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E-subdifferential of E-convex Functions and its Applications to Minimization Problem

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Abstract. In this paper, a new concept of the subdifferential is defined for nondifferentiable (not necessarily) locally Lipschitz functions. Namely, the concept of E-subdifferential and the notion of E-subconvexity are introduced for E-convex functions. Thus, the notion of an E-subdifferentiable E-convex function is introduced and some properties of this class of nondifferentiable nonconvex functions are studied. The necessary optimality conditions in E-subdifferentials terms of the involved functions are established for a new class of nondifferentiable optimization problems. The introduced concept of E-subconvexity is used to prove the sufficiency of the aforesaid necessary optimality conditions for nondifferentiable optimization problems in which the involved functions are E-subdifferentiable E-convex.

1. Introduction

It is commonly accepted the opinion that nonsmoothness arises naturally in optimization. Even if one considers a smooth data model of real-world processes, several operations associated with control or optimization destroy the initial differentiability and lead to the need of employing nonsmooth techniques. A typical example of such a situation is the minimization of a nonsmooth function. In such a case, it has been observed that, in general, the minimum occurs at a point of nondifferentiability of such a function.

The concept of subdifferential was introduced for studying the minima of convex functions which are important for practical reasons as well as for their theoretical interest (see, for example, [2, 10, 18, 25, 26, 28, 30, 34], and others). It is one of the most important branches of convex analysis in the case of nondifferentiable convex functions. From the perspective of optimization, the subdifferential of a convex function has many of the useful properties of the derivative.

The term "nonsmooth analysis" was coined in the 1970s by Francis Clarke [7, 8], who performed pioneering work in this area for fairly general nonsmooth objects. Hence, he

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extended the domain of nonsmooth analysis from convex to locally Lipschitz functions. Namely, Clarke [8] introduced the concept of generalized gradient of a locally Lipschitz function. This concept has first been introduced in the finite-dimensional case for locally Lipschitzian functions, in an analytic form, and for lower semi-continuous functions in a geometric form.

In recent years, there was a growing realization that a large number of optimization problems which appeared in applications involved minimization of nondifferentiable functions. The subject of nonsmooth analysis arose out of the need to develop a theory to deal with the minimization of nonsmooth functions.

During the last half-century, there has been an extremely rapid development in subdifferential calculus of nonsmooth analysis and which is well recognized for its many applications to optimization theory. There are several natural ways to define general subdifferentials satisfying useful calculus rules. Therefore, after introducing by Clarke a generalized gradient for a locally Lipschitz function, several other subdifferentials (convex as well as nonconvex) were proposed in the literature (see, e.g., [3–6, 9, 11–17, 19, 21–24, 27, 29, 31, 33, 35, 36], and many others). They are useful for the study of some properties of specific classes of nonsmooth functions. Their common property is that the given function is investigated by means of one set (convex or nonconvex). Moreover, it is also a consequence of the fact that, in recent years, the study of nonsmooth optimization problems with constraints has led many authors to introduce various notions of subdifferential or generalized differential for nondifferentiable functions.

In this paper, we introduce and study a new subdifferential for not necessarily locally Lipschitz functions. We analyze some properties of the introduced E-subdifferential which is the set of E-subgradients and which is based on the effect of an operator $E\colon R^n \to R^n$ on the domain of a function for which is defined. We compare some of the analyzed properties of the E-subdifferential to the analogous properties of the classical subdifferential of a convex function. It turns out that the classical subdifferential is a special case of the introduced E-subdifferential. Further, we also introduce a new class of nondifferentiable generalized convex functions which is an extension and a generalization of the class of differentiable E-convex functions which included also convex functions. Namely, we introduce the definition of E-subdifferentiable E-convex functions, called E-subconvex functions, for short. As it follows from this definition, E-subconvex functions are such a class of generalized convex functions for which E-subdifferential is nonempty. We prove the necessary optimality conditions for an E-minimizer of an E-subdifferentiable E-convex function which is formulated in term of its E-subdifferential.

As applications of the E-subdifferential, both the necessary optimality conditions of Fritz John type and the necessary optimality conditions of Karush–Kuhn–Tucker type for a feasible solution to be an E-minimizer of the considered nonsmooth optimization problem with E-subdifferentiable functions are established. In order to prove the so-called E-Karush–Kuhn–Tucker necessary optimality conditions, the generalized E-constraint qualification is introduced in the paper. Further, the sufficiency of the E-Karush–Kuhn–Tucker necessary optimality conditions is also established under assumptions that the involved functions are E-subconvex.

The paper is organized as follows. In Section 2, we briefly recall some fundamental definitions of the classical subdifferential of a convex function and also the Clarke subgradient and the Clarke subdifferential for locally Lipschitz functions. Moreover, we also re-call the definition of an E-convex set and the definition of an E-convex function introduced by Youness [37]. In Section 3, we introduce the definition of an E-subdifferential of an E-convex function and we analyze its properties. We compare some of the properties of the E-subdifferential to analogous properties of the classical subdifferential of a convex function and the Clarke subdifferential of a locally Lipschitz function. Further, we introduce the definition of an E-subdifferentiable E-convex function, called E-subconvex, and we also analyze property of such a new concept of generalized convexity of (not necessarily) locally Lipschitz functions. In the next section, Applications, we use the E-subdifferential to formulate new necessary optimality conditions for nondifferentiable optimization problems with E-subdifferentiable functions. We introduce a new constraint qualification formulated with the help of E-subdifferentials of inequality constraints in proving one of such necessary optimality conditions. Moreover, we also prove the sufficient optimality conditions under assumptions that the functions constituting the considered nonsmooth constrained extremum problem are E-subconvex. The last section contains conclusive remarks.

2. Preliminaries

Let R^n be the n-dimensional Euclidean space and R^n_+ be its nonnegative orthant. It is well-known that a function $f\colon R^n \to R$ is convex on R^n if the inequality

$$f(\lambda x + (1-\lambda)u) \leq \lambda f(x) + (1-\lambda) f(u)$$

holds for all $x, u \in R^n$ and any $\lambda \in [0,1]$. It is well-known that the concept of a subgradient for nondifferentiable convex functions is a simple generalization of the gradient defined for differentiable functions.

Definition 2.1. [25, 28] It is said that a vector $\xi \in R^n$ is a subgradient of a convex function $f\colon R^n \to R$ at $u \in R^n$ if

$$f(x) - f(u) \geq \langle \xi, x-u \rangle, \quad \forall\, x \in R^n.$$

The above definition has a simple geometric interpretation. Since f is a convex function, we can find a supporting hyperplane at the boundary point $(u, f(u))$ that supports the epigraph of f. The slope of the hyperplane is a subgradient ξ of f at the point u.

Definition 2.2. [25, 28] The subdifferential $\partial f(u)$ of a convex function $f\colon R^n \to R$ at $u \in R^n$ is the set of subderivatives, that is, vectors $\xi \in R^n$ belonging to the following set

$$\partial f(u) = \{\xi \in R^n : f(x) - f(u) \geq \langle \xi, x - u\rangle, \forall\, x \in R^n\},$$

where the symbol $\langle \cdot, \cdot \rangle$ denotes the inner product of two vectors.

Several classes of functions have been defined for the purpose of weakening the limitations of convexity in mathematical programming. One of such classes of generalized convex functions are E-convex functions defined on E-convex sets. The definitions of an E-convex set and the definition of an E-convex function were introduced by Youness [37]. Now, for convenience, we recall the aforesaid definitions.

Definition 2.3. [37] Let $E\colon R^n \to R^n$ be a given operator. A set $S \subseteq R^n$ is said to be E-convex if and only if the following relation

$$E(u) + \lambda(E(x) - E(u)) \in S$$

holds for all $x, u \in S$ and any $\lambda \in [0, 1]$.

It is clear that every convex set is E-convex if E is the identity map. If a set $S \subset R^n$ is E-convex, then $E(S) \subseteq S$. Further, if $E(S)$ is a convex set and $E(S) \subseteq S$, then S is an E-convex set (see [32, 37]).

Definition 2.4. [37] Let $E\colon R^n \to R^n$ be a given operator and S be a nonempty E-convex subset of R^n. A real-valued function $f\colon S \to R$ is said to be E-convex on S if and only if the following inequality

$$f(\lambda E(x) + (1 - \lambda)E(u)) \leq \lambda f(E(x)) + (1 - \lambda) f(E(u))$$

holds for all $x, u \in S$ and any $\lambda \in [0, 1]$.

Remark 2.5. In the case when $E(x) \equiv x$, the above definition of an E-convex function reduces to the definition of a convex function.

Proposition 2.6. [1] *Let $E\colon R^n \to R^n$, S be an E-convex subset of R^n and $f\colon S \to R$ be an E-convex function at $u \in S$ on S. Further, assume that f is differentiable at u. Then, the inequality*

$$f(E(x)) - f(E(u)) \geq \langle \nabla f(E(u)), E(x) - E(u)\rangle \tag{2.1}$$

holds for all $x \in S$, where $\nabla f(E(u))$ denotes the gradient of f at $E(u)$.

Note that, in general, the gradient of a function is not the unique element in R^n which satisfies the inequality (2.1) with respect to the given operator $E\colon R^n \to R^n$. We illustrate this fact in the following example.

Example 2.7. Let $f\colon R \to R$ be a function defined by $f(x) = x^3$. Let $E\colon R \to R$ be an operator defined by $E(x) = x^2$. It can be shown by Definition 2.4 that f is an E-convex at $u = 0$ on R. Note that f is differentiable function at $u = 0$ and $\nabla f(E(u)) = 0$. However, it can be shown that not only $\nabla f(E(u))$ satisfies the inequality $f(E(x)) - f(E(u)) \geq \langle \nabla f(E(u)), E(x) - E(u)\rangle$. Indeed, note that any real number belonging to $(-\infty, 0]$ satisfies the aforesaid inequality.

The property of E-convex functions illustrated in Example 2.7 is similar to that one for convex functions. It makes, therefore, that we introduce the concept of E-subdifferential for E-convex functions, similarly to the concept of subdifferential which exists in convex analysis for convex functions.

3. E-subdifferential and its properties

Based on the definition of a subgradient of a convex function (see Definition 2.1) and the property of differentiable E-convex functions illustrated in Proposition 2.6, we introduce the following concepts of an E-subgradient and E-subdifferential for an E-convex function.

Definition 3.1. Let $E\colon R^n \to R^n$ be a given operator and $f\colon R^n \to R$ be an E-convex function at $u \in R^n$. It is said that $\xi \in R^n$ is an E-subgradient of f at $u \in R^n$ if the inequality

$$f(E(x)) - f(E(u)) \geq \langle \xi, E(x) - E(u)\rangle$$

holds for all $x \in R^n$.

Definition 3.2. Let $E\colon R^n \to R^n$ be a given operator and $f\colon R^n \to R$ be an E-convex function at $u \in R^n$. The set of all E-subgradients of $f\colon R^n \to R$ at $u \in R^n$, that is, the set which is denoted by $\partial_E f(u)$ and defined by

$$\partial_E f(u) := \{\xi \in R^n : f(E(x)) - f(E(u)) \geq \langle \xi, E(x) - E(u)\rangle, \forall\, x \in R^n\} \tag{3.1}$$

is said to be the E-subdifferential of f at u.

Definition 3.3. A function f is called E-subdifferentiable at u if it is E-subdifferential at u, that is, the set $\partial_E f(u)$ is nonempty.

In the next example, we consider the function for which the classical subdifferential of a convex function (and, thus, its Clarke's subdifferential) and the E-subdifferential are not the same.

Example 3.4. Consider the function $f\colon R \to R$ defined by $f(x) = |x|$. Let the operator $E\colon R \to R$ be defined by $E(x) = |x|$. Note that, by Definition 2.4, f is an E-convex function at $u = 0$ on R. Hence, by Definition 3.2, E-subdifferential of f at $u = 0$ is $\partial_E f(0) = (-\infty, 1]$, whereas the subdifferential of a convex function f at $u = 0$ is $\partial f(0) = [-1, 1]$.

Remark 3.5. Of course, if we take another operator E, then we can obtain a different E-subdifferential. We consider again the function f from Example 3.4. However, this time, we take the operator $E_1\colon R \to R$ defined by $E_1(x) = -|x|$. Then, $\partial_{E_1} f(0) = [-1, \infty)$.

Now, we consider a function for which the Clarke subdifferential is contained in its E-subdifferential.

Example 3.6. Consider the function $f\colon R \to R$ defined by $f(x) = x^2$. Let the operator $E\colon R \to R$ be defined by

$$E(x) = \begin{cases} 0 & \text{if } x \geq 0, \\ -x^2 & \text{if } x < 0. \end{cases}$$

Hence, E-subdifferential of f at $u = 0$ is $\partial_E f(0) = [0, \infty)$, whereas the Clarke's subdifferential of f at $u = 0$ is $\partial f(0) = \{0\}$. Therefore, $\partial f(0) \subset \partial_E f(0)$.

It is possible also that an E-subdifferential is contained in the Clarke subdifferential. Now, we illustrate such a case in the next example.

Example 3.7. Consider the function $f\colon R \to R$ defined by $f(x) = |x|$. Let the operator $E\colon R \to R$ be defined by

$$E(x) = \begin{cases} -1 & \text{if } x < 0, \\ 1/2 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$

Hence, it can be shown by Definition 2.4 that f is an E-convex function at $u = 0$ with respect to the given operator E. Further, the E-subdifferential of f at $u = 0$ is $\partial_E f(0) = [-1/3, 1]$, whereas the Clarke's subdifferential of f at $u = 0$ is $\partial f(0) = [-1, 1]$. Then, $\partial_E f(0) \subset \partial f(0)$.

Also it is possible that the Clarke's subdifferential of f at a given point is an empty set, whereas the E-subdifferential of f at this point is a nonempty subset of R^n.

Example 3.8. Let the function $f\colon R \to R$ be defined by $f(x) = \sqrt[3]{x}$. We define the operator $E\colon R \to R$ by

$$E(x) = \begin{cases} x^3 & \text{if } x \leq -1, \\ 0 & \text{if } -1 < x < 1, \\ -x^3 & \text{if } x \geq 1. \end{cases}$$

Note that f is not a Lipschitz function and, therefore, its Clarke's subdifferential is empty. However, by Definition 3.2, we have that $\partial_E f(0) = [1, \infty)$.

Remark 3.9. The classical subdifferential of a convex function f at the given point u can be treated as a special case of the E-subdifferential of f at this point. Namely, the subdifferential of a convex function f at the given point u is the E-subdifferential of f at this point with respect to the operator $E\colon R^n \to R^n$ defined by $E(x) = x$.

Note that the classical subdifferential can be empty at the given point, whereas an E-subdifferential can be nonempty.

Example 3.10. Let $f\colon R \to R$ be a function defined by $f(x) = -|x|$. Note that the classical subdifferential of f at $u = 0$ is $\partial f(0) = \emptyset$. Let the operator $E\colon R \to R$ be defined by $E(x) = -|x|$. Hence, by Definition 3.2, we have that $\partial_E f(0) = [1, \infty)$.

Note that even though the E-subdifferential $\partial_E f(u)$ is a single point set, then f does not need to be a differentiable function at this point. We now give an example of such a function to illustrate this property.

Example 3.11. Let the function $f\colon R \to R$ be defined by $f(x) = \begin{cases} \sqrt[3]{x} & \text{if } x < 0, \\ x & \text{if } x \geq 0. \end{cases}$ Further, let the operator $E\colon R \to R$ be defined by

$$E(x) = \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$

Then, by Definition 3.2, we have that $\partial_E f(0) = \{1\}$. Although the E-subdifferential of f at 0 is a single point, however, f is not differentiable at this point.

Another interesting property of the E-subdifferential is the fact that the gradient of a differentiable function cannot be its element.

Example 3.12. Let $f\colon R \to R$ be a function defined by $f(x) = x^2$ and $E\colon R \to R$ be a map defined by $E(x) = x^2$. We show that f is an E-subconvex function at any u on R. First, we prove, by Definition 2.4, that f is an E-convex function on R. In fact, we show that the inequality

$$f(\lambda E(x) + (1-\lambda)E(u)) \leq \lambda f(E(x)) + (1-\lambda) f(E(u))$$

holds for all $x, u \in S$ and any $\lambda \in [0,1]$. Then, for all $x, u \in S$ and any $\lambda \in [0,1]$, one has

$$(\lambda x^2 + (1-\lambda)u^2)^2 \leq \lambda x^4 + (1-\lambda)u^4. \tag{3.2}$$

Note that (3.2) is fulfilled for $\lambda = 0$ and $\lambda = 1$. Hence, (3.2) is equivalent to the inequality $(x^2-u^2)^2 \geq 0$, which is fulfilled for all $x, u \in S$ and any $\lambda \in (0,1)$. Then, by Definition 2.4, f is an E-convex function on R.

Further, note that $\partial_E f(u)$ is nonempty for each $u \in R$. Indeed, note that $\xi = 2u^2 \in \partial_E f(u)$ for each $u \in R$. This means that $\partial_E f(u)$ is nonempty for each $u \in R$. However, it is not difficult to see that $\nabla f(2) \notin \partial_E f(2)$.

It is known that the Clarke's subdifferential is defined for locally Lipschitz functions. However, as it follows even from the example below, the E-subdifferential may be a nonempty set, also for a function which is not locally Lipschitz. In other words, there are E-subdifferentiable functions which are not locally Lipschitz. In the next example, we present such a function.

Example 3.13. Consider the following function $f\colon R \to R$ defined by

$$f(x) = \begin{cases} 1/\sqrt[3]{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Note that, in fact, f is not locally Lipschitz at 0, since it is discontinuous at this point. Let the operator $E\colon R \to R$ be defined by

$$E(x) = \begin{cases} x & \text{if } x < -1, \\ 1/x^3 & \text{if } -1 \leq x < 0, \\ 0 & \text{if } x = 0, \\ -1/x^3 & \text{if } 0 < x \leq 1, \\ -x & \text{if } x > 1. \end{cases}$$

Then, it can be shown, by Definition 3.2, that $\partial_E f(0) = \{\xi \in R : \xi \geq 1\}$. This illustrates the fact that although f is not a locally Lipschitz function at 0, however, its E-subdifferential at this point may be a nonempty set.

In order to show the existence of E-subgradients of some nondifferentiable functions or, in other words, that their E-subdifferentials are nonempty, we introduce the concept of an E-subdifferentiable E-convex function, also called an E-subconvex function.

Definition 3.14. Let $E\colon R^n \to R^n$ be a given operator. A function $f\colon R^n \to R$ is said to be E-subdifferentiable E-convex at $u \in R^n$ or, in short, E-subconvex, if there exists $\xi \in R^n$ such that the inequality

$$f(E(x)) - f(E(u)) \geq \langle \xi, E(x) - E(u) \rangle$$

holds for all $x \in R^n$. The function f is said to be E-subconvex on R^n if its is E-subconvex at each $u \in R^n$.

Remark 3.15. We use the wording in the definition of an E-subconvex function f that this function is E-subdifferentiable. Indeed, as it follows from Definition 3.3, any E-subconvex function is, in fact, E-subdifferentiable since its E-subdifferential is a nonempty set.

Remark 3.16. If $E(x) \equiv x$, then the definition of an E-subconvex function reduces to the well-known definition of a subdifferentiable convex function (see [25]).

Remark 3.17. Note that the class of differentiable E-convex functions is a subclass of the class of E-subconvex. In fact, if f is a differentiable E-convex function, then the inequality

$$f(E(x)) - f(E(u)) \geq \langle \nabla f(E(u)), E(x) - E(u) \rangle$$

holds for all $x \in R^n$. Hence, the subdifferential $\partial_E f(u)$ of f is nonempty in such a case since $\nabla f(E(u)) \in \partial_E f(u)$. Then, by Definition 3.14, f is an E-subconvex at u.

Example 3.18. Let the function $f\colon R \to R$ be defined by $f(x) = x^2$. Further, let the operator $E\colon R \to R$ be defined by

$$E(x) = \begin{cases} 0 & \text{if } x \neq 0, \\ 1/2 & \text{if } x = 0. \end{cases}$$

It is not difficult to show, by Definition 2.4, that f is an E-convex function on R. Further, by Definition 3.14, f is an E-subconvex function on R and, moreover, by Definition 3.2, $\partial_E f(0) = [1/2, \infty)$. Then, note that $\nabla f(0) = 0 \notin \partial_E f(0)$. This means that the gradient of f at 0 is not element of the E-subdifferential of f at this point, although f is an E-subconvex at this point, that is, f has a nonempty E-subdifferential at 0.

Now, we prove some properties of E-subdifferential.

Proposition 3.19. *Let $E\colon R^n \to R^n$ be a given operator. If a function $f\colon R^n \to R$ is E-subconvex at $u \in R^n$, then $\partial_E f(u)$ is a nonempty closed convex subset of R^n.*

Proof. The result that $\partial_E f(u)$ is a nonempty subset of R^n if a function $f\colon R^n \to R$ is E-subconvex at $u \in R^n$ follows directly from Definition 3.14.

Closedness $\partial_E f(u)$ is evident—(3.1) is an infinite system of nonstrict linear inequalities with respect to ξ, the inequalities being indexed by x.

We now prove that $\partial_E f(u)$ is a convex subset of R^n. Let ξ_1 and ξ_2 be two elements of $\partial_E f(u)$. Hence, by Definition 3.2, we have that two inequalities

$$f(E(x)) - f(E(u)) \geq \langle \xi_1, E(x) - E(u) \rangle,$$
$$f(E(x)) - f(E(u)) \geq \langle \xi_2, E(x) - E(u) \rangle$$

hold for all $x \in R^n$. Let $\lambda \in [0,1]$. Multiplying the first inequality above by λ and the second one by $1-\lambda$, we get

$$\lambda[f(E(x)) - f(E(u))] \geq \langle \lambda\xi_1, E(x) - E(u)\rangle,$$
$$(1-\lambda)[f(E(x)) - f(E(u))] \geq \langle (1-\lambda)\xi_2, E(x) - E(u)\rangle.$$

Adding both sides of the above inequalities, we get

$$f(E(x)) - f(E(u)) \geq \langle \lambda\xi_1 + (1-\lambda)\xi_2, E(x) - E(u)\rangle.$$

Hence, by Definition 3.2, $\lambda\xi_1 + (1-\lambda)\xi_2$ belong to $\partial_E f(u)$. This means that $\partial_E f(u)$ is a convex subset of R^n. □

Remark 3.20. Note that, in general, $\partial_E f(u)$ is not a compact subset of R^n as it follows even from Example 3.4 (and also from Remark 3.5).

Proposition 3.21. *Let $E\colon R^n \to R^n$ be a given open map. Further, assume that $f\colon R^n \to R$ is an E-subconvex function at a given $u \in R^n$. If there exists a neighborhood $U(u)$ of u such that the inequality*

$$|f(E(z)) - f(E(x))| \leq K_u\|E(z) - E(x)\| \tag{3.3}$$

holds for all $x, z \in U(u)$, where $K_u > 0$, then $\partial_E f(u)$ is a compact subset of R^n.

Proof. Let $E\colon R^n \to R^n$ be a given operator and $u \in R^n$ be given. By assumption, $f\colon R^n \to R$ is E-subdifferentiable E-convex at $u \in R^n$. Hence, by Definition 3.2, the inequality

$$f(E(z)) - f(E(x)) \geq \langle \xi, E(z) - E(x)\rangle$$

holds for all $x, z \in U(u)$. Let x be any point of $U(u)$. Then, by (3.3), the inequality

$$f(E(z)) - f(E(x)) \leq K_u\|E(z) - E(x)\|$$

holds for all $z \in U(u)$. Let us denote $\varphi(z,x) = E(z) - E(x)$. By assumption, E is an open map. Hence, the map $z \to \varphi(z,x)$ is also an open map. Therefore, the map $z \to \varphi(z,x)$ maps the open set $U(u)$ onto an open neighborhood $V(0)$ of the origin. Hence, we have that, at each $x \in U(u)$, the inequality

$$\langle \xi, z - 0\rangle \leq K_u\|z - 0\| \tag{3.4}$$

holds for any $\xi \in \partial_E f(u)$ and all $z \in B(0,r) \subset V(0)$, where $B(0,r)$ denotes the open ball centered at 0 with the radius $r > 0$. Hence, (3.4) implies that the inequality

$$|\langle \xi, z\rangle| \leq K_u\|z\|$$

holds. This means that $\|\xi\| \leq K_u$. Thus, we conclude that the map $x \to \partial_E f(x)$ is locally bounded at u. Therefore, the set $\partial_E f(u)$ is compact. □

Remark 3.22. Note that if $E\colon R^n \to R^n$ is not an open map and $f\colon R^n \to R$ is an E-subconvex function at $u \in R^n$, then $\partial_E f(u)$ is not necessarily a compact subset of R^n. Indeed, we consider the function $f\colon R \to R$ which is defined by $f(x) = x^3$. Moreover, let the operator $E\colon R \to R$ be defined by

$$E(x) = \begin{cases} 1 & \text{if } x < -1 \vee x > 1, \\ -x^3 & \text{if } -1 \le x \le 0, \\ x^3 & \text{if } 0 \le x \le 1. \end{cases}$$

It can be shown, by Definition 2.4, that f is an E-convex function at $u = 0$. Also it is not difficult to note that E is not an open map. It can be shown, by Definition 3.2, that $\partial_E f(0) = (-\infty, 0]$. Since E is not an open map, $\partial_E f(0)$ is not a compact subset of R.

Proposition 3.23. *Let $f_1\colon R^n \to R$ and $f_2\colon R^n \to R$ be E-subconvex functions. Then, $f_1 + f_2$ is also an E-subconvex function and, moreover, the following relation*

$$\partial_E f_1(u) + \partial_E f_2(u) \subseteq \partial_E(f_1 + f_2)(u) \tag{3.5}$$

holds for each $u \in R^n$.

Proof. The proof of this proposition follows directly from Definitions 3.2 and 3.14. □

Note that the equality does not hold in a general case. Now, we give an example of such E-subconvex functions for which there is no the equality in (3.5).

Example 3.24. Let $f_1\colon R \to R$ and $f_2\colon R \to R$ be functions defined by

$$f_1(x) = \begin{cases} x & \text{if } x < 0, \\ x/2 & \text{if } x \ge 0 \end{cases} \quad \text{and} \quad f_2(x) = \begin{cases} 0 & \text{if } x < 0, \\ x/2 & \text{if } x \ge 0. \end{cases}$$

Further, let $E\colon R \to R$ be an operator defined by $E(x) = x^2$. Then, it can be shown by Definition 2.4, that f_1 and f_2 are E-convex functions at $u = 0$ on R. Moreover, by Definition 3.14 both f_1 and f_2 are E-subconvex functions at $u = 0$ and, by Definition 3.2, $\partial_E f_1(0) = \{\xi \in R : \xi \le 1/2\}$ and $\partial_E f_2(0) = \{\xi \in R : \xi \le 0\}$. Then, by Proposition 3.23, $f_1 + f_2$ is also an E-subconvex function at $u = 0$ and, by Definition 3.2, $\partial_E(f_1 + f_2)(0) = \{\xi \in R : \xi \le 1\}$. Thus, $\partial_E f_1(0) + \partial_E f_2(0) \subset \partial_E(f_1 + f_2)(0)$.

Proposition 3.25. *If a function $f\colon R^n \to R$ has a nonempty E-subdifferential at any $u \in R^n$, then f is an E-subconvex function.*

Proof. For arbitrary $x, u \in R^n$ and $\lambda \in [0,1]$, there exists an E-subgradient ξ at $\lambda x + (1 - \lambda)u$. Hence, by Definition 3.1, it follows that

$$f(E(x)) \ge f(\lambda E(x) + (1-\lambda)E(u)) + \langle \xi, (1-\lambda)(E(x) - E(u)) \rangle,$$
$$f(E(u)) \ge f(\lambda E(x) + (1-\lambda)E(u)) + \langle \xi, \lambda(E(u) - E(x)) \rangle.$$

Multiplying the first inequality above by λ and the second one by $1-\lambda$, we get, respectively,

$$\lambda f(E(x)) \geq \lambda f(\lambda E(x) + (1-\lambda)E(u)) + \lambda(1-\lambda)\langle \xi, E(x) - E(u)\rangle,$$
$$(1-\lambda) f(E(u)) \geq (1-\lambda) f(\lambda E(x) + (1-\lambda)E(u)) - \lambda(1-\lambda)\langle \xi, E(x) - E(u)\rangle.$$

Adding both sides of the above inequalities, we obtain that the inequality

$$\lambda f(E(x)) + (1-\lambda) f(E(u)) \geq f(\lambda E(x) + (1-\lambda)E(u))$$

holds for any $\lambda \in [0,1]$. Since x and u are arbitrary points and λ is any number from $[0,1]$, by Definition 2.4, f is an E-convex function. By assumption, f has a nonempty E-subdifferential at any $u \in R^n$. Hence, by Definition 3.14, f is an E-subconvex function. □

Now, we prove the sufficient condition for $\xi \in R^n$ to belong to the E-subdifferential of a differentiable E-convex function. Therefore, we introduce the definition of the so-called E-normal cone at $u \in X$.

Definition 3.26. Let X be a subset of R^n and $E\colon R^n \to R^n$ be a given operator. The E-normal cone $N_X^E(u)$ at $u \in X$ is defined by

$$N_X^E(u) = \{\xi \in R^n : \langle \xi, E(x) - E(u)\rangle \leq 0, \forall\, x \in X\}.$$

Remark 3.27. It is clear that $0 \in N_X^E(u)$. Further, note that if $X = R^n$, then the E-normal cone doesn't need to be a singleton set containing zero element alone, unlike the usual normal cone in convex analysis. Indeed, if $X = R$, we define an operator $E\colon R \to R$ by $E(x) = x^2$. Then, by Definition 3.26, it follows that $N_X^E(0) = \{\xi \in R : \xi \leq 0\}$.

Remark 3.28. As it follows from Definition 3.26 and the definition of the usual normal cone in convex analysis, the usual normal cone, that is, the set $N_X(u) = \{\xi \in R^n : \langle \xi, x - u\rangle \leq 0, \forall\, x \in X\}$ can be considered as a special case of $N_X^E(u)$ if $E(x) \equiv x$.

Theorem 3.29. *Let $E\colon R^n \to R^n$ be a differentiable mapping and $f\colon R^n \to R$ be a differentiable function. If ξ is an element R^n such that*

$$\xi - \nabla f(E(u)) \in N_{R^n}^E(u), \tag{3.6}$$

then $\xi \in \partial_E f(u)$.

Proof. Let $f\colon R^n \to R$ be an differentiable E-convex function and $u \in R^n$ be given. Moreover, assume that $\xi \in R^n$ satisfies (3.6). Then, by Definition 3.26, it follows that

$$\langle \xi - \nabla f(E(u)), E(x) - E(u)\rangle \leq 0, \quad \forall\, x \in R^n.$$

Thus,

$$\langle \nabla f(E(u)), E(x) - E(u) \rangle \geq \langle \xi, E(x) - E(u) \rangle, \quad \forall\, x \in R^n. \tag{3.7}$$

Since E and f are differentiable, $f \circ E$ is differentiable function. Hence, by the Taylor's formula, we have

$$\begin{aligned} f(E(u) + \lambda(E(x) - E(u))) = f(E(u)) &+ \langle \nabla(f \circ E)(u), \lambda(E(x) - E(u)) \rangle \\ &+ o(\lambda \|E(x) - E(u)\|), \end{aligned} \tag{3.8}$$

where $\frac{o(\lambda\|E(x)-E(u)\|)}{\lambda} \to 0$ as $\lambda \to 0^+$. Using (3.7) in (3.8), we get that the inequality

$$f(E(u) + \lambda(E(x) - E(u))) \geq f(E(u)) + \lambda\langle \xi, E(x) - E(u) \rangle + o(\lambda\|E(x) - E(u)\|)$$

holds for all $x \in R^n$. By assumption, f is E-convex on R^n. Then, by Definition 2.4, the inequality

$$f(\lambda E(x) + (1-\lambda)E(u)) \leq \lambda f(E(x)) + (1-\lambda) f(E(u)) \tag{3.9}$$

holds for all $x \in R^n$ and any $\lambda \in [0,1]$. Combining (3.8) and (3.9), we obtain that

$$\lambda f(E(x)) + (1-\lambda) f(E(u)) \geq f(E(u)) + \lambda\langle \xi, E(x) - E(u) \rangle + o(\lambda\|E(x) - E(u)\|).$$

Thus,

$$\lambda[f(E(x)) - f(E(u))] \geq \lambda\langle \xi, E(x) - E(u) \rangle + o(\lambda\|E(x) - E(u)\|).$$

Dividing by $\lambda > 0$, we have

$$f(E(x)) - f(E(u)) \geq \langle \xi, E(x) - E(u) \rangle + \frac{o(\lambda\|E(x) - E(u)\|)}{\lambda}.$$

Letting $\lambda \to 0^+$ and taking into account that $\frac{o(\lambda\|E(x)-E(u)\|)}{\lambda} \to 0$ as $\lambda \to 0^+$, we get that the inequality

$$f(E(x)) - f(E(u)) \geq \langle \xi, E(x) - E(u) \rangle$$

holds for all $x \in R^n$. This means, by Definition 3.2, that $\xi \in \partial_E f(u)$ and completes the proof of this theorem. □

Now, we give the definition of an E-minimizer of a function.

Definition 3.30. Let $E\colon R^n \to R^n$ be a given operator. It is said that $\overline{x} \in R^n$ is a global E-minimizer of $f\colon R^n \to R$ if the inequality

$$f(E(x)) \geq f(E(\overline{x}))$$

holds for all $x \in R^n$.

The following result follows directly from the definition of the E-subdifferential.

Proposition 3.31. *Let $E\colon R^n \to R^n$ be a given operator and $f\colon R^n \to R$ be an E-subdifferentiable function. If $\overline{x} \in R^n$ is a global E-minimizer of f, then $0 \in \partial_E f(\overline{x})$.*

Proof. This result follows directly from Definitions 3.2 and 3.30. □

Note that the result established in Proposition 3.31 is only the necessary condition for a global E-minimizer of an E-subdifferentiable function. In order to prove the sufficient condition for a global E-minimizer of E-subdifferentiable function, we assume that this function is E-subconvex at such a point.

Proposition 3.32. *Let $E\colon R^n \to R^n$ be a given operator and $f\colon R^n \to R$ be an E-subconvex function. If $0 \in \partial_E f(\overline{x})$, then $\overline{x}$ is an E-minimizer of f.*

Proof. Let $E\colon R^n \to R^n$ be a given operator. Further, assume that $f\colon R^n \to R$ is an E-subconvex function on R^n. Hence, by Definition 3.2, the inequality

$$f(E(x)) - f(E(\overline{x})) \geq \langle \xi, E(x) - E(\overline{x}) \rangle \tag{3.10}$$

holds for all $x \in R^n$ and any $\xi \in \partial_E f(\overline{x})$. Since $0 \in \partial_E f(\overline{x})$, (3.10) implies that the inequality

$$f(E(x)) \geq f(E(\overline{x}))$$

holds for all $x \in R^n$. This means that $\overline{x}$ is an E-minimizer of f. □

The result established in Proposition 3.31 is true for a global E-minimizer only. But if $\overline{x}$ is a local E-optimal solution, then it is possible that $0 \notin \partial_E f(\overline{x})$, even if f is a subconvex at $\overline{x}$. We illustrate such a case in the next example.

Example 3.33. Consider the function $f\colon [a, \infty) \to R$, where a is any negative number such that $a \leq -3$, be defined by

$$f(x) = \begin{cases} -2 & \text{if } a \leq x < -2, \\ 3x + 4 & \text{if } -2 \leq x < -1, \\ -\sqrt[3]{x} & \text{if } -1 \leq x < 0, \\ x^2 & \text{if } x \geq 0. \end{cases}$$

Further, let the operator $E\colon R^n \to R^n$ be defined by

$$E(x) = \begin{cases} -x^3 & \text{if } a \leq x < -1, \\ -\sqrt[3]{x} & \text{if } -1 \leq x < 0, \\ \sqrt{x} & \text{if } x \geq 0. \end{cases}$$

It is not difficult to show by Definition 2.4, that f is an E-convex function at $\overline{x} = 0$ on $[a, \infty)$. Further, note that $\overline{x} = 0$ is a local E-minimizer of f. However, by Definition 3.2, we have that $\partial_E f(0) = (-\infty, -a^6]$. This means that the condition $0 \in \partial_E f(0)$ is not necessary condition for $\overline{x}$ to be a local E-minimizer of a subconvex function.

Lemma 3.34. *Let $f\colon R^n \to R$ be an E-subconvex function such that $0 \notin \partial_E f(\overline{x})$. Then the set $F^E(\xi) = \{z \in R^n : \langle \xi, z\rangle < 0\}$ is nonempty for any $\xi \in \partial_E f(\overline{x})$.*

Proof. By assumption, $0 \notin \partial_E f(\overline{x})$. Hence, by Proposition 3.31, $\overline{x}$ is not an E-minimizer of f over R^n. Then, by Definition 3.30, there exists another $\widetilde{x} \in R^n$ such that

$$f(E(\widetilde{x})) < f(E(\overline{x})).$$

Since f is an E-subconvex at $\overline{x}$, the inequality

$$f(E(x)) - f(E(\overline{x})) \geq \langle \xi, E(x) - E(\overline{x})\rangle$$

holds for all $x \in R^n$ and any $\xi \in \partial_E f(\overline{x})$. Therefore, it is also satisfied for $x = \widetilde{x} \in R^n$. Combining two last inequalities, we get the result. □

4. Applications

In this section, as an application of the concept of an E-subdifferential and the E-subconvexity notion, we prove both necessary and sufficient optimality conditions for a nondifferentiable optimization problem with E-subconvex functions.

Let $E\colon R^n \to R^n$ be a given. In the paper, consider the following nonlinear nondifferentiable constrained optimization problem (4.1) defined by

$$\text{minimize } f(x) \text{ subject to } g_j(x) \leq 0,\ j \in J = \{1, \ldots, m\}, \tag{4.1}$$

where $f\colon R^n \to R$, $g_j\colon R^n \to R$, $j \in J$, are real-valued E-subdifferentiable functions defined on R^n.

For the purpose of simplifying our presentation, we will next introduce some notations which will be used frequently throughout this section. Let

$$D := \{x \in R^n : g_j(x) \leq 0, j \in J\}$$

be the set of all feasible solutions of (4.1). Throughout this section, we shall assume that a map $E\colon R^n \to R^n$ is given. Now, we introduce the definition of an E-optimal solution for the considered optimization problem (4.1).

Definition 4.1. It is said that $\overline{x} \in D$ is an E-optimal solution of (4.1) if and only if there exists no other feasible point x such that

$$f(E(x)) < f(E(\overline{x})).$$

Further, let us denote by $J_E(\overline{x})$ the set of inequality constraint indices that are active at $E(\overline{x})$, that is, $J_E(\overline{x}) = \{j \in J : g_j(E(\overline{x})) = 0\}$. If we assume that the functions constituting the considered optimization problem (4.1) are E-convex, then the following result is true.

Proposition 4.2. [37] *Let the constraint functions g_j, $j \in J$, be E-convex on D. Then, the set D of all feasible solutions of* (4.1) *is E-convex.*

To prove the necessary optimality conditions for (4.1) by using E-subdifferentials of the involved functions, we shall assume that the objective function f and the constraint functions g_j, $j \in J$, satisfy the following condition at any E-optimal solution $\overline{x}$.

Condition (E). Let $E\colon R^n \to R^n$ be a given operator. It said that the function $\varphi\colon R^n \to R$ satisfies Condition (E) at $\overline{x}$ if the inequality

$$\limsup_{\lambda \downarrow 0} \frac{\varphi(E(\overline{x}) + \lambda(E(x) - E(\overline{x}))) - \varphi(E(\overline{x}))}{\lambda} \leq \langle \xi, E(x) - E(\overline{x}) \rangle$$

holds for some $\xi \in \partial_E \varphi(\overline{x})$ and all $x \in R^n$.

Theorem 4.3 (The necessary optimality conditions of E-Fritz John type)**.** *Let $\overline{x} \in D$ be an E-minimizer of the considered optimization problem* (4.1)*. Further, assume that the objective function f and the constraint functions g_j, $j \in J$, constituting* (4.1) *are E-subdifferentiable and they satisfy Condition* (E) *at this point. Then there exist $\overline{\vartheta} \in R_+$ and $\overline{\mu} \in R^m_+$ such that*

$$0 \in \overline{\vartheta}\partial_E f(\overline{x}) + \sum_{j=1}^{m} \overline{\mu}_j \partial_E g_j(\overline{x}), \tag{4.2}$$

$$\overline{\mu}_j g_j(\overline{x}) = 0, \quad j \in J, \tag{4.3}$$

$$(\overline{\vartheta}, \overline{\mu}) \neq 0. \tag{4.4}$$

Proof. Consider the following cases:

(i) If either $0 \in \partial_E f(\overline{x})$, then we take $\overline{\vartheta} \neq 0$ and $\overline{\mu}_j = 0$, $j \in J$, then (4.2) is fulfilled. If $0 \in \partial_E g_{j^*}(\overline{x})$ for some $j^* \in J_E(\overline{x})$, then we set $\overline{\vartheta} = 0$, $\overline{\mu}_{j^*} \neq 0$ and $\overline{\mu}_j = 0$, $j \in J \setminus \{j^*\}$. Hence, (4.2) is also fulfilled in such a case.

(ii) Let us assume that $0 \notin \partial_E f(\overline{x})$ and $0 \notin \partial_E g_j(\overline{x})$, $j \in J(\overline{x})$. By assumption, f and g_j, $j \in J$, satisfy Condition (E) at $\overline{x}$. Hence, there exist $\xi \in \partial_E f(\overline{x})$ and $\xi_j \in \partial_E g_j(\overline{x})$, $j \in J$ such that the inequalities

$$\limsup_{\lambda \downarrow 0} \frac{f(E(\overline{x}) + \lambda(E(x) - E(\overline{x}))) - f(E(\overline{x}))}{\lambda} \leq \langle \xi, E(x) - E(\overline{x}) \rangle, \tag{4.5}$$

$$\limsup_{\lambda \downarrow 0} \frac{g_j(E(\overline{x}) + \lambda(E(x) - E(\overline{x}))) - g_j(E(\overline{x}))}{\lambda} \leq \langle \xi_j, E(x) - E(\overline{x}) \rangle, \quad j \in J(\overline{x}) \tag{4.6}$$

hold for all $x \in R^n$. By assumption, $0 \notin \partial_E f(\overline{x})$ and $0 \notin \partial_E g_j(\overline{x})$, $j \in J$. Hence, $\xi \neq 0$, $\xi \in \partial_E f(\overline{x})$, and $\xi_j \neq 0$, $\xi_j \in \partial_E g_j(\overline{x})$, $j \in J$. Let us denote

$$F^{E(\overline{x})}(\xi) = \{x \in R^n : \langle \xi, E(x) - E(\overline{x}) \rangle < 0\}, \tag{4.7}$$

$$F_j^{E(\overline{x})}(\xi_j) = \{x \in R^n : \langle \xi_j, E(x) - E(\overline{x}) \rangle < 0\}, \quad j \in J. \tag{4.8}$$

Since $0 \notin \partial_E f(\overline{x})$ and $0 \notin \partial_E g_j(\overline{x})$, $j \in J$, by Lemma 3.34, it follows that $F^{E(\overline{x})}(\xi) \neq \emptyset$ and $F_j^{E(\overline{x})}(\xi_j) \neq \emptyset$, $j \in J$.

Now, we show that $F^{E(\overline{x})}(\xi) \cap \bigcap_{j=1}^m F_j^{E(\overline{x})}(\xi) = \emptyset$. We proceed by contradiction. Suppose, contrary to the result, that there exists $x \in F^{E(\overline{x})}(\xi) \cap \bigcap_{j=1}^m F_j^{E(\overline{x})}(\xi_j)$. Then, combining (4.5)–(4.8), we get that, for sufficiently small $\lambda > 0$, $f(E(\overline{x})+\lambda(E(x)-E(\overline{x}))) < f(E(\overline{x}))$ $\lambda \in (0, \lambda_f]$ and $g_j(E(\overline{x}) + \lambda(E(x) - E(\overline{x}))) < g_j(E(\overline{x}))$, $j \in J$, $\lambda \in (0, \lambda_{g_j}]$. Note that $g_j(E(\overline{x})) = 0$, $j \in J_E(\overline{x})$. Hence, $g_j(E(\overline{x}) + \lambda(E(x) - E(\overline{x}))) < 0$, $j \in J_E(\overline{x})$, $\lambda \in (0, \lambda_{g_j}]$. Moreover, we have $g_j(E(\overline{x})) < 0$, $j \notin J_E(\overline{x})$. Therefore, $g_j(E(\overline{x}) + \lambda(E(x) - E(\overline{x}))) < 0$, $j \notin J(\overline{x})$, $\lambda \in (0, \lambda_{g_j}]$. Hence, $g_j(E(\overline{x}) + \lambda(E(x) - E(\overline{x}))) < 0$, $j \in J$, $\lambda \in (0, \lambda_{g_j}]$.

Let us set $\overline{\lambda} = \min\{\lambda_f, \lambda_{g_j}, j \in J\}$. Then, we have that $f(E(\overline{x}) + \lambda(E(x) - E(\overline{x}))) < f(E(\overline{x}))$ and $g_j(E(\overline{x}) + \lambda(E(x) - E(\overline{x}))) < 0$, $j \in J$, for any $\lambda \in (0, \overline{\lambda}]$. Hence, this is a contradiction to the assumption that $\overline{x}$ is an E-minimizer of (4.1). Hence, the system of inequalities

$$\langle \xi, E(x) - E(\overline{x}) \rangle < 0 \quad \text{and} \quad \langle \xi_j, E(x) - E(\overline{x}) \rangle < 0, \quad j \in J_E(\overline{x})$$

has no a solution $x \in R^n$. Hence, by Gordan's theorem of the alternative (see [20]), there exist $\overline{\vartheta} \in R_+$ and $\overline{\mu} \in R_+^m$ such that

$$\overline{\vartheta}\xi + \sum_{j \in J_E(\overline{x})} \overline{\mu}_j \xi_j = 0. \tag{4.9}$$

If we set $\overline{\mu}_j = 0$, $j \notin J_E(\overline{x})$, then (4.9) implies that the necessary optimality condition of E-Fritz John type (4.2) is fulfilled. Also it is not difficult to see that the necessary optimality conditions of E-Fritz John type (4.3) and (4.4) are fulfilled. This completes the proof of this theorem. □

Example 4.4. Consider the following nondifferentiable optimization problem defined by

$$\text{minimize } f(x) = |x|^3 \text{ subject to } g(x) = -x^3 \leq 0. \tag{4.10}$$

Note that $D = \{x \in R : x \geq 0\}$ and $\overline{x} = 0$ is a feasible solution of (4.10). Let $E: R \to R$ be a mapping defined by $E(x) = -x^2$. It is not difficult to show, by Definition 2.4, that the functions constituting (4.10) are E-convex at $\overline{x} = 0$ on R (therefore, they are E-convex at $\overline{x} = 0$ on D). Further, note that, by Definition 3.2, the E-subdifferentials at $\overline{x} = 0$ of the functions constituting the considered nonlinear constrained extremum problem (4.10) are as follows: $\partial_E f(\overline{x}) = [0, \infty)$ and $\partial_E g(\overline{x}) = [0, \infty)$. Since the E-subdifferentials of f and g are nonempty at $\overline{x}$, by Definition 3.3, the functions constituting (4.10) are E-subdifferentiable at $\overline{x} = 0$. Also it can be noticed that each function constituting (4.10) satisfies Condition (E). Thus, the necessary optimality conditions of E-Fritz John type (see Theorem 4.3) are fulfilled for the considered nondifferentiable optimization problem (4.10).

In order to prove the necessary optimality conditions of E-Karush–Kuhn–Tucker type, we introduce the generalized E-constraint qualification.

Definition 4.5. It is said that the generalized E-constraint qualification (E-GCQ) is satisfied at $\overline{x} \in D$ for the problem (4.1) if

$$0 \notin \operatorname{co} \sum_{j \in J_E(\overline{x})} \partial_E g_j(\overline{x}). \tag{4.11}$$

Theorem 4.6 (The necessary optimality conditions of E-Karush–Kuhn–Tucker type)**.** *Let $\overline{x} \in D$ be an E-minimizer of the considered optimization problem* (4.1) *and all hypotheses of Theorem* 4.3 *be fulfilled. Further, assume that the generalized E-constraint qualification (E-GCQ) holds at $\overline{x}$. Then there exists $\overline{\mu} \in R^m$ such that*

$$0 \in \partial_E f(\overline{x}) + \sum_{j=1}^{m} \overline{\mu}_j \partial_E g_j(\overline{x}), \tag{4.12}$$

$$\overline{\mu}_j g_j(\overline{x}) = 0, \quad j \in J, \tag{4.13}$$

$$\overline{\mu} \geq 0. \tag{4.14}$$

Proof. Let $\overline{x} \in D$ be an E-minimizer of the considered optimization problem (4.1). Further, assume that all hypotheses of Theorem 4.3 are fulfilled. Then, the necessary optimality conditions of E-Fritz John type (4.2)–(4.4) are fulfilled. Therefore, it is sufficient to show that $\overline{\vartheta} \neq 0$ in order to prove that the necessary optimality conditions of E-Karush–Kuhn–Tucker type hold at $\overline{x}$.

We proceed by contradiction. Suppose, contrary to the result, that $\overline{\vartheta} = 0$. Hence, by the necessary optimality conditions of E-Fritz John type (4.2), it follows that

$$0 \in \sum_{j=1}^{m} \overline{\mu}_j \partial_E g_j(\overline{x}).$$

Since $\overline{\vartheta} = 0$, by the necessary optimality conditions of E-Fritz John type (4.3) and (4.4), we have that $\overline{\mu}_j > 0$ for at least one $j \in J_E(\overline{x})$. This means that the set $J_E(\overline{x})$ is not empty. Thus,

$$0 \in \sum_{j \in J_E(\overline{x})} \overline{\mu}_j \partial_E g_j(\overline{x}). \tag{4.15}$$

Hence, (4.15) gives

$$0 \in \sum_{j \in J_E(\overline{x})} \frac{\overline{\mu}_j}{\sum_{t \in J(\overline{x})} \overline{\mu}_t} \partial_E g_j(E(\overline{x})).$$

Denote $\overline{\mu}_j^0 = \frac{\overline{\mu}_j}{\sum_{t \in J(\overline{x})} \overline{\mu}_t}$. Thus, $\overline{\mu}_j^0 \geq 0$, $j \in J_E(\overline{x})$, and, moreover, $\sum_{j \in J_E(\overline{x})} \frac{\overline{\mu}_j}{\sum_{t \in J_E(\overline{x})} \overline{\mu}_t} = 1$. Then, we have

$$0 \in \sum_{j \in J_E(\overline{x})} \overline{\mu}_j^0 \partial_E g_j(\overline{x}).$$

Hence, by the definition of a convex hull, it follows that

$$0 \in \operatorname{co} \sum_{j \in J_E(\overline{x})} \partial_E g_j(\overline{x}). \tag{4.16}$$

By assumption, the generalized E-constraint qualification (E-GCQ) is satisfied at $\overline{x} \in D$ for the considered optimization problem (4.1). Therefore, (4.16) contradicts (4.11). Then, $\overline{\vartheta} \neq 0$, which completes the proof of this theorem. □

We now prove the sufficiency of the necessary optimality conditions of E-Karush–Kuhn–Tucker type under appropriate E-subconvexity hypotheses.

Theorem 4.7. *Let $\overline{x} \in D$ be a feasible solution of* (4.1) *such that the necessary optimality conditions of E-Karush–Kuhn–Tucker type* (4.12)–(4.14) *are fulfilled. Further, assume that f and g_j, $j \in J_E(\overline{x})$, are E-subconvex at $\overline{x}$ on D. Then $\overline{x} \in D$ is an E-minimizer in* (4.1).

Proof. By assumption, $\overline{x} \in D$ and there exists $\overline{\mu} \in R^m$ such that the necessary optimality conditions of E-Karush–Kuhn–Tucker type (4.12)–(4.14) are fulfilled. Then, there exist $\overline{\mu}_j$, $j \in J$, for which the conditions (4.12)–(4.14) are fulfilled. Also by assumption, f and g_j, $j \in J$, are E-subconvex at $\overline{x}$ on D. Then, by Definition 3.14, the following inequalities

$$f(E(x)) - f(E(\overline{x})) \geq \langle \xi, (E(x) - E(\overline{x})) \rangle, \tag{4.17}$$

$$g_j(E(x)) - g_j(E(\overline{x})) \geq \langle \xi_j, (E(x) - E(\overline{x})) \rangle, \quad j \in J_E(\overline{x}) \tag{4.18}$$

hold for all $x \in D$ and any $\xi \in \partial_E f(\overline{x})$ and $\xi_j \in \partial_E g_j(\overline{x})$, $J_E(\overline{x})$. Multiplying (4.18) by the corresponding Lagrange multiplier $\overline{\mu}_j$, $j \in J_E(\overline{x})$, we get

$$\overline{\mu}_j g_j(E(x)) - \overline{\mu}_j g_j(E(\overline{x})) \geq \langle \overline{\mu}_j \xi_j, (E(x) - E(\overline{x})) \rangle, \quad j \in J_E(\overline{x}).$$

Using $x \in D$ together with the necessary optimality condition of E-Karush–Kuhn–Tucker type (4.13), we obtain

$$\langle \overline{\mu}_j \xi_j, (E(x) - E(\overline{x})) \rangle \leq 0, \quad j \in J_E(\overline{x}). \tag{4.19}$$

Adding both sides of (4.19), taking into account $\overline{\mu}_j = 0$, $j \notin J_E(\overline{x})$, we get

$$\sum_{j=1}^{m} \langle \overline{\mu}_j \xi_j, (E(x) - E(\overline{x})) \rangle \leq 0. \tag{4.20}$$

By (4.17) and (4.20), we have

$$f(E(x)) - f(E(\overline{x})) \geq \left\langle \xi + \sum_{j=1}^{m} \overline{\mu}_j \xi_j, (E(x) - E(\overline{x})) \right\rangle.$$

By the necessary optimality condition of E-Karush–Kuhn–Tucker type (4.12), we obtain that the inequality

$$f(E(x)) \geq f(E(\overline{x}))$$

holds for all $x \in D$. This means, by Definition 3.30, that $\overline{x}$ is an E-minimizer of (4.1) and completes the proof of this theorem. □

In order to illustrate the optimality conditions established for the considered nondifferentiable optimization problem with E-subconvex functions.

Example 4.8. Consider the following nonconvex nondifferentiable optimization problem

$$\text{minimize } f(x) = \sqrt[3]{x} \text{ subject to } g(x) = -x \leq 0. \tag{4.21}$$

Note that $D = \{x \in R : x \geq 0\}$ and $\overline{x} = 0$ is a feasible solution of (4.21). Let $E\colon R \to R$ be a mapping defined by

$$E(x) = \begin{cases} 0 & \text{if } x \geq 0, \\ -1 & \text{if } -1 \leq x < 0, \\ x^3 & \text{if } x < -1. \end{cases}$$

Note that, by Definition 3.2, the E-subdifferentials at $\overline{x} = 0$ of the functions constituting the considered nonlinear constrained extremum problem are as follows $\partial_E f(0) = [1, \infty)$ and $\partial_E g(0) = [-1, \infty)$. Since the E-subdifferentials of f and g are nonempty, by Definition 3.3, the functions constituting (4.21) are E-subdifferentiable at $\overline{x} = 0$. Moreover, it can be shown that each function constituting (4.21) satisfies Condition (E). Then, if we set $\overline{\mu} = 1$, then the Karush–Kuhn–Tucker necessary optimality conditions (4.12)–(4.14) are fulfilled at $\overline{x} = 0$. Further, it follows, by Definition 3.14, that f and g are E-subconvex functions at $\overline{x} = 0$ on R (the more so, on D). Therefore, the sufficient optimality conditions from Theorem 4.7 are also satisfied which means that $\overline{x} = 0$ is an E-minimizer of (4.21).

Remark 4.9. Note that it is not possible to use for (4.21) the optimality conditions with Clarke's generalized gradients (see, for example, [8]). This is a consequence of the fact that Clarke generalized gradient doesn't exist for the objective function in (4.21) because it is not locally Lipschitz.

5. Conclusion

In the paper, we have introduced the concept of an E-subdifferential which is the set of E-subgradients and is based on the effect of an operator $E\colon R^n \to R^n$ on the the domain of a function. Further, we have analyzed some of its properties. Then we have shown that the Clarke generalized gradient is a special case of the introduced E-subdifferential. However, the E-subdifferential can be nonempty even for some not locally Lipschitz functions for which Clarke generalized gradient is not defined. Admittedly, the E-subdifferential is a convex closed set, but, in opposition to the Clarke generalized gradient, it can be a noncompact set for some nondifferentiable functions and some operators E. However, we have presented in the paper the condition under which the E-subdifferential is compact. In order to show the existence of the E-subdifferential, we have introduced the class of E-subdifferentiable E-convex functions which are called, for short, E-subconvex. This class of nondifferentiable functions is a generalization and extension of the class of differentiable E-convex functions to the case when E-functions are not necessarily differentiable. Then we have presented the relationship between the gradient of a differentiable E-convex function and an E-subgradient which is an element of the E-subdifferential. It is an interesting case that there are cases of differentiable functions for which their gradients can not be elements of their E-subdifferentials. Also it has been proved that the necessary condition for a point to be a global E-minimizer of E-subdifferentiable function is that 0 is an element of its E-subdifferential at this point. However, it turned out that this result is true only in the case of global E-optimal solutions. In the case of local E-optimal solutions of some E-subdifferentiable functions, it may be not true—this interesting result has been illustrated in the paper. Further, we use the introduced E-subdifferential in formulating the necessary optimality conditions of Fritz John type and Karush–Kuhn–Tucker type which have been proved for the nonsmooth constrained extremum problem considered in the paper. For proving the necessary optimality conditions Karush–Kuhn–Tucker type, we have introduced a new constraint qualification which is based on E-subdifferentials of active constraints. Finally, the sufficiency of the aforesaid necessary optimality conditions have been established for such nonsmooth optimization problem under assumption that the involved functions are E-subconvex. This result has been illustrated by an E-subdifferentiable optimization problem with E-subconvex functions in which not all functions are locally Lipschitz.

However, some interesting topics for further research remain. It would be of interest to investigate whether it is possible to prove similar results for other classes of E-subdifferentiable optimization problems. We shall investigate these questions in subsequent papers.

References

[1] T. Antczak and N. Abdulaleem, *E-optimality conditions and Wolfe E-duality for E-differentiable vector optimization problems with inequality and equality constraints*, J. Nonlinear Sci. Appl. **12** (2019), no. 11, 745–764.

[2] A. Bagirov, N. Karmitsa and M. M. Mäkelä, *Introduction to Nonsmooth Optimization*, Springer, Cham, 2014.

[3] J. Benoist, *The size of the Dini subdifferential*, Proc. Amer. Math. Soc. **129** (2001), no. 2, 525–530.

[4] Z. Boros and Z. Páles, *$\mathbb{Q}$-subdifferential of Jensen-convex functions*, J. Math. Anal. Appl. **321** (2006), no. 1, 99–113.

[5] J. M. Borwein and Q. J. Zhu, *A survey of subdifferential calculus with applications*, Nonlinear Anal. **38** (1999), no. 6, 687–773.

[6] J. V. Burke and L. Q. Qi, *Weak directional closedness and generalized subdifferentials*, J. Math. Anal. Appl. **159** (1991), no. 2, 485–499.

[7] F. H. Clarke, *Generalized gradients and applications*, Trans. Amer. Math. Soc. **205** (1975), 247–262.

[8] ______, *Optimization and Nonsmooth Analysis*, Canad. Math. Soc. Ser. Monogr. Adv. Texts Wiley-Intersci. Publi., John Wiley & Sons, New York, 1983.

[9] V. F. Demyanov and V. Jeyakumar, *Hunting for a smaller convex subdifferential*, J. Global Optim. **10** (1997), no. 3, 305–326.

[10] J.-B. Hiriart-Urruty and C. Lemaréchal, *Convex Analysis and Minimization Algorithms I*, Grundlehren Math. Wiss. **305**, Springer-Verlag, Berlin, 1993.

[11] H. Huang and C. Sun, *Sigma-subdifferential and its application to minimization problem*, Positivity **24** (2020), no. 3, 539–551.

[12] A. Ioffe, *A Lagrange multiplier rule with small convex-valued subdifferentials for non-smooth problems of mathematical programming involving equality and nonfunctional constraints*, Math. Programming **58**, Ser. A (1993), no. 1, 137–145.

[13] R. Kasimbeyli and G. Inceoğlu, *The properties of the weak subdifferentials*, Gazi Univ. J. Sci. **23** (2010), no. 1, 49–52.

[14] R. Kasimbeyli and M. Mammadov, *Optimality conditions in nonconvex optimization via weak subdifferentials*, Nonlinear Anal. **74** (2011), no. 7, 2534–2547.

[15] P. Q. Khanh, H. T. Quyen and J.-C. Yao, *Optimality conditions under relaxed quasiconvexity assumptions using star and adjusted subdifferentials*, European J. Oper. Res. **212** (2011), no. 2, 235–241.

[16] A. Y. Kruger, *Properties of generalized differentials*, Sibirsk. Mat. Zh. **26** (1985), no. 6, 54–66, 189.

[17] ______, *On Fréchet subdifferentials*, J. Math. Sci. (N.Y.) **116** (2003), no. 3, 3325–3358.

[18] A. G. Kusraev and S. S. Kutateladze, *Subdifferentials: Theory and applications*, Math. Appl. **323**, Kluwer Academic Publishers Group, Dordrecht, 1995.

[19] M. Lassonde, *Links between subderivatives and subdifferentials*, J. Math. Anal. Appl. **457** (2018), no. 2, 1478–1491.

[20] O. L. Mangasarian, *Nonlinear Programming*, McGraw-Hill, New York, 1969.

[21] P. Michel and J.-P. Penot, *Calcus sous-differential pour les fonctions lipschitzienness etnon-lipschitziennes*, C. R. Acad. Sc. Paris Ser. I **298** (1984), 269–272.

[22] B. S. Mordukhovich, *Nonsmooth analysis with nonconvex generalized differentials and conjugate mappings*, Dokl. Akad. Nauk BSSR **28** (1984), no. 11, 976–979.

[23] J.-P. Penot, *The directional subdifferential of the difference of two convex functions*, J. Global Optim. **49** (2011), no. 3, 505–519.

[24] H. X. Phú, *γ-subdifferential and γ-convexity of functions on the real line*, Appl. Math. Optim. **27** (1993), no. 2, 145–160.

[25] R. T. Rockafellar, *Convex Analysis*, Princeton Math. Ser. **28**, Princeton University Press, Princeton, NJ, 1970.

[26] ______, *Directionally Lipschitzian functions and subdifferential calculus*, Proc. London Math. Soc. (3) **39** (1979), no. 2, 331–355.

[27] ______, *Proximal subgradients, marginal values, and augmented Lagrangians in nonconvex optimization*, Math. Oper. Res. **6** (1981), no. 3, 424–436.

[28] ______, *The Theory of Subgradients and its Applications to Problems of Optimization: Convex and nonconvex functions*, Heldermann Verlag, Berlin, 1981.

[29] ______, *Generalized subgradients in mathematical programming*, in: *Mathematical Programming: The state of the art (Bonn, 1982)*, 368–390, Springer-Verlag, Berlin, 1983.

[30] W. Schirotzek, *Nonsmooth Analysis*, Universitext, Springer, Berlin, 2007.

[31] M. Soleimani-damaneh, *Nonsmooth optimization using Mordukhovich's subdifferential*, SIAM J. Control Optim. **48** (2010), no. 5, 3403–3432.

[32] ______, *E-convexity and its generalizations*, Int. J. Comput. Math. **88** (2011), no. 16, 3335–3349.

[33] M. Soleimani-damaneh and G. R. Jahanshahloo, *Nonsmooth multiobjective optimization using limiting subdifferentials*, J. Math. Anal. Appl. **328** (2007), no. 1, 281–286.

[34] T. Q. Son and C. F. Wen, *Weak-subdifferentials for vector functions and applications to multiobjective semi-infinite optimization problems*, Appl. Anal. **99** (2020), no. 5, 840–855.

[35] S. Suzuki and D. Kuroiwa, *Characterizations of the solution set for quasiconvex programming in terms of Greenberg–Pierskalla subdifferential*, J. Global Optim. **62** (2015), no. 3, 431–441.

[36] X.-Q. Yang and B. D. Craven, *Necessary optimality conditions with a modified subdifferential*, Optimization **22** (1991), no. 3, 387–400.

[37] E. A. Youness, *E-convex sets, E-convex functions, and E-convex programming*, J. Optim. Theory Appl. **102** (1999), no. 2, 439–450.

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