$ and $e_\phi(\eta_j)$ for $j = r'+1, \dots, r$ form an SMB of $\phi[u^n]$. By Proposition 2.10(2), this contradicts $|e_\phi(\eta_i)| < |\lambda_i|$. □

Proposition 4.11. (cf. Proposition 3.10) *If n is large enough so that $|u^n|_\infty \ge |\omega^0_r|/|\omega^0_{r'+1}|$, then we have*

$$K(u^{-n}\Lambda) = K(\phi[u^n]),$$

where $K(u^{-n}\Lambda)$ (resp. $K(\phi[u^n])$) is the extension of K generated by all elements in $u^{-n}\Lambda$ (resp. in $\phi[u^n]$).

Proof. Note that e_ϕ is given by a power series with coefficients in K and it induces an isomorphism $\mathcal{E}_\phi\colon u^{-n}\Lambda/\Lambda \to \phi[u^n]$. Similar to the proof of Proposition 3.10, one can show $K(\phi[u^n]) \subset K(u^{-n}\Lambda)$.

Note that $\log_\phi$ is given by a power series with coefficients in K. For any $y \in C \cap K^{\mathrm{sep}}$, we have $\log_\phi(y) \in K(y)$. Let $\{\lambda_i\}_{i=1,\dots,r}$ be an SMB of $\phi[u^n]$. As $|u^n|_\infty > |\omega^0_r|/|\omega^0_{r'+1}|$, by Theorem 4.10, the families $\{\omega_i'\}_{i=1,\dots,r'}$ and $\{u^n \cdot_\psi \omega_i'\}_{i=r'+1,\dots,r}$ are respectively an SMB of $\psi[u^n]$ and Λ, where $\omega_i' = \log_\phi(\lambda_i)$. Since $K(\omega_i') \subset K(\lambda_i)$ for each i, it suffices to show that ω_i' for all i form a generating set of $u^{-n}\Lambda$. For any $\omega \in u^{-n}\Lambda$, it is a root of $\psi_{u^n}(X) - u^n \cdot_\psi \omega$. Note $u^n \cdot_\psi \omega \in \Lambda$. Since $\{u^n \cdot_\psi \omega_i'\}_{i=r'+1,\dots,r}$ is an SMB of Λ, we have

$u^n \cdot_\psi \omega = \sum_{i>r'} a_i \cdot_\psi (u^n \cdot_\psi \omega_i')$ for some $a_i \in A$. Hence $\sum_{i>r'} a_i \cdot_\psi \omega_i'$ is also a root of $\psi_{u^n}(X) - u^n \cdot_\psi \omega$. Since $\{\omega_i'\}_{i=1,\ldots,r'}$ is an SMB of $\psi[u^n]$, we have $\sum_{i>r'} a_i \cdot_\psi \omega_i' - \omega = \sum_{i\le r'} a_i \cdot_\psi \omega_i'$ for some $a_i \in A \mod u^n$ and the claim follows. $\square$

Combining Corollary 4.7(2), Theorem 4.10 and Proposition 4.11, we have

Corollary 4.12. (cf. Corollary 3.11) *Let l be a positive integer and $\{\eta_i\}_{i=1,\ldots,r}$ an SMB of $\phi[u^l]$. Let $\{\lambda_i\}_{i=1,\ldots,r}$ be an SMB of $\phi[u^n]$. If n is large enough so that $|u^n|_\infty > |\eta_r|/|\eta_{r'+1}|$, then we have*

(1) *the family $\{\log_\phi(\lambda_i)\}_{i=1,\ldots,r'}$ is an SMB of $\psi[u^n]$;*

(2) *the family $\{u^n \cdot_\psi \log_\phi(\lambda_i)\}_{i=r'+1,\ldots,r}$ is an SMB of Λ;*

(3) *$K(u^{-n}\Lambda) = K(\phi[u^n])$.*

Proposition 4.13. (cf. Proposition 3.12) *Assume $w(u) = 0$, i.e., u is not divisible by the prime w. Let $\{\lambda_i\}_{i=1,\ldots,r}$ be an SMB of $\phi[u^n]$. Then we have*

$$\left|\sum_i a_i \cdot_\phi \lambda_i\right| = \max_i \{|a_i \cdot_\phi \lambda_i|\}$$

for any $a_i \in A \mod u^n$.

Proof. Assume first that n is large enough so that $|u^n|_\infty > |\omega_r^0|/|\omega_{r'+1}^0|$. By Theorem 4.10, the families $\{\omega_i'\}_{i=1,\ldots,r'}$ and $\{u^n \cdot_\psi \omega_i'\}_{i=r'+1,\ldots,r}$ are respectively an SMB of $\psi[u^n]$ and Λ, where $\omega_i' = \log_\phi(\lambda_i)$. Without loss of generality, we assume $\deg(a_i) < \deg(u^n)$. Assume that a_i is nonzero for some $i > r'$. By Corollary 4.5(3), we have

$$\left|e_\phi\left(\sum_i a_i \cdot_\psi \omega_i'\right)\right| = \max_i \{|a_i \cdot_\phi e_\phi(\omega_i')|\}.$$

As $e_\phi\big(\sum_i a_i \cdot_\psi \omega_i'\big) = \sum_i a_i \cdot_\phi \lambda_i$, the claim follows. If $a_i = 0$ for all $i > r'$, then $\sum_{i\le r'} a_i \cdot_\psi \omega_i'$ belongs to $\psi[u^n]$. By Lemma 4.1(1), we have $\left|\sum_{i\le r'} a_i \cdot_\psi \omega_i'\right| = 0$ if some $a_i \ne 0$ and $|a_i \cdot_\psi \omega_i'| = 0$ for all $i \le r'$ if $a_i \ne 0$. The desired equality follows from Lemma 4.3. Similar to the proof of Proposition 3.12, the case where n is arbitrary follows from the case where n is large enough. $\square$

Proposition 4.14. (cf. Proposition 3.13) *Let u be a finite prime of A not divisible by the prime w. Let $\{\lambda_i\}_{i=1,\ldots,r}$ be an SMB of $\phi[u^n]$ so that $|\lambda_1| = \cdots = |\lambda_s| < |\lambda_{s+1}|$ for some positive integer s.*

(1) *The extension of K generated by $\lambda_1, \ldots, \lambda_s$ is unramified.*

(2) *For an element σ in the ramification subgroup* $\mathrm{Gal}(K(\phi[u^n])/K)_0$*, we have* $\sigma(\lambda_j) = \lambda_j$ *for* $j = 1, \dots, s$.

Proof. (2) straightforwardly follows from (1). We show (1). Notice $w(u) = 0$. Following the proof of Lemma 4.9, we know that $s = r'$ and $0 = w(\lambda_1) = \cdots = w(\lambda_{r'}) > w(\lambda_{r'+1})$. Let n' be an integer satisfying $n' \geq n$ and $|u^{n'}|_\infty > |\lambda_r|/|\lambda_{r'+1}|$. By Proposition 2.13(1), we know that there exists an SMB $\{\lambda'_i\}_{i=1,\dots,r}$ of $\phi[u^{n'}]$ such that $u^{n'-n} \cdot_\phi \lambda'_i = \lambda_i$ for all i. By Corollary 4.12, with $\omega'_i = \log_\phi(\lambda'_i)$ for all i, we have that the families $\{\omega'_i\}_{i=1,\dots,r'}$ and $\{u^{n'} \cdot_\psi \omega'_i\}_{i=r'+1,\dots,r}$ are respectively an SMB of $\psi[u^{n'}]$ and an SMB of Λ. As $e_\phi(u^{n'-n} \cdot_\psi \omega'_i) = \lambda_i$, we have $K(\lambda_i) \subset K(u^{n'-n} \cdot_\psi \omega'_i)$ for all i. Note that $\{u^{n'-n} \cdot_\psi \omega'_i\}_{i=1,\dots,r'}$ is an SMB of $\psi[u^n]$ by Proposition 2.13(2). By [14, Theorem 6.3.1] (initially proved by Takahashi), the extension of K generated by the elements in $\psi[u^n]$ is unramified. The result follows. $\square$

We assumed above that each Drinfeld module has stable reduction over K. For a Drinfeld A-module ϕ over K which does not have stable reduction, it turns out that ϕ is isomorphic to a Drinfeld module having stable reduction over an at worst tamely ramified subextension of $K(\phi[u])/K$ with u not divisible by w.

Proposition 4.15. *Let ϕ be a rank r Drinfeld A-module over K which does not necessarily have stable reduction. Let u be a finite prime of A with $w \nmid u$. Let r' be the positive integer $\leq r$ so that ϕ is isomorphic to a Drinfeld module having stable reduction over some extension of K and the reduction has rank $r' \leq r$. Let $\{\lambda_i\}_{i=1,\dots,r}$ be an SMB of $\phi[u]$. Then $b\phi b^{-1}$ has stable reduction over $K(\lambda_1)$ for $b = \lambda_1^{-1}$ and the extension $K(\lambda_1)/K$ is at worst tamely ramified.*

Proof. Assume $r' < r$. Put $tX + \sum_{i=1}^r a_i X^{q^i} := \phi_t(X)$. Let M be a tamely ramified extension of K of degree $q^{r'} - 1$. Let b be an element in M with valuation $w(b) = \frac{w(a_{r'})}{q^{r'}-1}$. Then $b\phi b^{-1}$ has stable reduction over M. The family $\{b\lambda_i\}_{i=1,\dots,r}$ is an SMB of $b\phi b^{-1}[u]$ (see Remark 2.9). By Proposition 4.14, the extension $M(\lambda_1)/M = M(b\lambda_1)/M$ is unramified. Hence $M(\lambda_1)/K$ is tamely ramified and its subextension $K(\lambda_1)/K$ is at worst tamely ramified.

Following the proof of Lemma 4.9, we know that $w(b\lambda_i) = 0$ for $i = 1, \dots, r'$. Hence $w(\lambda_1) = -\frac{w(a_{r'})}{q^{r'}-1}$ and we may take b to be λ_1^{-1}.

As for the case $r' = r$, for a tamely ramified extension M/K of degree $q^r - 1$ and $b \in M$ with $w(b) = \frac{w(a_r)}{q^r-1}$, we have that $b\phi b^{-1}$ has good reduction over M. By [14, Theorem 6.3.1], the extension $M(b\lambda_1)/M$ is unramified. The result for the case $r' = r$ follows similarly. $\square$

5. Applications to rank 2 Drinfeld modules, infinite prime case

Throughout this section, let w be an infinite prime, u a finite prime of A having degree d, and n a positive integer. Let ϕ be a rank 2 Drinfeld A-module over K determined by $\phi_t(X) = tX + a_1X^q + a_2X^{q^2} \in K[X]$. Let $\boldsymbol{j}$ denote the j-invariant a_1^{q+1}/a_2 of ϕ. Put $w_0 := w(t)$, $w_1 := w(a_1)$ and $w_2 := w(a_2)$. For each positive integer j, let $\{\xi_{i,j}\}_{i=1,2}$ be an SMB of $\phi[t^j]$ obtained as in Corollary 2.14.

5.1. The valuations of SMBs

Our goal is to determine the valuations of elements of SMBs of the lattice Λ and the module $\phi[u^n]$ in terms of w_0, w_1 and w_2. If $w(\boldsymbol{j}) < w_0q$, let m be the integer satisfying $w(\boldsymbol{j}) \in (w_0q^{m+1}, w_0q^m]$. By [1, Lemma 2.1], we have

$$w(\xi_{1,n}) = -\left(w_0(n-1) + \frac{w_1 - w_0}{q-1}\right) \quad \text{for } n \geq 1 \text{ and}$$
$$w(\xi_{2,n}) = \begin{cases} -\frac{w_2+w_1(q^n-q-1)}{(q-1)q^n}, & 0 < n \leq m; \\ -\left(w_0(n-m) + \frac{w_2+w_1(q^m-q-1)}{(q-1)q^m}\right), & n \geq m. \end{cases}$$

Now the condition $|t^n| \geq |\xi_{2,n}|/|\xi_{1,n}|$ in Remark 3.6 reads $-w_0n \geq -w(\xi_{2,n}) + w(\xi_{1,n})$. For $n \geq m$, this inequality is equivalent to

$$-w_0n \geq -w_0(m-1) + \frac{w_0}{q-1} - \frac{w(\boldsymbol{j})}{(q-1)q^m}.$$

For any $n \geq m$, the inequality $|t^n| \geq |\xi_{2,n}|/|\xi_{1,n}|$ holds. If $w(\boldsymbol{j}) \geq w_0q$, by [1, Proposition 2.4], we have

$$w(\xi_{1,n}) = w(\xi_{2,n}) = -\left(w_0(n-1) + \frac{w_2 - w_0}{q^2-1}\right).$$

Hence the condition $|t^n| \geq |\xi_{2,n}|/|\xi_{1,n}|$ is fulfilled for any positive integer n.

Proposition 5.1. *Let $\{\omega_i\}_{i=1,2}$ be an SMB of Λ and $\{\lambda_i\}_{i=1,2}$ an SMB of $\phi[u^n]$.*

(1) *If $w(\boldsymbol{j}) < w_0q$ and m is the integer so that $w(\boldsymbol{j}) \in (w_0q^{m+1}, w_0q^m]$, we have*

$$w(\omega_1) = w_0 + \frac{w_0}{q-1} - \frac{w_1}{q-1},$$
$$w(\omega_2) = w_0m + \frac{w(\boldsymbol{j})}{(q-1)q^m} - \frac{w_1}{q-1}.$$

For $n \geq m/d$, we have $|u^n| > |\omega_2|/|\omega_1|$, $w(\lambda_1) = w(\xi_{1,nd})$ and $w(\lambda_2) = w(\xi_{2,nd})$.

(2) *If $w(\boldsymbol{j}) \geq w_0 q$, we have*

$$w(\omega_1) = w(\omega_2) = w_0 + \frac{w_0}{q^2-1} - \frac{w_2}{q^2-1}.$$

For $n \geq 1$, we have $w(\lambda_1) = w(\lambda_2) = w(\xi_{1,nd}) = w(\xi_{2,nd})$.

We note that the valuations $w(\omega_1)$ and $w(\omega_2)$ above have been obtained by Chen–Lee in [5, Theorem 3.1 and Corollary 3.1]. One may also recover the rank $r = 2$ case of Gekeler's formula [9, Proposition 3.2] (see also [14, Proposition 5.5.8]).

Proof of Proposition 5.1. The claims of $w(\omega_1)$ and $w(\omega_2)$ follow from Remark 3.6, Corollary 3.5(1), and the arguments before the proposition. Then the claims of $w(\lambda_1)$ and $w(\lambda_2)$ are proved by Corollary 3.5(1). □

Remark 5.2. Let r be a positive integer and ϕ a rank r Drinfeld A-module over K such that $\phi_t(X) = tX + a_s X^{q^s} + a_r X^{q^r} \in K[X]$. Here s is an positive integer $< r$. Let $\{\omega_i\}_{i=1,\ldots,r}$ be an SMB of Λ (associated to ϕ) and $\{\lambda_i\}_{i=1,\ldots,r}$ an SMB of $\phi[u^n]$ for u and n as above. Put

$$\boldsymbol{j} := a_s^{\frac{q^r-1}{q-1}} / a_r^{\frac{q^s-1}{q-1}}.$$

We obtain the following generalization of Proposition 5.1. Its proof is similar to [1, Lemma 2.1 and Proposition 2.4] and Proposition 5.1:

(1) If $w(\boldsymbol{j}) < w_0 q^s \frac{q^{r-s}-1}{q-1}$ and m is the integer such that

$$w(\boldsymbol{j}) \in \left(w_0 q^{(m+1)s} \frac{q^{r-s}-1}{q-1}, w_0 q^{ms} \frac{q^{r-s}-1}{q-1} \right],$$

we have

$$w(\omega_i) = w_0 + \frac{w_0}{q^s-1} - \frac{w_s}{q^s-1} \quad \text{for } i = 1, \ldots, s,$$
$$w(\omega_i) = w_0 m + \frac{w(\boldsymbol{j})(q-1)}{q^{ms}(q^s-1)(q^{r-s}-1)} - \frac{w_s}{q^s-1} \quad \text{for } i = s+1, \ldots, r.$$

For $n \geq m/d$, we have $|u^n| > |\omega_r|/|\omega_1|$, $w(\lambda_i) = -w_0 nd + w(\omega_i)$ for $i = 1, \ldots, s$, and $w(\lambda_i) = -w_0 nd + w(\omega_i)$ for $i = s+1, \ldots, r$.

(2) If $w(\boldsymbol{j}) \geq w_0 q^s \frac{q^{r-s}-1}{q-1}$, we have

$$w(\omega_i) = w_0 + \frac{w_0}{q^r-1} - \frac{w_r}{q^r-1} \quad \text{for } i = 1, \ldots, r.$$

For $i = 1, \ldots, r$ and $n \geq 1$, we have $w(\lambda_i) = -w_0 nd + w(\omega_i)$.

5.2. The action of the wild ramification subgroup on the division points

Let $K(\Lambda)$ (resp. $K(\phi[u^n])$) denote the extension of K generated by all elements in Λ (resp. in $\phi[u^n]$). If $w(\boldsymbol{j}) < w_0 q$ and m is the integer such that $w(\boldsymbol{j}) \in (w_0 q^{m+1}, w_0 q^m]$, then by Propositions 5.1(1) and 3.10, we have for any integer $n \geq m/d$ that (cf. [1, Lemma 3.3])

$$K(\phi[u^n]) = K(\Lambda) = K(\phi[t^m]). \tag{5.1}$$

If $w(\boldsymbol{j}) \geq w_0 q$, then by Propositions 5.1(2) and 3.10, we have for any positive integer n that (cf. [1, Lemma 3.14])

$$K(\phi[u^n]) = K(\Lambda) = K(\phi[t]). \tag{5.2}$$

Put $G(\Lambda) := \mathrm{Gal}(K(\Lambda)/K)$. Let $G(\Lambda)_i$ and $G(\Lambda)^y$ denote respectively the i-th lower and y-th higher ramification subgroups. We are to study the action of the wild ramification subgroup $G(\Lambda)_1$ on the SMBs of $\phi[u^n]$ for n to be large enough. Let us recall two lemmas.

Lemma 5.3. [1, Lemma 3.8] *Assume $w(\boldsymbol{j}) < w_0 q$ and $p \nmid w(\boldsymbol{j})$. Let m be the integer satisfying $w(\boldsymbol{j}) \in (w_0 q^{m+1}, w_0 q^m)$. Then we have the (Herbrand) ψ-function of the extension $K(\Lambda)/K$ to be*

$$\psi_{K(\Lambda)/K}(y) = \begin{cases} y, & -1 \leq y \leq 0; \\ Ey, & 0 \leq y \leq r_m; \\ q^j E y + w(\boldsymbol{j}) E \frac{q^j - 1}{q-1} - w_0 j E q^m, & r_{m-j+1} \leq y \leq r_{m-j} \text{ for } j = 1, \ldots, m-1; \\ q^m E y + w(\boldsymbol{j}) E \frac{q^m - 1}{q-1} - w_0 m E q^m, & r_1 \leq y, \end{cases}$$

where

$$r_n := \frac{-w(\boldsymbol{j}) + w_0 q^n}{q-1}$$

for any positive integer $n \leq m$ and E is some positive integer not divisible by p.

Lemma 5.4. [1, Lemma 3.14] *Assume $w(\boldsymbol{j}) \geq w_0 q$. Then the extension $K(\phi[t])/K$ is at worst tamely ramified.*

In Lemma-Definition A.1, the conductor of ϕ at w is defined to be

$$\mathfrak{f}_w(\phi) := \int_0^\infty \left(2 - \mathrm{rank}_{A_u} T_u^{G^y}\right) dy,$$

where T_u is the u-adic Tate module of ϕ and G^y is the y-th upper ramification subgroup of the Galois group $\mathrm{Gal}(K^{\mathrm{sep}}/K)$. In the next result, we calculate $\mathfrak{f}_w(\phi)$ explicitly. This calculation generalizes the infinite prime case of [1, Lemma-Definition 4.1].

Lemma 5.5. *Assume that one of the following two cases happens*

(C1) $w(\boldsymbol{j}) < w_0 q$ *and* $p \nmid w(\boldsymbol{j})$*;*

(C2) $w(\boldsymbol{j}) \geq w_0 q$*.*

Then we have

$$\mathfrak{f}_w(\phi) = \begin{cases} \frac{-w(\boldsymbol{j})+w_0 q}{q-1} & \textit{if } (\mathrm{C1}) \textit{ happens,} \\ 0 & \textit{if } (\mathrm{C2}) \textit{ happens.} \end{cases}$$

Proof. By Corollary 2.14, there is an SMB $\{\lambda_{i,n}\}_{i=1,2}$ of $\phi[u^n]$ for each integer $n \geq 1$ such that $u \cdot_\phi \lambda_{i,n+1} = \lambda_{i,n}$ for $i = 1, 2$. Recall that T_u is defined to be $\varprojlim_n \phi[u^n]$ using the morphisms $\phi_u \colon \phi[u^{n+1}] \to \phi[u^n]$ for all integers $n \geq 1$. Hence the tuples $(\lambda_{1,n})_{n\geq 1}$ and $(\lambda_{2,n})_{n\geq 1}$ form an A_u-basis of T_u.

Assume (C1) happens. By (5.1), the action of G^y on $\phi[u^n]$ for any $n \geq m/d$ and any $y > 0$ factors through $G(\Lambda)^y$. Notice $G(\Lambda)_1 = \bigcup_{y>0} G(\Lambda)^y$. By Proposition 3.13, any element $\sigma \in G(\Lambda)^y$ for $y > 0$ fixes $\lambda_{1,n}$ and fixes $u^j \cdot_\phi \lambda_{1,n} = \lambda_{1,n-j}$ for any positive integer $j < n$. Hence σ fixes $(\lambda_{1,n})_{n\geq 1}$. As $\lambda_{1,n}$ and $\lambda_{2,n}$ generate $K(\phi[u^n])/K = K(\Lambda)/K$ for $n \geq m/d$, we also have that if σ is not the unit, then it nontrivially acts on $\lambda_{2,n}$ and hence nontrivially acts on $(\lambda_{2,n})_{n\geq 1}$. Therefore Lemma 5.3 implies $\mathrm{rank}_{A_u} T_u^{G^y} = 1$ if $0 < y \leq r_1$ and $= 2$ if $r_1 < y$. We have

$$\mathfrak{f}_w(\phi) = \int_0^{r_1} 1\, dy = \frac{-w(\boldsymbol{j}) + w_0 q}{q-1}.$$

For the case (C2), by (5.2), the action of G^y on $\phi[u^n]$ for any $n \geq 1$ and any $y > 0$ factors through $G(\Lambda)^y$. By Lemma 5.4, we have $G(\Lambda)^y = \{e\}$ if $y > 0$. The result for the case (C2) immediately follows. $\square$

For an SMB $\{\lambda_{i,n}\}_{i=1,2}$ of $\phi[u^n]$ and an element $\sigma \in G(\Lambda)_1$ which is not the unit, we work out $\sigma(\lambda_{2,n})$ in the remainder of this subsection.

Lemma 5.6. *Assume* $w(\boldsymbol{j}) \in (w_0 q^{m+1}, w_0 q^m)$ *for a positive integer* m*. Let* n *be an integer* $\geq m/d$ *and* $\{\lambda_i\}_{i=1,2}$ *an SMB of* $\phi[u^n]$*. Then we have*

$$w(t^i \cdot_\phi \lambda_1) = w(\xi_{1,nd-i}) \quad \textit{and} \quad w(t^i \cdot_\phi \lambda_2) = w(\xi_{2,nd-i}) \quad \textit{for } 1 \leq i < nd.$$

Proof. We show the result for λ_2. The proof of the result for λ_1 is similar. By Proposition 5.1(1), we have $w(\lambda_2) = w(\xi_{2,nd})$. To know $w(t \cdot_\phi \lambda_2) = w(t\lambda_2 + a_1\lambda_2^q + a_2\lambda_2^{q^2})$, we calculate

$$w(t\lambda_2) - w(a_1\lambda_2^q) = \frac{-w(\boldsymbol{j}) + w_0 q^m((q-1)(nd-m)+1)}{q^m},$$

$$w(a_1\lambda_2^q) - w(a_2\lambda_2^{q^2}) = \frac{w(\boldsymbol{j})(q^{m-1}-1) + w_0(q-1)(nd-m)q^m}{q^{m-1}} < 0.$$

We have

$$w(t\lambda_2) - w(a_1\lambda_2^q) \begin{cases} > 0 & \text{if } nd = m, \\ < 0 & \text{if } nd > m. \end{cases}$$

Hence we have

$$w(t \cdot_\phi \lambda_2) = \begin{cases} w(a_1\lambda_2^q) = w(\xi_{2,m-1}) & \text{if } nd = m, \\ w(t\lambda_2) = w(\xi_{2,nd-1}) & \text{if } nd > m. \end{cases}$$

We assume that the result for $i-1$ is valid. Put $\lambda_2' := t^{i-1} \cdot_\phi \lambda_2$. If $i \le nd - m$, to know $w(t \cdot_\phi \lambda_2')$, we calculate

$$w(t\lambda_2') - w(a_1\lambda_2'^q) = \frac{-w(\boldsymbol{j}) + w_0 q^m((q-1)(nd-i-m)+q)}{q^m} < 0,$$
$$w(t\lambda_2') - w(a_2\lambda_2'^{q^2}) = \frac{w(\boldsymbol{j})(q^m - q - 1) + w_0 q^m((q^2-1)(nd-i-m)+q^2)}{q^m} < 0.$$

Hence we have $w(t \cdot_\phi \lambda_2') = w(t\lambda_2') = w(\xi_{2,nd-i})$. Assume $i > nd - m$. To know $w(t \cdot_\phi \lambda_2')$, we calculate

$$w(t\lambda_2') - w(a_1\lambda_2'^q) = \frac{-w(\boldsymbol{j}) + w_0 q^{nd-i+1}}{q^{nd-i+1}} > 0,$$
$$w(a_1\lambda_2'^q) - w(a_2\lambda_2'^{q^2}) = \frac{w(\boldsymbol{j})(q^{nd-i}-1)}{q^{nd-i}} < 0.$$

Hence $w(t \cdot_\phi \lambda_2') = w(a_1\lambda_2'^q) = w(\xi_{2,nd-i})$ and the result for λ_2 follows. □

Corollary 5.7. *Resume the assumptions in the lemma.*

(1) *For any $a \in A$ with* $\deg(a) < nd$, *we have*

$$w(a \cdot_\phi \lambda_1) = w(t^{\deg(a)} \cdot_\phi \lambda_1) = w(\xi_{1,nd-\deg(a)}), \tag{5.3}$$
$$w(a \cdot_\phi \lambda_2) = w(t^{\deg(a)} \cdot_\phi \lambda_2) = w(\xi_{2,nd-\deg(a)}). \tag{5.4}$$

(2) *For $\lambda \in \phi[u^n]$ having valuation $\ge w(\xi_{1,nd-m+1})$, there exists some $b \in A$ with* $\deg(b) < m$ *such that $b \cdot_\phi \lambda_1 = \lambda$.*

Proof. By [1, Proposition 2.2], we have

$$w(\xi_{1,j}) > w(\xi_{2,nd}) \quad \text{for } j = nd, nd-1, \ldots, nd-m+1, \tag{5.5}$$
$$w(\xi_{i,j+1}) > w(\xi_{i,j}) \quad \text{for } i = 1,2 \text{ and positive integers } j < nd. \tag{5.6}$$

For (1), by (5.6) and the lemma, we have $w(t^{\deg(a)} \cdot_\phi \lambda_1) < w(t^i \cdot_\phi \lambda_1)$ for any positive integer $i < \deg(a)$. Hence the desired equality follows from the ultrametric inequality. The equation for λ_2 follows in the same way.

For (2), by (5.5), we have $w(\lambda) \geq w(\xi_{1,nd-m+1}) > w(\xi_{2,nd}) = w(\lambda_2)$. As $\{\lambda_i\}_{i=1,2}$ is an SMB of $\phi[u^n]$, there exist $b, b' \in A \mod u^n$ such that $\lambda = b \cdot_\phi \lambda_1 + b' \cdot_\phi \lambda_2$. We may assume that b and b' have degrees $< \deg(u^n) = nd$. Assume conversely $b' \neq 0$. By (5.4) and (5.6), we have

$$w(b' \cdot_\phi \lambda_2) = w(t^{\deg(b')} \cdot_\phi \lambda_2) = w(\xi_{2,nd-\deg(b')}) \leq w(\lambda_2).$$

By Proposition 3.12, we have $w(\lambda) = \min\{w(b \cdot_\phi \lambda_1), w(b' \cdot_\phi \lambda_2)\}$. Hence $w(\lambda) \leq w(b' \cdot_\phi \lambda_2) \leq w(\lambda_2)$, a contradiction. By (5.3), we have

$$w(b \cdot_\phi \lambda_1) = w(t^{\deg(b)} \cdot_\phi \lambda_1) = w(\xi_{1,nd-\deg(b)}).$$

Then $w(b \cdot_\phi \lambda_1) \geq w(\xi_{1,nd-m+1})$ and (5.6) imply $\deg(b) < m$. □

Remark 5.8. Resume the assumptions in the lemma.

(1) The elements $t^j \cdot_\phi \lambda_i$ for $i = 1, 2$ and $0 \leq j < nd$ form an $\mathbb{F}_q$-basis of $\phi[u^n]$ as a vector space. Indeed, by the lemma and [1, Proposition 2.2], the valuations $w(t^j \cdot_\phi \lambda_i)$ for all i and j are different from each other. Hence all elements $t^j \cdot_\phi \lambda_i$ are $\mathbb{F}_q$-linearly independent and form a $2nd$-dimensional vector subspace of $\phi[u^n]$. Since $\phi[u^n]$ has dimension $2nd$ as an $\mathbb{F}_q$-vector space, the claim follows.

(2) For a positive integer $j \leq n$, let $\{\lambda'_i\}_{i=1,2}$ be an SMB of $\phi[u^j]$. By Corollary 5.7(1), we have

$$w(\lambda'_1) = w(\xi_{1,jd}) \quad \text{and} \quad w(\lambda'_2) = w(\xi_{2,jd}).$$

Under the assumptions in Lemma 5.3, we put $R_i := \psi_{K(\Lambda)/K}(r_i)$ for $i = 1, \ldots, m$ and we have

$$R_i = -w(\boldsymbol{j})E\frac{1}{q-1} - w_0 E q^m \left(m - i - \frac{1}{q-1}\right).$$

Theorem 5.9. *Assume $w(\boldsymbol{j}) < w_0 q$ and $p \nmid w(\boldsymbol{j})$. Let m be the integer so that $w(\boldsymbol{j}) \in (w_0 q^{m+1}, w_0 q^m)$. Let n be an integer $\geq m/d$ and $\{\lambda_i\}_{i=1,2}$ an SMB of $\phi[u^n]$. Put $G(\Lambda) :=$ $\mathrm{Gal}(K(\Lambda)/K)$. For a positive integer i, let $A^{<i}$ denote the subgroup of A consisting of elements with degrees $< i$.*

(1) *Any element in $G(\Lambda)_1$ fixes λ_1;*

(2) *The map*

$$g \colon G(\Lambda)_1 \to A^{<m} \cdot_\phi \lambda_1, \quad \sigma \mapsto \sigma(\lambda_2) - \lambda_2$$

is an isomorphism.

(3) *Put* $r_i := \frac{-w(\boldsymbol{j})+w_0q^i}{q-1}$ *for* $1 \leq i \leq m$ *as in Lemma* 5.3. *Let* $G(\Lambda)^{r_i}$ *denote the* r_i*-th upper ramification subgroup of* $G(\Lambda)$. *Then for each* $i = 1, \ldots, m$, *the restriction*

$$g\colon G(\Lambda)^{r_i} \to A^{<i} \cdot_\phi \lambda_1$$

is an isomorphism.

Proof. (1) has been shown in Proposition 3.13.

(2) We show $\sigma(\lambda_2) - \lambda_2 \in A^{<m} \cdot_\phi \lambda_1$ for an element σ in $G(\Lambda)_1 = G(\Lambda)_{R_m}$. Clearly $\sigma(\lambda_2) - \lambda_2 \in \phi[u^n]$. By Corollary 5.7(2), an element of $\phi[u^n]$ having valuation $\geq w(\xi_{1,nd-m+1})$ belongs to the $\mathbb{F}_q$-vector space $A^{<m} \cdot_\phi \lambda_1$. Hence it suffices to show $w(\sigma(\lambda_2) - \lambda_2) \geq w(\xi_{1,nd-m+1})$. By Proposition 5.1, we have $w(\lambda_i) = w(\xi_{i,nd})$. Let w_Λ denote the normalized valuation associated to $K(\Lambda)$. We have $w_\Lambda = Eq^m w$. Consider

$$\begin{aligned}
w_\Lambda(\sigma(\lambda_2) - \lambda_2) &= w_\Lambda(\sigma(\lambda_2)\lambda_2^{-1} - 1) + w_\Lambda(\lambda_2) \\
&\geq R_m + w_\Lambda(\lambda_2) \\
&= -w(\boldsymbol{j})E\frac{1}{q-1} - w_0Eq^m\left(-\frac{1}{q-1}\right) \\
&\quad - Eq^m\left(w_0(nd-m) + \frac{w_1}{q-1} - \frac{w(\boldsymbol{j})}{q^m(q-1)}\right) \\
&= -Eq^m\left(w_0(nd-m) + \frac{w_1 - w_0}{q-1}\right) = w_\Lambda(\xi_{1,nd-m+1}).
\end{aligned}$$

Hence the image $\sigma(\lambda_2) - \lambda_2$ of σ under the map g belongs to $A^{<m} \cdot_\phi \lambda_1$.

Next, we show that g is an isomorphism. The map is injective since λ_1 and λ_2 generate $K(\Lambda)/K$ and $\sigma(\lambda_1) = \lambda_1$ for any $\sigma \in G(\Lambda)_1$. By [1, Theorem 3.9], we know $G(\Lambda)_1 \cong \mathbb{F}_q^m$. As q^m is also the cardinal of $A^{<m} \cdot_\phi \lambda_1$, the map is bijective. It suffices to show that this map is a morphism. For any $\sigma \in G(\Lambda)_1$, we have that σ fixes λ_1 and $\sigma(\lambda_2) - \lambda_2 = b \cdot_\phi \lambda_1$ for some $b \in A$. Hence for any $\sigma', \sigma \in G(\Lambda)_1$, we have

$$\sigma'(\sigma(\lambda_2) - \lambda_2) = \sigma(\lambda_2) - \lambda_2.$$

This implies

$$\begin{aligned}
\sigma'(\sigma(\lambda_2)) - \lambda_2 &= \sigma'(\sigma(\lambda_2)) - \sigma'(\lambda_2) + \sigma'(\lambda_2) - \lambda_2 \\
&= \sigma'(\sigma(\lambda_2) - \lambda_2) + \sigma'(\lambda_2) - \lambda_2 \\
&= \sigma(\lambda_2) - \lambda_2 + \sigma'(\lambda_2) - \lambda_2,
\end{aligned}$$

which shows that the map is a morphism.

(3) Note $G(\Lambda)^{r_i} = G(\Lambda)_{R_i}$. We show that $g\colon G(\Lambda)_{R_i} \to A^{<i} \cdot_\phi \lambda_1$ is an isomorphism for each $i = 1, \ldots, m$. By Corollary 5.7(1), (5.5) and (5.6), the vector space $A^{<i} \cdot_\phi \lambda_1$

consists of elements of $\phi[u^n]$ having valuations $\geq w(\xi_{1,nd-i+1})$. For i to be one of $1,\dots,m$ and σ to be a nontrivial element in $G(\Lambda)_{R_i}$, we have

$$\begin{aligned}
w_\Lambda(\sigma(\lambda_2)-\lambda_2) &= w_\Lambda(\sigma(\lambda_2)\lambda_2^{-1}-1)+w_\Lambda(\lambda_2)\\
&\geq R_i + w_\Lambda(\lambda_2)\\
&= -w(\boldsymbol{j})E\frac{1}{q-1} - w_0Eq^m\left(m-i-\frac{1}{q-1}\right)\\
&\quad - Eq^m\left(w_0(nd-m)+\frac{w_1}{q-1}-\frac{w(\boldsymbol{j})}{q^m(q-1)}\right)\\
&= -Eq^m\left(w_0(nd-i)+\frac{w_1-w_0}{q-1}\right) = w_\Lambda(\xi_{1,nd-i+1}).
\end{aligned}$$

This implies that $g(G(\Lambda)_{R_i}) \subset A^{<i}\cdot_\phi \lambda_1$. As the cardinals of $G(\Lambda)_{R_i}$ and $A^{<i}\cdot_\phi\lambda_1$ are both q^i, the restriction

$$g\colon G(\Lambda)_{R_i} \to A^{<i}\cdot_\phi \lambda_1$$

is an isomorphism for each i. $\square$

Example 5.10. (with the help of T. Asayama and Y. Taguchi) Let C denote the Carlitz $\mathbb{F}_q[t]$-module over $\mathbb{F}_q(t)$ determined by $C_t(X) = tX + X^q$. Put $T := t^2 + a$ for some $a \in \mathbb{F}_q$. Let ϕ denote the Drinfeld $\mathbb{F}_q[T]$-module over $\mathbb{F}_q(t)$ determined by $\phi_T(X) = C_{t^2+a}(X)$. Let K denote the completion of $\mathbb{F}_q(t)$ at the infinite prime and w the associated normalized valuation so that $w(t) = -1$. The j-invariant $\boldsymbol{j}$ of ϕ has valuation $w(\boldsymbol{j}) = -q(q+1) < w(t)q$. However, by [14, Theorem 7.1.13] (initially given by Hayes), the extension $K(\phi[T^n]) = K(C[(t^2+a)^n])$ of K for any positive integer n is tamely ramified. By this example, to remove the condition $p \nmid w(\boldsymbol{j})$ might be hard.

6. Applications to rank 2 Drinfeld modules, finite prime case

Let w be a finite prime of K. Throughout this section, let u be a finite prime of A having degree d, and n a positive integer. Let ϕ be a rank 2 Drinfeld A-module over K determined by $\phi_t(X) = tX + a_1X^q + a_2X^{q^2} \in K[X]$. Let $\boldsymbol{j}$ denote the j-invariant a_1^{q+1}/a_2.

6.1. The valuations of SMBs

Throughout this subsection, assume that ϕ has bad reduction over K, i.e., ϕ has stable reduction over K and the reduction has rank 1. We have $w(a_1) = 0$ and $w(a_2) > 0$ such that $w(\boldsymbol{j}) < 0$. Let $\{\xi_{i,n}\}_{i=1,2}$ be an SMB of $\phi[t^n]$ obtained as in Corollary 2.14. By [1, Proposition 2.5 and Lemma A.1 (2)], we have

$$w(\xi_{1,n}) = \frac{w(t)}{(q-1)q^{n-1}} \quad \text{and} \quad w(\xi_{2,n}) = \frac{w(\boldsymbol{j})}{(q-1)q^n}.$$

Proposition 6.1. (cf. Proposition 5.1) *Let $\{\omega_1\}$ be an SMB of $\psi[u^n]$, $\{\omega_2^0\}$ an SMB of Λ, and $\{\lambda_i\}_{i=1,2}$ an SMB of $\phi[u^n]$. Then for any positive integer n, we have*

$$w(\omega_1) = w(\lambda_1) = \frac{w(u)}{(q^d-1)q^{(n-1)d}}, \quad w(\omega_2^0) = \frac{w(\boldsymbol{j})}{q-1} \quad \textit{and} \quad w(\lambda_2) = \frac{w(\boldsymbol{j})}{(q-1)q^{nd}}.$$

Proof. Note that the condition "$|u^n|_\infty > |\omega_r^0|/|\omega_{r'+1}^0|$" in Section 4 is trivial. The results for ω_2^0 and λ_2 follow from the value $w(\xi_{2,n})$ and Corollary 4.7(1).

By Lemma 4.3 and Proposition 2.10(2), it remains to calculate $w(\omega_1)$. The case $w \nmid u$ is straightforward. Assume $w \mid u$. We have $\psi_t(X) = tX + b_1X^q \in K[X]$ such that the valuation of b_1 is 0. Let K' denote the extension of K generated by some $b \in K^{\mathrm{sep}}$ with $b^{q-1} = b_1$. Then we have $C = b\psi b^{-1}$ as Drinfeld A-modules over K' where C denotes the Carlitz module. Let $\{\eta_{1,j}\}$ be an SMB of $C[u^j]$ for each positive integer j as in Corollary 2.14. As $b\omega_1$ forms an SMB of $b\psi b^{-1}[u^n]$, we have $w(\omega_1) = w(\eta_{1,n})$ by Proposition 2.10(2).

To calculate $w(\eta_{1,n})$, we proceed by induction. We first calculate $w(\eta_{1,1})$. Put $u_0 := u$, $\sum_{i=0}^d u_iX^{q^i} := C_u(X)$ and $P_i := (q^i, w(u_i))$ for $i = 0, \ldots, d$. By the explicit formula of u_i in [14, Corollary 5.4.4] (initially given by Carlitz), we have $w(u_i) = w(u)$ for $i = 0, \ldots, d-1$. The Newton polygon of $C_u(X)$ is P_0P_d having exactly one segment. Hence we have $w(\eta_{1,1}) = \frac{w(u)}{q^d-1}$.

Assume $w(\eta_{1,i-1}) = \frac{w(u)}{(q^d-1)q^{(i-2)d}}$. Put $Q_{i-1} := (0, w(\eta_{1,i-1}))$. The Newton polygon of $C_u(X)-\eta_{1,i-1}$ is $Q_{i-1}P_d$ having exactly one segment. Hence we have $w(\eta_{1,i}) = \frac{w(u)}{(q^d-1)q^{(i-1)d}}$, as desired. $\square$

6.2. The action of the wild ramification subgroup on the division points

Assume $w \nmid u$, i.e., $w(u) = 0$ throughout this subsection. Assume that ϕ has bad reduction over K. Let L be the extension of K generated by the elements in Λ. For a positive integer n, let L_n denote the extension of L generated by the elements in $u^{-n}\Lambda$. As the condition "$|u^n|_\infty > |\omega_r^0|/|\omega_{r'+1}^0|$" in Section 4 is fulfilled for any positive integer n, by Proposition 4.11, we have $K(\phi[u^n]) = L_n$ for any positive integer n. We put $G(n) := \mathrm{Gal}(K(\phi[u^n])/K)$.

In this subsection, we first study the action of the wild ramification subgroup $G(n)_1$ on $u^{-n}\Lambda/\Lambda$. Next, using the isomorphism $\mathcal{E}_\phi\colon u^{-n}\Lambda/\Lambda \to \phi[u^n]$, we know the action of $G(n)_1$ on $\phi[u^n]$. Let $\{\omega_1\}$ be an SMB of $\psi[u^n]$, $\{\omega_2^0\}$ an SMB of Λ, and ω_2 a root of $\psi_{u^n}(X) - \omega_2^0$.

Lemma 6.2. *The extension L/K is at worst tamely ramified.*

Proof. We know that Λ is an A-lattice via ψ and is $\mathrm{Gal}(K^{\mathrm{sep}}/K)$-invariant. As L/K is a subextension of L_1/K and $L_1 = K_1$ is Galois over K, we have that L/K is separable. Then the desired claim follows from Lemma 2.6. □

Theorem 6.3. (cf. Theorem 5.9) *Let ϕ be a rank 2 Drinfeld A-module over K having bad reduction such that $w(\boldsymbol{j}) < 0$. Put $R := -w(\omega_2^0) = \frac{-w(\boldsymbol{j})}{q-1}$. Assume $p \nmid w(\boldsymbol{j})$.*

(1) *Let $L(\psi[u^n])$ be the extension of L generated by the elements in $\psi[u^n]$. There is an isomorphism*

$$\mathrm{Gal}(L_n/L(\psi[u^n])) \to \psi[u^n], \quad \sigma \mapsto \sigma(\omega_2) - \omega_2.$$

(2) *Let E be the ramification index of L/K. The (Herbrand) ψ-function of the extension L_n/K is*

$$\psi_{L_n/K}(y) = \begin{cases} y, & -1 \leq y \leq 0; \\ Ey, & 0 \leq y \leq R; \\ q^{nd}Ey - (q^{nd}-1)ER, & R \leq y. \end{cases}$$

Proof. Let w_L denote the normalized valuation associated to L. We have $w_L = Ew$. As the extension $L(\psi[u^n])/L$ is unramified, we may let w_L denote the normalized valuation associated to $L(\psi[u^n])$. The field L_n is the splitting field of $\psi_{u^n}(X) - \omega_2^0$ over $L(\psi[u^n])$. As E is not divisible by p (see Lemma 6.2), we have $p \nmid ER = -w_L(\omega_2^0)$. Note $w_L(\omega_2^0) < 0$. We can apply Proposition B.2(2) to $\psi_{u^n}(X) - \omega_2^0 \in L(\psi[u^n])[X]$. Note that the difference between two roots of $\psi_{u^n}(X) - \omega_2^0$ belongs to $\psi[u^n]$. The extension $L_n/L(\psi[u^n])$ is totally ramified and is generated by ω_2. The map $\mathrm{Gal}(L_n/L(\psi[u^n])) \to \psi[u^n]$, $\sigma \mapsto \sigma(\omega_2) - \omega_2$ is an isomorphism.

(2) By Lemma 6.2, we have the ψ-function of L/K to be

$$\psi_{L/K}(y) = \begin{cases} y, & -1 \leq y \leq 0; \\ Ey, & 0 \leq y. \end{cases}$$

The ψ-function of $L(\psi[u^n])/L$ is $\psi_{L(\psi[u^n])/L}(y) = y$. Applying Proposition B.2(3) to $\psi_{u^n}(X) - \omega_2^0 \in L(\psi[u^n])[X]$, we have

$$\psi_{L_n/L(\psi[u^n])}(y) = \begin{cases} y, & -1 \leq y \leq ER; \\ q^{nd}y - (q^{nd}-1)ER, & ER \leq y, \end{cases}$$

and the desired ψ-function follows from Lemma B.1. □

Let ϕ be a rank 2 Drinfeld A-module over K which does not necessarily have stable reduction. Assume that $w(\boldsymbol{j}) < 0$ such that ϕ is isomorphic to a Drinfeld module having bad reduction over some extension of K. By Proposition 4.15, we may take this extension of K to be $K(\lambda_{1,1})$, where $\{\lambda_{i,1}\}_{i=1,2}$ is an SMB of $\phi[u]$ and $K(\lambda_{1,1})/K$ is at worst tamely ramified. Let ψ and Λ denote respectively the Drinfeld module having good reduction and the lattice associated to the Drinfeld module having stable reduction via the Tate uniformization. Let L denote the extension of $K(\lambda_{1,1})$ generated by the elements in Λ. By Lemma 2.6, the extension L/K is at worst tamely ramified. For a positive integer n, let L_n denote the extension of L generated by the elements in $u^{-n}\Lambda$. We have $K(\phi[u^n]) = L_n$.

Corollary 6.4. *Let ϕ be a rank 2 Drinfeld A-module over K which does not necessarily have stable reduction. Assume $w(\boldsymbol{j}) < 0$ and $p \nmid w(\boldsymbol{j})$.*

(1) *Let E be the ramification index of L/K. Put $R = \frac{-w(\boldsymbol{j})}{q-1}$. The ψ-function of the extension $K(\phi[u^n])/K$ is*

$$\psi_{K(\phi[u^n])/K}(y) = \begin{cases} y, & -1 \le y \le 0; \\ Ey, & 0 \le y \le R; \\ q^{nd}Ey - (q^{nd}-1)ER, & R \le y. \end{cases}$$

(2) *Let $\{\lambda_i\}_{i=1,2}$ be an SMB of $\phi[u^n]$. Then each element in $G(n)_1$ fixes λ_1 and there is an isomorphism*

$$G(n)_1 \to A \cdot_\phi \lambda_1, \quad \sigma \mapsto \sigma(\lambda_2) - \lambda_2.$$

Proof. Apply Theorem 6.3(2) with K in the theorem being $K(\lambda_{1,1})$ and we obtain the ψ-function of $K(\phi[u^n])/K(\lambda_{1,1})$. As $K(\lambda_{1,1})/K$ is at worst tamely ramified, its ψ-function is clear. Then (1) follows from Lemma B.1.

We show (2). Note that $L(\psi[u^n])/K$ is at worst tamely ramified. By the ψ-function of L_n/K, we have the following equation of the higher ramification subgroups

$$G(n)_1 = \mathrm{Gal}(L_n/K)_1 = \mathrm{Gal}(L_n/K)_{ER} = \mathrm{Gal}(L_n/L(\psi[u^n])).$$

By Proposition 4.15, the Drinfeld module $b\phi b^{-1}$ for $b = \lambda_{1,1}^{-1}$ has stable reduction over $K(\lambda_{1,1})$. Let $\log_\phi$ denote $\log_{b\phi b^{-1}}$. By Theorem 4.10, the element $\log_\phi(b\lambda_1)$ forms an SMB of $\psi[u^n]$ and $u^n \cdot_\psi \log_\phi(b\lambda_2)$ forms an SMB of Λ. Apply Theorem 6.3(1) with $\omega_1 = \log_\phi(b\lambda_1)$ and $\omega_2 = \log_\phi(b\lambda_2)$. We have $\sigma(\log_\phi(b\lambda_1)) = \log_\phi(b\lambda_1)$ for any $\sigma \in G(n)_1$ and an isomorphism

$$G(n)_1 \to \psi[u^n], \quad \sigma \mapsto \sigma(\log_\phi(b\lambda_2)) - \log_\phi(b\lambda_2).$$

Note $\psi[u^n] = A \cdot_\psi \log_\phi b\lambda_1$. The map $\mathcal{E}_{b\phi b^{-1}}|_{\psi[u^n]} \colon \psi[u^n] \to A \cdot_{b\phi b^{-1}} b\lambda_1$ induced by the exponential map e_ϕ is an isomorphism. Indeed, it is injective as $\psi[u^n] \cap \Lambda = \{0\}$. Since the sets $\psi[u^n]$ and $A \cdot_{b\phi b^{-1}} b\lambda_1$ both have cardinal q^{nd}, we have the surjectivity. Notice that $\mathcal{E}_{b\phi b^{-1}}$ is compatible with the $\mathrm{Gal}(K^{\mathrm{sep}}/K)$-actions and $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ acts on $u^{-n}\Lambda/\Lambda$ and

$\phi[u^n]$ via $G(n)$. We obtain that $\sigma(\log_\phi(b\lambda_1)) = \log_\phi(b\lambda_1)$ and $\sigma(\log_\phi(b\lambda_2)) - \log_\phi(b\lambda_2)$ map to respectively $\sigma(b\lambda_1) = b\lambda_1$ and $\sigma(b\lambda_2) - b\lambda_2$. The desired isomorphism is the composition

$$G(n)_1 \to \psi[u^n] \xrightarrow{\mathcal{E}_{b\phi b^{-1}}} A \cdot_{b\phi b^{-1}} b\lambda_1 \xrightarrow{b^{-1}\cdot -} A \cdot_\phi \lambda_1, \quad \sigma \mapsto \sigma(\lambda_2) - \lambda_2. \qquad \square$$

The next result generalizes the finite prime case of [1, Lemma-Definition 4.1].

Lemma-Definition 6.5. (cf. Lemma 5.5) *Let ϕ be a rank 2 Drinfeld A-module over K which does not necessarily have stable reduction. Assume one of the following two cases happens*

(C1) $w(\boldsymbol{j}) < 0$ *and* $p \nmid w(\boldsymbol{j})$*;*

(C2) $w(\boldsymbol{j}) \geq 0$.

Let G^y denote the y-th upper ramification subgroup of the Galois group $\mathrm{Gal}(K^{\mathrm{sep}}/K)$*. For any finite prime u of A not divisible by w, let T_u denote the u-adic Tate module of ϕ. Put*

$$\mathfrak{f}_w(\phi) := \int_0^\infty \left(2 - \mathrm{rank}_{A_u} T_u^{G^y}\right) dy.$$

Then we have

(1) *the value $\mathfrak{f}_w(\phi)$ is independent of the choice of u.*

(2) $\mathfrak{f}_w(\phi) = \begin{cases} \frac{-w(\boldsymbol{j})}{q-1} & \textit{if } (\mathrm{C1}) \textit{ happens}, \\ 0 & \textit{if } (\mathrm{C2}) \textit{ happens}. \end{cases}$

Define the conductor *of ϕ at w to be the integral $\mathfrak{f}_w(\phi)$.*

Proof. We will show (2) for any finite prime u of A with $w \nmid u$ and (1) straightforwardly follows.

Assume the case (C1) happens. By Corollary 2.14, there is an SMB $\{\lambda_{i,n}\}_{i=1,2}$ of $\phi[u^n]$ for each integer $n \geq 1$ such that $u \cdot_\phi \lambda_{i,n+1} = \lambda_{i,n}$ for $i = 1, 2$. By Corollary 6.4(1), we have $G(n)^y = G(n)_1$ for any $0 < y \leq \frac{-w(\boldsymbol{j})}{q-1}$ and $= \{e\}$ for $y > \frac{-w(\boldsymbol{j})}{q-1}$. By Corollary 6.4(2), for any $n \geq 1$ and $0 < y \leq \frac{-w(\boldsymbol{j})}{q-1}$, any element in $G(n)^y$ fixes $\lambda_{1,i}$ for all $i \leq n$, and any nontrivial element $\sigma \in G(n)^y$ nontrivially acts on $\lambda_{2,n}$.

As $u \cdot_\phi \lambda_{1,n+1} = \lambda_{1,n}$ and $u \cdot_\phi \lambda_{2,n+1} = \lambda_{2,n}$ for any $n \geq 1$, the tuples $(\lambda_{1,n})_{n\geq 1}$ and $(\lambda_{2,n})_{n\geq 1}$ form an A_u-basis of T_u. Note that G^y acts on T_u via $G(\infty)^y = \varprojlim_n G(n)^y$. Any nontrivial element of $G(\infty)^y$ for $0 < y \leq \frac{-w(\boldsymbol{j})}{q-1}$ fixes $(\lambda_{1,n})_{n\geq 1}$ and nontrivially acts on $(\lambda_{2,n})_{n\geq 1}$. Hence $\mathrm{rank}_{A_u} T_u^{G^y} = 1$ if $0 < y \leq \frac{-w(\boldsymbol{j})}{q-1}$ and $= 2$ if $\frac{-w(\boldsymbol{j})}{q-1} < y$. We have

$$\mathfrak{f}_w(\phi) = \int_0^{\frac{-w(\boldsymbol{j})}{q-1}} 1\, dy = \frac{-w(\boldsymbol{j})}{q-1}.$$

For the case (C2), we know that ϕ is isomorphic to a Drinfeld module having good reduction over some extension of K. By Proposition 4.15, we may take the extension of K to be $K(\lambda_{1,1})$ and the extension $K(\lambda_{1,1})/K$ is at worst tamely ramified, where $\{\lambda_{i,1}\}_{i=1,2}$ is an SMB of $\phi[u]$. For $b = \lambda_{1,1}^{-1}$, as the Drinfeld module $b\phi b^{-1}$ has good reduction, the extension $K(b\phi b^{-1}[u^n])/K(\lambda_{1,1})$ is unramified. Hence the extension $K(\phi[u^n])/K$ is at worst tamely ramified and the conductor vanishes. $\square$

6.3. A function field analogue of Szpiro's conjecture for rank 2 Drinfeld modules

Let ϕ be a rank 2 Drinfeld A-module over F (F is the global function field defined in Section 1.1). For a prime w of F, consider ϕ as a Drinfeld module over F_w and let $\mathfrak{f}_w(\phi)$ be the conductor calculated in Lemma 5.5 and Lemma-Definition 6.5. Similar to [1, Section 4.2], we can obtain a relation between the J-height of ϕ and the conductors of ϕ.

Let M_F denote the set of primes of F. For a prime w of F, let $\deg(w)$ denote the degree of the residue field of F_w over $\mathbb{F}_q$. The J-height of ϕ is defined to be (see [4, Section 2.2] or [1, Section 4.2])

$$h_J(\phi) := \frac{1}{[F : \mathbb{F}_q(t)]} \sum_{w \in M_F} \deg(w) \cdot \max\{-w(\boldsymbol{j}), 0\},$$

where $\boldsymbol{j}$ is the j-invariant of ϕ. Following [1, Section 4.2], we may define the (*global*) *conductor* of the Drinfeld module ϕ to be

$$\mathfrak{f}(\phi) := \sum_{w \in M_F} \deg(w) \cdot \mathfrak{f}_w(\phi).$$

Similar to the proof of [1, Theorem 4.3], we have the following statement by Lemma 5.5 and Lemma-Definition 6.5. It is a function field analogue of Szpiro's conjecture.

Theorem 6.6. *Put $w_0 = w(t)$ if w is an infinite prime of F. Let ϕ be a rank 2 Drinfeld A-module over F such that for each prime w of F, its j-invariant $\boldsymbol{j}$ satisfies*

$$\begin{cases} \text{either } \big(w(\boldsymbol{j}) < w_0 q \text{ and } p \nmid w(\boldsymbol{j})\big), \text{ or } w(\boldsymbol{j}) \geq w_0 q & \text{if } w \text{ is infinite,} \\ \text{either } \big(w(\boldsymbol{j}) < 0 \text{ and } p \nmid w(\boldsymbol{j})\big), \text{ or } w(\boldsymbol{j}) \geq 0 & \text{if } w \text{ is finite.} \end{cases}$$

Then

$$h_J(\phi) \leq \mathfrak{f}(\phi) \cdot \frac{q-1}{[F : \mathbb{F}_q(t)]} + q.$$

A. The conductors of Drinfeld modules at infinite prime

Let K be the completion of a global function field at an infinite prime w. Let ϕ denote a rank r Drinfeld A-module over K for an integer $r \geq 2$. Let T_u be the u-adic Tate module of ϕ. Let G^y denote the y-th upper ramification subgroup of the absolute Galois group G of K.

Lemma-Definition A.1. *The value of the integral*

$$\int_0^\infty \left(r - \operatorname{rank}_{A_u} T_u^{G^y}\right) dy$$

is convergent and independent of u. Define the conductor of ϕ at w to be this integral.

Proof. The result follows from the following two claims:

(1) $\operatorname{rank}_{A_u} T_u^{G^y} = \operatorname{rank}_A \Lambda^{G(\Lambda)^y}$ for any finite prime u of A, where $G(\Lambda)^y$ denotes the y-th upper ramification subgroup of the Galois group of the extension $K(\Lambda)$ of K generated by all elements in Λ.

(2) The following integral
$$\int_0^\infty \left(r - \operatorname{rank}_A \Lambda^{G(\Lambda)^y}\right) dy$$
is convergent

As for (1), note that the isomorphism $\mathcal{E}_\phi \colon u^{-n}\Lambda/\Lambda \to \phi[u^n]$ induced by the exponential map e_ϕ is G-equivariant. We have a G-equivariant isomorphism $T_u \cong \Lambda \otimes_A A_u$. Note that the action of G^y on Λ factors through $G(\Lambda)^y$. It suffices to see $\operatorname{rank}_{A_u}(\Lambda \otimes_A A_u)^{G(\Lambda)^y} = \operatorname{rank}_A \Lambda^{G(\Lambda)^y}$. As $(\Lambda \otimes_A A_u)^{G(\Lambda)^y}$ is free over A_u and is identified with $\Lambda^{G(\Lambda)} \otimes_A A_u$, we know that $\Lambda^{G(\Lambda)^y}$ is projective over A according to the following two facts:

(1) The flatness is a local property [16, 00HT];

(2) Each finitely generated flat module over a Noetherian ring is projective [16, 00NX].

Then the desired equality follows from the definition of rank and the Nakayama lemma.

As for (2), notice that the extension $K(\Lambda)/K$ is finite. There is an integer i so that the i-th lower ramification subgroup of $\operatorname{Gal}(K(\Lambda)/K)$ is trivial. Hence there is a rational number $\overline{y}$ so that $\overline{y}$-th upper ramification subgroup of $\operatorname{Gal}(K(\Lambda)/K)$ is trivial. This shows that
$$\int_0^\infty \left(r - \operatorname{rank}_A \Lambda^{G(\Lambda)^y}\right) dy \le \overline{y}(r-1),$$
i.e., the monotone function

$$f(x) := \int_0^x \left(r - \operatorname{rank}_A \Lambda^{G(\Lambda)^y}\right) dy$$

is bounded. Hence the limit $\lim_{x \to +\infty} f(x)$ exists, i.e., the integral is convergent. $\square$

Remark A.2. (1) Let n be an integer so that $K(\Lambda) = K(\phi[u^n])$. Hence one may expect that

$$\operatorname{rank}_{A_u} T_u^{G^y} = \operatorname{rank}_{A/u^n} \phi[u^n]^{G^y} = \operatorname{rank}_{A/u^n} (\Lambda/u^n\Lambda)^{G^y} = \operatorname{rank}_A \Lambda^{G(\Lambda)^y}.$$

Here the A/u^n-submodule $\phi[u^n]^{G^y}$ of $\phi[u^n]$ is free by [3, VII.14, Theorem 1].

(2) Let w be a finite prime, u a finite prime of A with $w \nmid u$. Let G^y denote the y-th upper ramification subgroup of the absolute Galois group K. M. Mornev has proved that [13, Theorem 1] there is some rational number $\overline{y}$ so that $G^{\overline{y}}$ trivially acts on T_u. Using this result, similar to the proof above, one can show that the integral

$$\int_0^\infty \left(r - \operatorname{rank}_{A_u} T_u^{G^y}\right) dy$$

is convergent.

B. Basics of Herbrand ψ-functions

Throughout this section, let K be a complete discrete valuation field of characteristic p so that the residue field is a perfect field. Let us recall the definition of the (Herbrand) ψ-function $\psi_{L/K}$ for a finite Galois extension L/K of a complete valuation field of characteristic p. Let G^y denote the y-th upper ramification subgroup of the Galois group $\mathrm{Gal}(L/K)$ of L/K. By the ψ-function of L/K, we mean the real-valued function on the interval $[0, +\infty)$ defined as

$$\psi_{L/K}(y) = \int_0^y \frac{\#G^0}{\#G^r}\, dr.$$

We extend $\psi_{L/K}$ to $[-1, +\infty)$ by letting $\psi_{L/K}(y) = y$ if $-1 \leq y \leq 0$. Then $\psi_{L/K}$ is a continuous and piecewise linear function on $[-1, +\infty)$. If $\psi_{L/K}$ is linear on some interval $[a, b] \subset [-1, \infty)$, then we have $G^b = G^y = G_{\psi_{L/K}(y)}$ for $y \in (a, b]$. By the wild ramification subgroup of L/K, we mean the first lower ramification subgroup G_1, which is equal to the union of G^y for $y > 0$.

Lemma B.1. (see e.g., [7, Chapter III, (3.3)]) *Let L/M and M/K be finite Galois extensions. Then*

$$\psi_{L/K} = \psi_{L/M} \circ \psi_{M/K}.$$

Assume that K contains $\mathbb{F}_q$, where q is a power of p. Let v_K denote the normalized valuation associated to K so that $v_K(K^\times) = \mathbb{Z}$. For a positive integer s, put

$$f(X) = X^{q^s} + \sum_{k=1}^{s-1} a_k X^{q^k} + aX \in K[X]$$

such that $\frac{v_K(a_k)-v_K(a)}{q^k-1} \geq \frac{-v_K(a)}{q^s-1}$ for $k=1,\dots,s-1$, i.e., the Newton polygon of $f(X)/X$ has exactly one segment. The extension generated by the roots of the polynomial $f(X)-c$ for certain $c \in K$ plays a key role in Section 6.2. To obtain its ψ-function, we will need the following fact. It is a slight generalization of the function field case of [7, Chapter III, Proposition 2.5] (cf. [1, Proposition 3.2]).

Proposition B.2. *Let $f(X)-c$ be the polynomial above. Let F and L denote respectively the splitting field of $f(X)$ and that of $f(X)-c$. Put $v_c := v_K(c)$ and $v_a := v_K(a)$. Assume $p \nmid v_c$ and $\frac{-v_c}{q^s} < v_a - v_c$ so that the Newton polygon of $f(X)-c$ has exactly one segment and $R := \frac{v_a q^s}{q^s-1} - v_c > 0$. Then*

(1) *The extension of F/K is at worst tamely ramified.*

(2) *We have a composition of field extensions*

$$K — F — L.$$

Moreover, the extension L/F is totally ramified of degree q^s and generated by one root x of $f(X)-c$. We have an isomorphism

$$g\colon \operatorname{Gal}(L/F) \to V, \quad \sigma \mapsto \sigma(x) - x,$$

where $V \cong \mathbb{F}_q^s$ is the $\mathbb{F}_q$-vector space consisting of the roots of $f(X)$.

(3) *Let e denote the ramification index of F/K. The ψ-function of L/K is*

$$\psi_{L/K}(y) = \begin{cases} y, & -1 \leq y \leq 0; \\ ey, & 0 \leq y \leq R; \\ eq^s y - (q^s-1)eR, & R \leq y. \end{cases}$$

Proof. Let M be an extension of K with ramification index $q^s - 1$. We can take some $b \in M$ such that $v(b) = \frac{-v_a}{q^s-1}$. With $b' = b^{q^s}$, modify $f(X)$ to be

$$f_1(X) = X^{q^s} + \sum_{k=1}^{s-1} b_k X^{q^k} + b_0 X := b' f(X/b).$$

We have

$$v_K(b_0) = 0 \quad \text{and} \quad v_K(b_k) = v_K(a_k) - \frac{v_a(q^s - q^k)}{q^s - 1} \geq 0 \text{ for } k = 1, \dots, s-1.$$

Thus $f_1(X)$ is a monic polynomial whose reduction is separable. By Hensel's lemma [14, Corollary 2.4.5], the extension of M generated by the roots of $f(X)$ is unramified. Hence the extension of K generated by the roots of $f(X)$ is at worst tamely ramified. This shows (1).

For (2), note that the difference of any two roots of $f(X)-c$ is a root of $f(X)$. The field F is contained in L and L is the extension of F generated by one root of $f(X)-c$. As the polynomial $f(X)$ is additive, its roots form an $\mathbb{F}_q$-vector space of dimensional s, denoted V. Let x be a root of $f(X)-c$. For any $\sigma \in \mathrm{Gal}(L/F)$, the difference $\sigma(x)-x$ is a root of $f(X)$ and hence we obtain a map $g\colon \mathrm{Gal}(L/F) \to V$, $\sigma \mapsto \sigma(x)-x$. The element σ is determined by $\sigma(x)$ since x generates L/F. Hence the map g is injective. This implies that $\#\,\mathrm{Gal}(L/F) \leq q^s$. As the Newton polygon of $f(X)-c$ has exactly one segment, we have $v_F(x) = ev_c/q^s$, where v_F denotes the normalized valuation associated to F and e denotes the ramification index of F/K. As $p \nmid e$, $p \nmid v_c$, we have $\#\,\mathrm{Gal}(L/F) = q^s$. Therefore, the extension L/F is a totally ramified Galois extension of degree q^s. The map $\mathrm{Gal}(L/F) \to V$ is surjective as the cardinal of $\mathrm{Gal}(L/F)$ is equal to that of V. As each element $\mathrm{Gal}(L/F)$ fixes each element of V, the map g is a morphism.

We show (3). Let π_L be a uniformizer of L. For a nontrivial element σ in $\mathrm{Gal}(L/F)$, as $\sigma(x)/x$ is a unit of L (here x is a root of $f(X)-c$), we have

$$\sigma(x)/x = u_F\epsilon$$

for some $\epsilon \in 1+(\pi_L)$ (the first higher unit group of L) and some u_F in the unit group of F. Notice

$$\begin{aligned}&\sigma^2(x)/x = \sigma(xu_F\epsilon)/x = u_F\sigma(\epsilon)\sigma(x)/x = u_F^2\sigma(\epsilon)\epsilon,\\ &\sigma^3(x)/x = \sigma(xu_F^2\sigma(\epsilon)\epsilon)/x = u_F^2\sigma^2(\epsilon)\sigma(\epsilon)\sigma(x)/x = u_F^3\sigma^2(\epsilon)\sigma(\epsilon)\epsilon, \quad \text{and so on.}\end{aligned}$$

As the Galois group of L/F is isomorphic to the $\mathbb{F}_q$-vector space of dimensional s, the Galois group element σ has order p. We have

$$1 = \sigma^p(x)/x = u_F^p \prod_{k=0}^{p-1} \sigma^k(\epsilon).$$

This implies $u_F^p \equiv 1 \pmod{\pi_L}$. As p-th power map is injective on the residue field of L, we have $u_F \equiv 1 \pmod{\pi_L}$. Hence $u_F \in 1+(\pi_F)$, where π_F is a uniformizer of F. We know that $\sigma(x)/x \in 1+(\pi_L)$. Hence there exists some u_L in the unit group of L and some positive integer b such that

$$\sigma(x)/x \equiv (1+u_L\pi_L^b) \mod (\pi_L)^{b+1}. \tag{B.1}$$

From (2), we know $v_L(x) = ev_c$ and is prime to q^s (v_L denotes the normalized valuation associated to L). Hence there exist integers i, j satisfying $v_L(x^i\pi_F^j) = 1$. Here i is not divisible by p. The element $x^i\pi_F^j$ is a uniformizer of L. By [15, Chapter IV, Proposition 5], to know the ψ-function of L/F, we need to know $v_L(\sigma(x^i\pi_F^j)/x^i\pi_F^j - 1)$ for all nontrivial Galois group elements σ. By (B.1), we know

$$\sigma(x^i\pi_F^j)/x^i\pi_F^j \equiv (1+u_L\pi_L^b)^i \equiv 1 + iu_L^i\pi_L^b \mod (\pi_L)^{b+1}.$$

On the other hand, as $v_L(\sigma(x)-x) = \frac{v_a e q^s}{q^s-1}$ for any nontrivial σ, we know $b = v_L(\sigma(x) - x) - v_L(x) = eR$. The ψ-functions of F/K and L/F are respectively

$$\psi_{F/K}(y) = \begin{cases} y, & -1 \le y \le 0; \\ ey, & 0 \le y \end{cases} \quad \text{and} \quad \psi_{L/F}(y) = \begin{cases} y, & -1 \le y \le eR; \\ q^s y - (q^s-1)eR, & eR \le y. \end{cases}$$

By Lemma B.1, we obtain the ψ-function $\psi_{L/K}$ as the proposition describes. □

Acknowledgments

We are very grateful to Yuichiro Taguchi for his constant interest and encouragement. His comments have been very efficacious. We thank M. Papikian for informing us of the paper [9] and the initial idea so that we can formulate Remark 5.2. We thank the anonymous referee for this constructive comments.

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Maozhou Huang
Department of Mathematics, Tokyo Institute of Technology, Tokyo 152-8551, Japan
E-mail address: `huang.m.aa@m.titech.ac.jp`