

## Ground States of Nonlocal Fractional Schrödinger Equations with Potentials Well

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Abstract. Inspired by Alves and Souto [2], we investigate the existence of nontrivial solutions to a class of fractional Schrödinger equations with potentials well. Taking the superlinear nonlinearities into consideration, we obtain the existence of nontrivial solutions by losing monotonicity. Furthermore, a ground state solution is established.

### 1. Introduction

In this paper, we are concerned with the ground state solution of the following fractional Schrödinger equation

$$(1.1) \quad (-\Delta)^s u + V(x)u = K(x)g(u), \quad x \in \mathbb{R}^N,$$

where  $N \geq 2s$  with  $s \in (0, 1)$ ,  $V(x)$  is sign-changing,  $K(x) > 0$  and  $g(t)$  has super-linear growth with a loosen condition than the monotonicity of  $g(t)/t$ .

The fractional Schrödinger equation is a fundamental equation of fractional quantum mechanics, which was discovered by Laskin [12, 13]. As stated in [9], the fractional Schrödinger equation is important, as a result of extending the Feynman path integral, from the Brownian-like to Lévy-like quantum mechanical paths, where the Feynman path integral leads to the classical Schrödinger equation and the path integral over Lévy trajectories leads to the fractional Schrödinger equation. The fractional Schrödinger equation

$$(-\Delta)^s u + V(x)u = f(x, u), \quad x \in \mathbb{R}^N, \quad N \geq 2s,$$

arises in many fields of physics. For example, it is essentially related to looking for the standing waves  $\Psi(t, x) = e^{-i\omega t}u(x)$  for the time-dependent fractional nonlinear Schrödinger equation

$$i\hbar\partial_t\Psi + (-\Delta)^s\Psi + V(x)\Psi = f(x, \Psi), \quad x \in \mathbb{R}^N, \quad t \in \mathbb{R}.$$

This was proved in [7] that (1.1) can be reduced to the classical Schrödinger equation, since  $(-\Delta)^s$  can be changed to the standard Laplace  $-\Delta$  as  $s \rightarrow 1$ .

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Recently, the interest towards nonlinear equations has grown more and more, where several types of  $V(x)$  and  $f(x, t)$  are considered, see [1–5, 9, 14, 19, 21, 22, 25] and the references therein. For example, when  $V(x)$  is a nonnegative continuous function, Alves and Souto [2] proved the existence of ground state solutions for a class of elliptic equations under monotonicity and some other mild assumptions. For the nonnegative and vanishing potentials at infinity, Ambrosio and Isernia [4] have been established the existence of sign-changing solutions for a class of fractional Schrödinger equations. When  $V(x)$  is sign-changing with  $K(x) = 1$ , Li, Zhao and Wang [14] have gave a result about the existence of a positive ground state solution for (1.1) under certain assumptions. In that work,  $(f_2)$  is the key to obtain the boundedness of (PS) sequence, where

$$Q(t) := \frac{1}{2}f(t)t - F(t) \text{ satisfy } 0 < Q(t_1) \leq Q(t_2) \ (\forall 0 < t_1 \leq t_2) \text{ and } \lim_{t \rightarrow +\infty} Q(t) = +\infty.$$

For (1.1) with sign-changing and vanishing potentials  $V(x)$ , Wang and Zhou [22] and Xue et al. [25] deal with the existence of ground states based on monotonicity and some other proper conditions, respectively.

Known to all that the main difficulty in dealing with nonlinear problems on unbounded domains arises from the lack of the compactness of Sobolev embeddings. To obtain the compactness in  $\mathbb{R}^N$ , a class of Hardy-type inequalities were introduced in [2] under appropriate conditions. Then, the results were extended to the nonlocal fractional setting in [4] under the following conditions  $(V, K)$ :

$$(V_0) \ V \in C(\mathbb{R}^N) \text{ and } V(x) > 0;$$

$$(K_0) \ K \in C(\mathbb{R}^N), \ K(x) > 0 \text{ and } K \in L^\infty(\mathbb{R}^N);$$

$$(K_1) \ \text{if } \{A_n\} \subset \mathbb{R}^N \text{ is a sequence of Borel sets such that } |A_n| \leq R \text{ for all } n \text{ and some } R > 0, \text{ then } \lim_{r \rightarrow \infty} \int_{A_n \cap B_r^c(0)} K(x) \, dx = 0 \text{ uniformly in } n \in \mathbb{N};$$

(KV) one of the below conditions occurs:

$$\frac{K}{V} \in L^\infty(\mathbb{R}^N)$$

or there is  $\tau \in (2, 2_s^*)$  such that

$$\frac{K(x)}{[V(x)]^{(2_s^* - \tau)/(2_s^* - 2)}} \rightarrow 0 \text{ as } |x| \rightarrow +\infty,$$

where  $2_s^* = 2N/(N - 2s)$  is the fractional critical exponent.

This set of conditions  $(V, K)$  generalizes the condition states in [3]. Recently, using a kind of Sobolev embeddings introduced by Han [10], Toon and Ubilla [21] proved the existence

of at least one positive solution for a Schrödinger equation with a positive and vanishing potential at infinity and subcritical nonlinearity  $f$ . Their conditions allow to consider the examples of nonlinearities which do not verify the Ambrosetti–Rabinowitz condition, neither monotonicity conditions for the function  $f(x, t)/t$ .

Motivated by the works described above, for a sign-changing potential, the main aim of this paper is to study the ground state solution for (1.1) by loosing the monotonicity of  $g(t)/t$ . Firstly, for  $s \in (0, 1)$ , we define  $D^{s,2}(\mathbb{R}^N)$  as the completion of the set of  $C_0^\infty(\mathbb{R}^N)$  with respect to the so called Gagliardo semi-norm  $[u]_s = \left(\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dx dy\right)^{1/2}$ . Set  $V^+(x) = \max\{V(x), 0\}$  and  $V^-(x) = \max\{-V(x), 0\}$ . As this moment, we add the following conditions:

(V<sub>1</sub>)  $V = V^+ - V^-$ , where  $V^\pm \in L^{N/(2s)}(\mathbb{R}^N)$ .  $\Omega := \{x \in \mathbb{R}^N \mid V(x) < 0\} \neq \emptyset$  with  $\text{meas } \Omega > 0$  and there exists a large constant  $R_0$  such that  $V(x) > 0$  for a.e.  $|x| \geq R_0$ ;

(V<sub>2</sub>) there exists a constant  $\eta_0 > 1$  such that

$$\eta_1 := \inf_{u \in D^{s,2}(\mathbb{R}^N) \setminus \{0\}} \frac{[u]_s^2 + \int_{\mathbb{R}^N} V^+(x)u(x)^2 dx}{\int_{\mathbb{R}^N} V^-(x)u(x)^2 dx} \geq \eta_0;$$

(KV)  $K/V \rightarrow 0$  as  $|x| \rightarrow \infty$ ;

(g<sub>0</sub>)  $g \in C(\mathbb{R})$ ,  $g(t) = 0$  if  $t \leq 0$  and  $g(t) \geq 0$  if  $t > 0$ ;

(g<sub>1</sub>) there is  $q \in (1, 2_s^* - 1)$  such that  $g(t) \leq C(1 + |t|^q)$ ;

(g<sub>2</sub>)  $\lim_{t \rightarrow 0} \frac{g(t)}{t} = 0$ ;

(g<sub>3</sub>)  $\lim_{t \rightarrow \infty} \frac{G(t)}{t^2} \rightarrow \infty$ , where  $G(t) := \int_0^t g(\tau) d\tau$ ;

(g<sub>4</sub>) there exists a constant  $\theta_0 \in (0, 1)$  such that

$$\frac{1 - \theta^2}{2} g(t)t \geq \int_{\theta t}^t g(\tau) d\tau = G(t) - G(\theta t), \quad \forall \theta \in [0, \theta_0].$$

*Remark 1.1.* (I) The similar conditions of (V<sub>2</sub>) can be found in [6] and [15]. By (V<sub>2</sub>) and a simple calculation,

$$[u]_s^2 + \int_{\mathbb{R}^N} V^+|u|^2 dx \geq [u]_s^2 + \int_{\mathbb{R}^N} V|u|^2 dx \geq \frac{\eta_0 - 1}{\eta_0} \left( [u]_s^2 + \int_{\mathbb{R}^N} V^+|u|^2 dx \right).$$

For  $u \in D^{s,2}(\mathbb{R}^N)$ , set  $S := \inf_{\|u\|_{L^{2_s^*}}=1} [u]_s^2$ . By the conditions (V<sub>1</sub>), (V<sub>2</sub>) and the Hölder and Sobolev inequalities, we derive that

$$\begin{aligned} \frac{[u]_s^2 + \int_{\mathbb{R}^N} V^+(x)|u|^2 dx}{\int_{\mathbb{R}^N} V^-(x)|u|^2 dx} &\geq \frac{[u]_s^2}{\int_{\mathbb{R}^N} V^-(x)|u|^2 dx} \geq \frac{[u]_s^2}{\|V^-\|_{L^{N/(2s)}} \|u\|_{L^{2_s^*}}^2} \\ &\geq \frac{[u]_s^2}{S^{-1} \|V^-\|_{L^{N/(2s)}} [u]_s^2} = \frac{S}{\|V^-\|_{L^{N/(2s)}}}, \end{aligned}$$

which implies that if  $\|V^-\|_{L^{N/(2s)}} < S$ , then  $\eta_1 \geq \frac{S}{\|V^-\|_{L^{N/(2s)}}} > 1$ .

(II) Condition  $(g_4)$  is introduced by Tang in [20]. If the map  $t \mapsto g(t)/t$  is increasing in  $(0, +\infty)$  and  $g(t) = 0$  if  $t \leq 0$ , for any  $t \in \mathbb{R}$ , it follows that  $\frac{1}{2}g(t)t - G(t) \geq 0$  and  $\frac{\theta_1^2 - 1}{2}g(t)t + G(t) - G(\theta_1 t) \leq 0$  for  $\theta_1 \in [0, 1]$ , which yields  $(g_4)$ . In addition, if the version of (AR):

$$\exists \mu > 2, \quad 0 \leq \mu G(t) \leq g(t)t \quad \text{for } t \in \mathbb{R}$$

is satisfied, then set  $\theta_0 = \left(\frac{\mu-2}{\mu}\right)^{1/2}$ , hence we have

$$\frac{1 - \theta^2}{2}g(t)t \geq \frac{1}{\mu}g(t)t \geq G(t) \geq G(t) - G(\theta t) \quad \text{for } \theta \in [0, \theta_0].$$

This shows that  $(g_4)$  holds also.

Now, we are able to state our main theorems.

**Theorem 1.2.** *Suppose that conditions  $(K_0)$ ,  $(V_1)$ ,  $(KV)$  and  $(g_0)$ – $(g_4)$  hold. Then (1.1) has at least one nontrivial solution.*

**Theorem 1.3.** *Suppose that conditions  $(K_0)$ ,  $(V_1)$ ,  $(V_2)$ ,  $(KV)$  and  $(g_0)$ – $(g_4)$  hold. Then (1.1) has at least one ground state solution.*

This paper is organized as follows. In Section 2, we recall the variational setting and state some preliminary results. In Section 3, the proofs of our main results are given.

## 2. Preliminaries

In this paper, our work space is  $E = \{u \in D^{s,2}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V^+(x)u^2 dx < +\infty\}$ .  $E$  is a separable Hilbert space with the inner product

$$\langle u, v \rangle = \int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} dx dy + \int_{\mathbb{R}^N} V^+(x)u(x)v(x) dx$$

and the corresponding norm is  $\|u\|^2 = [u]_s^2 + \int_{\mathbb{R}^N} V^+(x)|u|^2 dx$ . More details about the space  $E$  can be found in [16, 17]. Let us define the weighted Lebesgue space

$$L_K^r(\mathbb{R}^N) = \left\{ u: \mathbb{R}^N \rightarrow \mathbb{R} : u \text{ is measurable and } \int_{\mathbb{R}^N} K(x)|u|^r dx < \infty \right\}$$

endowed with the norm  $\|\cdot\|_{L_K^r} = \left(\int_{\mathbb{R}^N} K(x)|u|^r dx\right)^{1/r}$ . We recall  $\|u\|_\infty = \text{ess sup}_{x \in \mathbb{R}^N} |u|$ .

It is known that problem (1.1) is variational and its solutions are the critical points of the functional defined in  $E$  by

$$\mathcal{J}(u) = \frac{1}{2}\|u\|^2 - \frac{1}{2} \int_{\mathbb{R}^N} V^-(x)|u|^2 dx - \int_{\mathbb{R}^N} K(x)G(u) dx,$$

which is well defined and of class  $C^1(E, \mathbb{R})$ . Moreover,

$$\begin{aligned} \langle \mathcal{J}'(u), v \rangle &= \int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} dx dy + \int_{\mathbb{R}^N} V^+(x)uv dx \\ &\quad - \int_{\mathbb{R}^N} V^-(x)uv dx - \int_{\mathbb{R}^N} K(x)g(u)v dx. \end{aligned}$$

**Definition 2.1** (Cerami condition). Let  $E$  be a Banach space,  $\mathcal{J} \in C^1(E, \mathbb{R})$  and  $c \in \mathbb{R}$ .  $\mathcal{J}$  satisfies the Cerami condition if any  $(C)_c$ -sequence  $\{u_n\}$  such that

$$\mathcal{J}(u_n) \rightarrow c \quad \text{and} \quad (1 + \|u_n\|)\|\mathcal{J}'(u_n)\| \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

has a strongly convergent subsequence.

**Lemma 2.2.** Assume that the conditions  $(K_0)$ ,  $(V_1)$  and  $(KV)$  hold. Then  $E$  is compactly embedded in  $L_K^\kappa(\mathbb{R}^N)$ ,  $\kappa \in [2, 2^*)$ .

*Proof.* Let  $\{u_n\} \subset E$  be a sequence such that  $u_n \rightharpoonup u$  in  $E$ . By  $(KV)$ ,  $\forall \varepsilon > 0$ , there is  $R_\varepsilon > R_0$  such that  $K(x)/V(x) < \varepsilon$  for all  $|x| \geq R_\varepsilon$ . For  $\kappa = 2$ , that  $\varepsilon$ , constants  $T_0, T$  and a.e.  $|x| \geq R_\varepsilon$ , we have

$$\begin{aligned} K(x)|t|^2 &\leq K(x)|t|^2 + \varepsilon C|t|^{2^*} + C_\varepsilon K(x)\chi_{(T_0, T)}|t|^{2^*} \\ &\leq \varepsilon V^+(x)|t|^2 + \varepsilon C|t|^{2^*} + C_\varepsilon K(x)\chi_{(T_0, T)}|t|^{2^*}, \quad \forall t \in \mathbb{R}. \end{aligned}$$

Hence,

$$\begin{aligned} (2.1) \quad \int_{B_{R_\varepsilon^c}^c(0)} K(x)|u|^2 dx &\leq C\varepsilon \left( \int_{B_{R_\varepsilon^c}^c(0)} V^+|u|^2 dx + \int_{B_{R_\varepsilon^c}^c(0)} |u|^{2^*} dx \right) \\ &\quad + C_\varepsilon T^{2^*} \int_{A \cap B_{R_\varepsilon^c}^c(0)} K(x) dx, \quad \forall u \in E, \end{aligned}$$

where  $A = \{x \in \mathbb{R}^N : T_0 \leq |u(x)| \leq T\}$ . Set  $A_n = \{x \in \mathbb{R}^N : T_0 \leq |u_n(x)| \leq T\}$ . Recalling that  $u_n \rightharpoonup u$  in  $E$ , we derive that  $T_0^{2^*}|A_n| \leq \int_{A_n} |u_n|^{2^*} dx \leq \int_{\mathbb{R}^N} |u_n|^{2^*} dx \leq C$ . Thus,  $\sup_{n \in \mathbb{N}} |A_n| < +\infty$ . Thereby, from  $(V_1)$  and  $(KV)$ , there is a large positive constant  $\widehat{R}_\varepsilon (> R_\varepsilon)$  such that

$$(2.2) \quad \int_{A_n \cap B_{\widehat{R}_\varepsilon^c}^c(0)} K(x) dx < \frac{\varepsilon}{C_\varepsilon T^{2^*}} \quad \text{for all } n \in \mathbb{N}.$$

Combining (2.1), (2.2) with  $u_n \rightharpoonup u$  in  $E$ , we have  $\int_{B_{\widehat{R}_\varepsilon^c}^c(0)} K(x)|u_n|^2 dx < C\varepsilon, \forall n \in \mathbb{N}$ . Furthermore, by Sobolev embeddings, it follows that  $\lim_{n \rightarrow \infty} \int_{B_{\widehat{R}_\varepsilon^c}^c(0)} K(x)|u_n|^2 dx = \int_{B_{\widehat{R}_\varepsilon^c}^c(0)} K(x)|u|^2 dx$ . Therefore,  $\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} K(x)|u_n|^2 dx = \int_{\mathbb{R}^N} K(x)|u|^2 dx$ , which yields  $u_n \rightarrow u$  in  $L_K^2(\mathbb{R}^N)$ .

For the case of  $\kappa \in (2, 2_s^*)$ ,  $\forall \varepsilon > 0$ , constants  $T'_0$ ,  $T'$  and  $|x| \geq R_\varepsilon > R_0$ , we have

$$K(x)|t|^\kappa \leq \varepsilon CK(x)|t|^2 + \varepsilon C|t|^{2_s^*} + C_\varepsilon K(x)\chi_{(T'_0, T')}(t)|t|^{2_s^*}, \quad \forall t \in \mathbb{R}.$$

Then, repeating the progress above, we complete the proof. □

**Lemma 2.3.** [23, Lemma 2.13] *Let  $V(x) \in L^{N/(2s)}(B_R)$ , where  $N \geq 2s$  and  $R > 0$  is a constant. Suppose that  $u_n \rightharpoonup u$  in  $E$ , then  $\int_{B_R} V(x)|u_n|^2 dx \rightarrow \int_{B_R} V(x)|u|^2 dx$ .*

**Lemma 2.4.** *Assume that the conditions  $(K_0)$ ,  $(V_1)$  and  $(g_0)$ – $(g_4)$  hold. Then the minimizing sequence  $\{u_n\}$  of  $\mathcal{J}$  is bounded.*

*Proof.* Let  $\{u_n\} \subset E$  be the sequence such that

$$(2.3) \quad \mathcal{J}(u_n) \rightarrow c \quad \text{and} \quad (1 + \|u_n\|)\mathcal{J}'(u_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

By contradiction, suppose that  $\{u_n\} \subset E$  is unbounded. We may assume that  $\|u_n\| \rightarrow \infty$  as  $n \rightarrow \infty$ . Set  $v_n = u_n/\|u_n\|$  and  $s_n = \varpi/\|u_n\|$ , where  $\varpi = (2c + 4)^{1/2} + 1$ . By a simple computation, it is not difficult to have

$$(2.4) \quad \begin{aligned} \mathcal{J}(\varpi v_n) - \mathcal{J}(u_n) &= \frac{\varpi^2}{2} - \frac{\varpi^2}{2} \int_{\mathbb{R}^N} V^-(x) \frac{u_n^2}{\|u_n\|^2} dx - \int_{\mathbb{R}^N} K(x)G(\varpi v_n) dx \\ &\quad - \frac{1}{2}\|u_n\|^2 + \frac{1}{2} \int_{\mathbb{R}^N} V^-(x)u_n^2 dx + \int_{\mathbb{R}^N} K(x)G(u_n) dx \\ &= \frac{1}{2} \left( \frac{\varpi^2}{\|u_n\|^2} - 1 \right) \|u_n\|^2 - \frac{1}{2} \left( \frac{\varpi^2}{\|u_n\|^2} - 1 \right) \int_{\mathbb{R}^N} V^-(x)u_n^2 dx \\ &\quad - \int_{\mathbb{R}^N} K(x)G(\varpi v_n) dx + \int_{\mathbb{R}^N} K(x)G(u_n) dx. \end{aligned}$$

Also,

$$\begin{aligned} \mathcal{J}'(u_n)(\varpi v_n - u_n) &= \varpi\|u_n\| - \frac{\varpi}{\|u_n\|} \int_{\mathbb{R}^N} V^-u_n^2 dx - \int_{\mathbb{R}^N} K(x)g(u_n)\varpi \frac{u_n}{\|u_n\|} dx \\ &\quad - \|u_n\|^2 + \int_{\mathbb{R}^N} V^-(x)u_n^2 dx + \int_{\mathbb{R}^N} K(x)g(u_n)u_n dx \\ &= \left( \frac{\varpi}{\|u_n\|} - 1 \right) \|u_n\|^2 - \left( \frac{\varpi}{\|u_n\|} - 1 \right) \int_{\mathbb{R}^N} V^-(x)u_n^2 dx \\ &\quad - \left( \frac{\varpi}{\|u_n\|} - 1 \right) \int_{\mathbb{R}^N} K(x)g(u_n)u_n dx, \end{aligned}$$

which yields that

$$(2.5) \quad \begin{aligned} \left( \frac{\varpi}{\|u_n\|} + 1 \right) \mathcal{J}'(u_n)(\varpi v_n - u_n) &= \left( \frac{\varpi^2}{\|u_n\|^2} - 1 \right) \|u_n\|^2 - \left( \frac{\varpi^2}{\|u_n\|^2} - 1 \right) \int_{\mathbb{R}^N} V^-u_n^2 \\ &\quad - \left( \frac{\varpi^2}{\|u_n\|^2} - 1 \right) \int_{\mathbb{R}^N} K(x)g(u_n)u_n dx. \end{aligned}$$

By (2.3), it is easy to know that  $\frac{1}{2}(\frac{\varpi}{\|u_n\|} + 1)\mathcal{J}'(u_n)(\varpi v_n - u_n) \rightarrow 0, n \rightarrow \infty$ . Therefore, by (2.5), for large  $n$ ,

$$\begin{aligned} & \frac{1}{2} \left( \frac{\varpi^2}{\|u_n\|^2} - 1 \right) \|u_n\|^2 - \frac{1}{2} \left( \frac{\varpi^2}{\|u_n\|^2} - 1 \right) \int_{\mathbb{R}^N} V^-(x)u_n^2 dx \\ & - \frac{1}{2} \left( \frac{\varpi^2}{\|u_n\|^2} - 1 \right) \int_{\mathbb{R}^N} K(x)g(u_n)u_n dx \leq 1, \end{aligned}$$

that is,

$$\begin{aligned} (2.6) \quad & \frac{1}{2} \left( \frac{\varpi^2}{\|u_n\|^2} - 1 \right) \|u_n\|^2 - \frac{1}{2} \left( \frac{\varpi^2}{\|u_n\|^2} - 1 \right) \int_{\mathbb{R}^N} V^-(x)u_n^2 dx \\ & \leq 1 + \frac{1}{2} \left( \frac{\varpi^2}{\|u_n\|^2} - 1 \right) \int_{\mathbb{R}^N} K(x)g(u_n)u_n dx. \end{aligned}$$

Combining (2.4), (2.6) and (g<sub>4</sub>), for large  $n$ , we derive that

$$\begin{aligned} (2.7) \quad & \mathcal{J}(\varpi v_n) - \mathcal{J}(u_n) \\ & \leq 1 + \frac{1}{2} \left( \frac{\varpi^2}{\|u_n\|^2} - 1 \right) \int_{\mathbb{R}^N} Kg(u_n)u_n dx - \int_{\mathbb{R}^N} KG(\varpi v_n) dx + \int_{\mathbb{R}^N} KG(u_n) dx \\ & = 1 + \frac{1}{2}(s_n^2 - 1) \int_{\mathbb{R}^N} Kg(u_n)u_n dx - \int_{\mathbb{R}^N} KG(s_n u_n) dx + \int_{\mathbb{R}^N} KG(u_n) dx \\ & \leq 1. \end{aligned}$$

Now we claim  $v_n \rightharpoonup v$  in  $E$  and  $v \neq 0$  for a.e.  $x \in \mathbb{R}^N$ . If not, by (2.7) and Lemma 2.3, for  $n$  large enough, we obtain that

$$\begin{aligned} c + 2 & \geq \mathcal{J}(u_n) + 1 \geq \mathcal{J}(\varpi v_n) \\ & = \frac{\varpi^2}{2} - \frac{\varpi^2}{2} \int_{\mathbb{R}^N} V^-(x)v_n^2 dx - \int_{\mathbb{R}^N} K(x)G(\varpi v_n) dx \\ & = \frac{\varpi^2}{2} + o(1), \end{aligned}$$

based on

$$\int_{\mathbb{R}^N} V^-(x)v_n^2 dx \rightarrow 0 \quad (\text{by } v_n \rightharpoonup 0 \text{ in } E, (V_1) \text{ and Lemma 2.3})$$

and  $|\int_{\mathbb{R}^N} K(x)G(\varpi v_n) dx| \leq \varepsilon \|\varpi v_n\|_{L_K^2}^2 + c_\varepsilon \|\varpi v_n\|_{L_K^{q+1}}^{q+1}$ . This is a contradiction to  $\varpi = (2c+4)^{1/2} + 1$ , hence we get the claim. By  $u_n = v_n \|u_n\|$ , it holds that  $|u_n| \rightarrow \infty$  as  $n \rightarrow \infty$ . By  $K(x) > 0$  and (g<sub>3</sub>), we obtain that  $\frac{K(x)G(u_n)}{|u_n|^2} |v_n|^2 \rightarrow \infty$  as  $n \rightarrow \infty$ . Consequently, using Fatou lemma, we get

$$\begin{aligned} c + o(1) & = \frac{1}{2} \|u_n\|^2 - \frac{1}{2} \int_{\mathbb{R}^N} V^-(x)u_n^2 dx - \int_{\mathbb{R}^N} K(x)G(u_n) dx \\ & \leq \frac{1}{2} \|u_n\|^2 - \int_{\mathbb{R}^N} K(x)G(u_n) dx \end{aligned}$$

and

$$\frac{1}{2} - \frac{c + o(1)}{\|u_n\|^2} \geq \int_{\mathbb{R}^N} \frac{K(x)G(u_n)}{\|u_n\|^2} dx \geq \int_{v \neq 0} \frac{K(x)G(u_n)}{u_n^2} v_n^2 dx \rightarrow \infty \quad \text{as } n \rightarrow \infty,$$

which is impossible. Therefore, our hypothesis is untenable. Hence, the sequence of  $\{u_n\}$  is bounded in  $E$ . This completes the proof. □

**Lemma 2.5.** *Assume that the conditions  $(K_0)$ ,  $(V_1)$ ,  $(KV)$  and  $(g_0)$ – $(g_4)$  hold. Then there exists a strong convergent subsequence of the minimizing sequence  $\{u_n\}$  in  $E$ .*

*Proof.* By Lemma 2.4, the minimizing sequence  $\{u_n\} \subset E$  of  $\mathcal{J}$  is bounded. Passing to a subsequence, one has  $u_n \rightharpoonup u$  in  $E$ ,  $u_n \rightarrow u$  in  $L^r_K(\mathbb{R}^N)$  ( $r \in [2, 2^*_s)$ ) and  $u_n(x) \rightarrow u(x)$  a.e.  $x \in \mathbb{R}^N$ . By a direct computation, we derive that

$$\begin{aligned} \|u_n - u\|^2 &= \langle \mathcal{J}'(u_n) - \mathcal{J}'(u), u_n - u \rangle - \int_{\mathbb{R}^N} V^-(x)|u_n - u|^2 dx \\ &\quad + \int_{\mathbb{R}^N} K(x)(g(u_n) - g(u))(u_n - u) dx. \end{aligned}$$

Combining Lemma 2.2 with Lemma 2.3, we can complete the proof. □

### 3. Proofs of Theorems 1.2 and 1.3

Now we focus on the eigenvalue problem

$$(3.1) \quad (-\Delta)^s u + V(x)u = \lambda K(x)u, \quad x \in \mathbb{R}^N.$$

In this paper, we study the case when the functional possesses the so-called linking geometric structure, so we assume that  $\lambda_1 (\leq 0)$  is the bottom of  $\sigma((-\Delta)^s + V)$ , and there exists  $k$  such that  $\lambda_k \leq 0$ . More details for  $\lambda_k$ , see the following lemma.

**Lemma 3.1.** *Let  $s \in (0, 1)$ ,  $N > 2s$ ,  $(K_0)$ ,  $(V_1)$  and  $(KV)$  hold. We claim that*

(a)  $\lambda_1$  is achieved, where

$$\lambda_1 := \inf_{\substack{u \in E \\ \|u\|_{L^2_K} = 1}} \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy + \int_{\mathbb{R}^N} V(x)u^2(x) dx.$$

Also,  $\lambda_1$  is the finite eigenvalue of (3.1) with a nonnegative eigenfunction  $e_1 \in E$ .

(b) *The spectrum of problem (3.1) has and only has eigenvalues which can be listed  $\lambda_1 < \lambda_2 \leq \dots \leq \lambda_k \leq 0 < \lambda_{k+1} \leq \dots$  with a finite multiplicity for each eigenvalue and the corresponding eigenfunctions  $\{e_k\}_{k \in \mathbb{N}}$  form a base of Hilbert spaces  $L^2(\mathbb{R}^N)$  and  $E$ .*



*Proof.* Denote  $\tilde{\mathcal{J}}(u) = \frac{1}{2} \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dx dy + \frac{1}{2} \int_{\mathbb{R}^N} V(x)u^2(x) dx$ , where  $u \in E$ . By Hölder and Sobolev inequalities, we derive that

$$\begin{aligned} \tilde{\mathcal{J}}(u) &= \frac{1}{2} \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dx dy + \frac{1}{2} \int_{\mathbb{R}^N} V^+(x)u^2(x) dx - \frac{1}{2} \int_{\mathbb{R}^N} V^-(x)u^2(x) dx \\ &\geq \frac{1}{2} \|u\|^2 - \varepsilon \left( \int_{\Omega} |u|^{2N/(N-2s)} dx \right)^{(N-2s)/N} - C(\varepsilon) \left( \int_{\Omega} |V^-(x)|^{N/(2s)} dx \right)^{2s/N} \\ &\geq \frac{1}{2} \|u\|^2 - C\varepsilon \|u\|^2 - C(\varepsilon) \|V^-\|_{L^{N/(2s)}} \end{aligned}$$

for every  $\varepsilon > 0$ . Thereby, fixing  $\varepsilon$  small, we obtain  $\tilde{\mathcal{J}}(u)$  is coercive on  $E$  and is bounded below. And so,  $\lambda_1$  is a finite number. Let  $\{u_n\} \subset E$  with  $\|u_n\|_{L_K^2} = 1$  being a minimizing sequence for  $\tilde{\mathcal{J}}$ , that is,  $\tilde{\mathcal{J}}(u_n) \rightarrow \lambda_1$ . Obviously,  $\{u_n\}$  is bounded in  $E$ . Up to a subsequence, suppose that  $u_n \rightharpoonup u$  in  $E$ . By the compact imbedding  $E \hookrightarrow L_K^2(\mathbb{R}^N)$ , we obtain  $u_n \rightarrow u$  in  $L_K^2(\mathbb{R}^N)$  as  $n \rightarrow \infty$ . Hence,  $\|u\|_{L_K^2} = 1$ . Furthermore, by (V<sub>1</sub>) and Lemma 2.3, it follows that  $\int_{\mathbb{R}^N} V^-(x)|u_n(x)|^2 dx \rightarrow \int_{\mathbb{R}^N} V^-(x)|u(x)|^2 dx$ . Hence, by the lower semi-continuity of the norm in  $E$ ,  $\tilde{\mathcal{J}}(u) \leq \liminf_{n \rightarrow +\infty} \tilde{\mathcal{J}}(u_n) = \lambda_1$ . Consequently,  $\tilde{\mathcal{J}}(u) = \lambda_1$ , and  $\lambda_1$  is achieved and is an eigenvalue of problem (3.1).

By (a) and based on the assumption that  $\lambda_1 \leq 0$ , we have

$$\begin{aligned} &\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dx dy + \int_{\mathbb{R}^N} V(x)u(x)^2 dx \\ &\geq \lambda_1 \int_{\mathbb{R}^N} K(x)u(x)^2 dx \geq \lambda_1 \|K\|_{\infty} \int_{\mathbb{R}^N} u(x)^2 dx. \end{aligned}$$

Since  $\lambda_1$  is a finite number, there is a constant  $V_0 = 2|\lambda_1| \|K\|_{\infty}$  such that

$$\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dx dy + \int_{\mathbb{R}^N} (V(x) + V_0)u(x)^2 dx > 0 \quad \text{for any } u \in E.$$

Let us consider the following equation

$$(-\Delta)^s u + (V(x) + V_0)u = \mu K(x)u \quad \text{in } \mathbb{R}^N.$$

And for any  $u, v \in E$ , set  $\langle u, v \rangle_{V_0} = \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dx dy + \int_{\mathbb{R}^N} (V(x) + V_0)u(x)v(x) dx$ . For any  $u \in E$  fixed, we have

$$\begin{aligned} \int_{\mathbb{R}^N} K(x)u(x)v(x) dx &\leq \left( \int_{\mathbb{R}^N} K(x)u(x)^2 dx \right)^{1/2} \left( \int_{\mathbb{R}^N} K(x)v(x)^2 dx \right)^{1/2} \\ &\leq C \left( \int_{\mathbb{R}^N} K(x)u(x)^2 dx \right)^{1/2} \|v\|. \end{aligned}$$

By Riesz representation theorem, we obtain that for any  $u \in L_K^2(\mathbb{R}^N)$ , there exists a unique  $w \in E$  such that

$$\int_{\mathbb{R}^N} K(x)u(x)v(x) dx = \langle w, v \rangle_{V_0}, \quad \forall v \in E.$$

For  $u \in E$ , define  $P_{V_0} : L^2_K(\mathbb{R}^N) \rightarrow E$  by  $w = P_{V_0}u$ .  $P_{V_0}$  is a bounded linear operator. Set  $Q : E \rightarrow L^2_K(\mathbb{R}^N)$ , which is the natural embedding operator. By Lemma 2.2, we know that  $Q$  is a compact operator. Hence, for any  $u, v \in E$ , we obtain

$$\langle P_{V_0} \cdot Q(u), v \rangle_{V_0} = \int_{\mathbb{R}^N} K(x)u(x)v(x) dx = \frac{1}{\mu} \langle u, v \rangle_{V_0}.$$

We know that  $P_{V_0} \cdot Q$  is a compact operator from  $E$  to  $E$  and  $\langle P_{V_0} \cdot Q(u), u \rangle_{V_0} > 0$  for  $u \neq 0$ . By Hilbert–Schmidt theorem, it follows that the sequence of all eigenvalues  $\{\mu_j\}_{j=1}^{+\infty}$  of  $P_{V_0} \cdot Q$  satisfies  $0 < \dots \leq \mu_n \leq \dots \leq \mu_2 < \mu_1$  and  $\mu_j \rightarrow 0$  as  $j \rightarrow +\infty$ . Therefore,  $\lambda_j = \frac{1}{\mu_j} - V_0$ , ( $j = 1, 2, \dots$ ) is the sequence of all eigenvalues of (3.1). More details about the problem of eigenvalue, see Proposition 9 in [18]. □

Suppose that the eigenvalues of problem (3.1) are listed as  $\lambda_1 < \lambda_2 \leq \dots \leq \lambda_k \leq 0 < \lambda_{k+1} \leq \dots$ . Set  $Y = \{e_1, \dots, e_k\}$ , where  $\{e_k\}$  are the corresponding eigenfunctions and  $Z = \{u \in E, \langle u, v \rangle_{L^2_K} = 0, \forall v \in Y\}$ .

**Lemma 3.2.** *Let  $s \in (0, 1)$ ,  $N > 2s$ ,  $(K_0)$ ,  $(V_1)$  and  $(KV)$  hold, then*

$$\Theta = \inf_{u \in Z, \|u\|=1} \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy + \int_{\mathbb{R}^N} V(x)u(x)^2 dx > 0.$$

*Proof.* The proof is standard, so we omit it. For details, see Lemma 3.2 of [11, 24]. □

*Proof of Theorem 1.2.* Firstly, we will verify the following result:

- (i)  $\exists r > 0$ , s.t.  $m = \inf \mathcal{J}(S_r^+) > 0$ , where  $S_r^+ = \partial B_r \cap Z$ ;
- (ii)  $\exists \tilde{r} > 0$ , s.t.  $\mathcal{J}(u) \leq 0, \forall u \in \partial Q$ , where  $Q = \{u = u^- + se_{k+1} : u^- \in Y, s \geq 0, \|u\| \leq \tilde{r}\}$ .

It is not difficult to obtain (i). Now we follow some arguments in [8, p. 72] and give the proof of (ii). It is sufficient to show that  $\mathcal{J}(u) \rightarrow -\infty$  as  $u \in Q$  with  $\|u\| \rightarrow \infty$ . Arguing indirectly, assume that for some sequence  $u_n \in Q$  with  $\|u_n\| \rightarrow +\infty$ , there exists  $M > 0$  such that  $\mathcal{J}(u_n) \geq -M$  for all  $n$ . We define  $w_n = u_n/\|u_n\| = w_n^- + w_n^{k+1}$  with the property  $\|w_n\| = 1$ . Noticing that the sequence lies in a finite dimensional space, without loss of generality, we assume that  $w_n^- \rightarrow w^-$ ,  $w_n^{k+1} \rightarrow w^{k+1}$ . It follows that

$$(3.2) \quad -\frac{M}{\|u_n\|^2} \leq \frac{\mathcal{J}(u_n)}{\|u_n\|^2} = \frac{1}{2} \left( \|w_n^{k+1}\|^2 - \|w_n^-\|^2 - \int_{\mathbb{R}^N} V^- w_n^2(x) \right) - \int_{\mathbb{R}^N} \frac{G(x, u_n)}{\|u_n\|^2} dx.$$

Take a number  $L$  such that  $L > \lambda_{k+1}\|K\|_\infty$ . By (g<sub>3</sub>), there exists an  $r > 0$  such that  $G(x, u) \geq L|u|^2$  if  $|u| > r$ . We have

$$(3.3) \quad \begin{aligned} \|w^{k+1}\|^2 - L \int_{\mathbb{R}^N} |w|^2 dx &\leq \lambda_{k+1}\|w^{k+1}\|_{L^2_K}^2 - L\|w^-\|_{L^2}^2 - L\|w^{k+1}\|_{L^2}^2 \\ &\leq -(L - \lambda_{k+1}\|K\|_\infty)\|w^{k+1}\|_{L^2}^2 - L\|w^-\|_{L^2}^2 < 0. \end{aligned}$$

Hence there is a bounded domain  $\Omega^* \subset \mathbb{R}^N$  such that  $\|w^{k+1}\|^2 - L \int_{\Omega^*} |w|^2 dx < 0$ . By the nonnegativity of  $G(x, u)$ , it follows that

$$\begin{aligned}
 \frac{\mathcal{J}(u_n)}{\|u_n\|^2} &\leq \frac{1}{2} \|w_n^{k+1}\|^2 - \frac{1}{2} \|w_n^-\|^2 - \int_{\Omega^*} \frac{G(x, u_n)}{\|u_n\|^2} dx \\
 (3.4) \qquad &\leq \frac{1}{2} \left( \|w_n^{k+1}\|^2 - L \int_{\Omega^*} |w_n|^2 dx \right) - \int_{\Omega^*} \frac{G(x, u_n) - \frac{L}{2}|u_n|^2}{\|u_n\|^2} dx \\
 &\leq \frac{1}{2} \left( \|w_n^{k+1}\|^2 - L \int_{\Omega^*} |w_n|^2 dx \right) + \frac{Lr^2|\Omega^*|}{2\|u_n\|^2},
 \end{aligned}$$

based on

$$- \int_{\Omega^*} \frac{G(x, u_n) - \frac{L}{2}|u_n|^2}{\|u_n\|^2} dx \leq \begin{cases} -\frac{Lr^2|\Omega^*|}{2\|u_n\|^2} & \text{if } |u_n| \geq r, \\ \frac{Lr^2|\Omega^*|}{2\|u_n\|^2} & \text{if } |u_n| < r. \end{cases}$$

By (3.2), (3.3) and (3.4), it follows that

$$0 \leq \lim_{n \rightarrow \infty} \frac{\mathcal{J}(u_n)}{\|u_n\|^2} \leq \lim_{n \rightarrow \infty} \left( \frac{1}{2} \|w_n^{k+1}\|^2 - \frac{1}{2} \|w_n^-\|^2 - \int_{\Omega^*} \frac{G(x, u_n)}{\|u_n\|^2} dx \right) < 0,$$

which is a contradiction.

Furthermore, combining Lemmas 2.4 and 2.5 with (i)–(ii), we have obtained both the compactness properties and the geometrical structure of the functional. Hence, by Linking theorem, we complete the proof of Theorem 1.2.  $\square$

Next, we will give the proof of ground state solution for (1.1) step by step.

**Lemma 3.3.** *Assume that the conditions  $(K_0)$ ,  $(V_1)$ ,  $(V_2)$ ,  $(KV)$  and  $(g_0)$ – $(g_4)$  hold. Then the zero function 0 is an isolated critical point.*

*Proof.* Let  $u$  be a critical point other than 0. Combining Sobolev inequality with  $(g_0)$ – $(g_2)$ , we derive that

$$\begin{aligned}
 \frac{\eta_0 - 1}{2\eta_0} \|u\|^2 &\leq \frac{1}{2} \|u\|^2 - \int_{\mathbb{R}^N} V^-(x)|u|^2 dx = \int_{\mathbb{R}^N} K(x)g(u)u dx \\
 &\leq \varepsilon \|u\|_{L_K^2}^2 + C_\varepsilon \|u\|_{L_K^{q+1}}^{q+1} \leq C\varepsilon \|u\|^2 + CC_\varepsilon \|u\|^{q+1}.
 \end{aligned}$$

Fix  $\varepsilon$  sufficiently small. Since  $2 < q + 1 < 2_s^*$ , we obtain that  $\|u\| \geq \kappa > 0$ , where  $\kappa$  is some positive constant. This implies that 0 is an isolated critical point.  $\square$

Let  $\mathcal{K}$  be the set of critical point of  $\mathcal{J}$ . By Theorem 1.2,  $\mathcal{K}$  is nonempty. Denote  $m = \inf\{\mathcal{J}(u) \mid u \in \mathcal{K}\}$ . Then, for any  $u \in \mathcal{K}$ , by  $(g_4)$ , set  $\theta = 0$  and we obtain that

$$\mathcal{J}(u) = \mathcal{J}(u) - \frac{1}{2} \langle \mathcal{J}'(u), u \rangle = \int_{\mathbb{R}^N} K(x) \left( \frac{g(u)u}{2} - G(u) \right) dx \geq 0.$$

This implies that  $0 \leq m \leq \mathcal{J}(u)$ .

**Lemma 3.4.** *Assume the conditions  $(K_0)$ ,  $(V_1)$ ,  $(V_2)$ ,  $(KV)$  and  $(g_0)$ – $(g_4)$  hold. Moreover,  $\{u_n\} \subset E$  is a  $C_c$  sequence. If  $u_n \rightharpoonup u \in E$ , then  $u$  is a ground state solution of (1.1).*

*Proof.* Suppose that  $\{u_n\} \subset \mathcal{K}$  such that  $\mathcal{J}(u_n) \rightarrow m$ . We know that  $\{u_n\}$  is bounded in  $E$ . We denote that  $\delta := \lim_{n \rightarrow \infty} \sup_{x \in \mathbb{R}^N} \int_{B(x,1)} K(x)|u_n|^2 dx$ . We claim that  $\delta > 0$ . If not,  $\delta = 0$ , by Lions lemma,  $u_n \rightarrow 0$  in  $L_K^{q+1}(\mathbb{R}^N)$ ,  $(q + 1) \in (2, 2_s^*)$ . By this, we get

$$\int_{\mathbb{R}^N} K(x)g(u_n)u_n dx \leq \varepsilon \|u_n\|_{L_K^2}^2 + C_\varepsilon \|u_n\|_{L_K^{q+1}}^{q+1}$$

and it follows that  $\overline{\lim}_{n \rightarrow \infty} \left| \int_{\mathbb{R}^N} K(x)g(u_n)u_n dx \right| = 0$ . Since  $\{u_n\} \subset \mathcal{K}$  and by  $(V_2)$ ,

$$0 \leq \frac{\eta_0 - 1}{\eta_0} \|u_n\|^2 \leq \|u_n\|^2 - \int_{\mathbb{R}^N} V^-(x)|u_n|^2 dx = \int_{\mathbb{R}^N} K(x)g(u_n)u_n dx.$$

Hence,  $\int_{\mathbb{R}^N} K(x)g(u_n)u_n dx \rightarrow 0$  as  $n \rightarrow \infty$ . Consequently,

$$\frac{\eta_0 - 1}{\eta_0} \|u_n\|^2 \leq \int_{\mathbb{R}^N} K(x)g(u_n)u_n dx \rightarrow 0$$

as  $n \rightarrow \infty$ , which contradicts with the fact that 0 is an isolated critical point. Therefore,  $\delta > 0$ . By  $u_n \rightharpoonup u$  in  $E$  and Lemma 2.2, we obtain  $u_n \rightarrow u$  in  $L_K^2(\mathbb{R}^N)$  and  $u_n(x) \rightarrow u(x)$  a.e. in  $\mathbb{R}^N$ . Thereby, we derive that

$$0 < \delta = \lim_{n \rightarrow \infty} \sup_{x \in \mathbb{R}^N} \int_{B(x,1)} K(x)|u_n|^2 dx \leq \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} K(x)|u_n|^2 dx = \int_{\mathbb{R}^N} K(x)|u|^2 dx,$$

which yields that  $u_n \rightharpoonup u \neq 0$  in  $E$ . By  $u_n \rightharpoonup u \neq 0$  in  $E$ , it is not difficult to obtain  $u$  is a nontrivial solution of (1.1). Next, we claim that  $u$  is a ground state solution of (1.1). Recall  $K(x) > 0$  and set  $\theta = 0$  in  $(g_4)$ , then  $K(x)(\frac{1}{2}g(u)u - G(u)) \geq 0$ . Hence,

$$\begin{aligned} \mathcal{J}(u) &= \mathcal{J}(u) - \frac{1}{2} \langle \mathcal{J}'(u), u \rangle = \int_{\mathbb{R}^N} K(x) \left( \frac{g(u)u}{2} - G(u) \right) dx \\ &\leq \varliminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} K(x) \left( \frac{g(u_n)u_n}{2} - G(u_n) \right) dx = \varliminf_{n \rightarrow \infty} \left( \mathcal{J}(u_n) - \frac{1}{2} \langle \mathcal{J}'(u_n), u_n \rangle \right) = m. \end{aligned}$$

This completes the proof. □

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