TAIWANESE JOURNAL OF MATHEMATICS
Vol. 26, No. 6, pp. 1219–1234, December 2022
DOI: 10.11650/tjm/220404

Boundedness in Asymmetric Oscillations at Resonance in a Critical Situation

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Abstract. In this article, by using Moser's twist theorem, we prove that all solutions of the equation $x'' + ax^+ - bx^- + \varphi(x) = p(t)$ are bounded in the critical situation, where p is a smooth periodic function, and φ is bounded one.

1. Introduction

In the past few decades, due to its background in applied sciences [7], the boundedness problems for the asymmetric equations

$$x'' + ax^+ - bx^- = f(x,t) \tag{1.1}$$

have been extensively studied; see for examples [2–5, 8, 11, 12, 15] and references therein, where $x^+ = \max\{x,0\}$, $x^- = \max\{-x,0\}$, a and b are different positive numbers. The function $f(x,t)$ is periodic in t.

Generally speaking, in the case of resonance, that is,

$$\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} = \frac{2}{n}, \quad n \in \mathbb{N}, \tag{1.2}$$

the boundedness problems for (1.1) are more difficult to study than in non-resonance case.

Liu [8] obtained the boundedness of the solutions for (1.1) when f dependents only on t and satisfies

$$\int_0^{2\pi} f(t)\mathbf{C}(\theta+t)\,dt \neq 0, \quad \theta \in \mathbb{R},$$

where $\mathbf{C}$ is the solution of the initial value problem

$$\begin{cases} x'' + ax^+ - bx^- = 0, \\ x(0) = 1, \quad x'(0) = 0. \end{cases} \tag{1.3}$$

In [12], Wang proved the boundedness of solutions for the equation

$$x'' + ax^+ - bx^- + \varphi(x) = p(t) \tag{1.4}$$

Received November 24, 2021; Accepted April 21, 2022.
Communicated by Cheng-Hsiung Hsu.
2020 *Mathematics Subject Classification.* 34C11, 34C27, 37E40.
Key words and phrases. asymmetric oscillations, boundedness, at resonance, critical situation, twist theorem.
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under the condition

$$\int_0^{2\pi} p(t)\mathbf{C}(\theta+t)\,dt \neq 2n\sqrt{a}\left(\frac{\varphi(+\infty)}{a} - \frac{\varphi(-\infty)}{b}\right), \quad \theta \in \mathbb{R} \tag{1.5}$$

with $\varphi(\pm\infty) = \lim_{x\to\pm\infty} \varphi(x)$.

Fabry and Mawhin [4] obtained the boundedness result for (1.1) with $f(x,t) = \varphi(x) + g(x) + p(t)$ under the condition (1.5), where g is a sublinear primitive and bounded.

If $a = b = n^2$, (1.4) becomes the symmetric equation

$$x'' + n^2 x + \varphi(x) = p(t), \quad n \in \mathbb{N}. \tag{1.6}$$

It is obviously in the resonant case. Lazer and Leach [6] proved that (1.6) has at least one periodic solution under the so-called Lazer–Leach condition

$$\left|\int_0^{2\pi} p(t)e^{-int}\,dt\right| < 2\left(\liminf_{x\to+\infty} \varphi - \limsup_{x\to-\infty} \varphi\right), \quad \forall\, \theta \in [0,2\pi]. \tag{1.7}$$

In 1999, Liu [9] proved that each solution of the equation (1.6) is bounded with $p \in C^7(\mathbb{R}/2\pi\mathbb{Z})$, $\varphi \in C^6(\mathbb{R})$ under the condition (1.7).

However, Alonso and Ortega [1] proved that if $\lim_{|x|\to\infty} \psi(x)/x = 0$ and φ is bounded, each solution of the semilinear equation

$$x'' + n^2 x + \varphi(x) + \psi'(x) = p(t) = p(t+2\pi) \tag{1.8}$$

is unbounded with a large initial condition if

$$\left|\int_0^{2\pi} p(t)e^{-int}\,dt\right| > 2(H-K),$$

where $H = \max\big\{\limsup_{x\to-\infty} \varphi, \limsup_{x\to+\infty} \varphi\big\}$ and $K = \min\big\{\liminf_{x\to-\infty} \varphi, \liminf_{x\to+\infty} \varphi\big\}$.

In 2016, Wang, Wang and Piao [13] showed that if ψ oscillates periodically in x, the Lazer–Leach condition (1.7) is sufficient and necessary for the boundedness of (1.8).

So we can ask a question: if the " $<$ " in (1.6) is changed to " $=$ " (critical situation), can one obtain boundedness results for (1.6)?

Recently, Xing, Wang and Wang [14] succeeded in answer the question. They obtained a certain sufficient and necessary condition for the boundedness for (1.6) in the critical situation, that is,

$$\left|\int_0^{2\pi} p(t)e^{-int}\,dt\right| = 2(\varphi(+\infty) - \varphi(-\infty)),$$

where $\varphi(\pm\infty)$ exit finitely and p is 2π-periodic in t.

In this article, we are going to study the analogical problem of [14] for the asymmetric equation (1.4). The corresponding critical situation should be

$$\int_0^{2\pi} p(t)\mathbf{C}(t-\theta)\,dt = 2n\sqrt{a}\left(\frac{\varphi(+\infty)}{a} - \frac{\varphi(-\infty)}{b}\right) \quad \text{for some } \theta \in \mathbb{R}. \tag{1.9}$$

We suppose that there exist two positive constants $c_\pm$ such that

$$\lim_{x\to\pm\infty} x^{k-1+d}\Gamma_\pm^{(k)}(x) = 0, \quad 0 < k \le 11 \tag{1.10}$$

with $0 < d < 1$ and

$$\Gamma_\pm(x) = \int_0^x (\varphi(x) - \varphi(\pm\infty))\,dx - c_\pm \cdot (1+x^2)^{(1-d)/2}. \tag{1.11}$$

Now we can state our main result as below.

Theorem 1.1. *Suppose that* $p \in C^6(\mathbb{R}/2\pi\mathbb{Z})$, $\varphi \in C^{10}(\mathbb{R})$ *satisfying*

$$\varphi(\pm\infty) := \lim_{x\to\pm\infty} \varphi(x)$$

are finite. Assume (1.2), (1.9)–(1.11) *hold true, then all solutions of* (1.4) *are bounded.*

Remark 1.2. Theorem 1.1 is applicable to many equations. Here we provide a concrete example.

Let $p(t) = 4\cos(nt)$, $\varphi(x) = \arctan x + 2x(1+x^2)^{-2/3}$. Then we have

$$\begin{aligned}
&\int_0^{2\pi} p(t)\mathbf{C}(t-\theta)\,dt \\
&= 4\cos(n\theta)\int_0^{2\pi} \cos(nt)\mathbf{C}(t)\,dt - 4\sin(n\theta)\int_0^{2\pi} \sin(nt)\mathbf{C}(t)\,dt \\
&= 8n\cos(n\theta)\left(\int_0^{\frac{\pi}{2\sqrt{a}}} \cos(nt)\cos(\sqrt{a}t)\,dt - \sqrt{\frac{a}{b}}\int_0^{\frac{\pi}{2\sqrt{b}}} \cos\left(n\left(t + \frac{\pi}{2\sqrt{a}}\right)\right)\sin\sqrt{b}t\,dt\right) \\
&= 8n\sqrt{a}\cos(n\theta)\cos\left(\frac{n\pi}{2\sqrt{a}}\right)\left(\frac{1}{a-n^2} - \frac{1}{b-n^2}\right).
\end{aligned}$$

Now we choose $a = 36$, $b = 144$, $n = 8$, which satisfy the condition (1.2). Meanwhile

$$\int_0^{2\pi} p(t)\mathbf{C}(t-\theta)\,dt = 8n\sqrt{a}\cos(n\theta)\cos\left(\frac{n\pi}{2\sqrt{a}}\right)\left(\frac{1}{a-n^2} - \frac{1}{b-n^2}\right) = \frac{324}{35}\cos(8\theta),$$

$$2n\sqrt{a}\left(\frac{\varphi(+\infty)}{a} - \frac{\varphi(-\infty)}{b}\right) = \frac{5\pi}{3}.$$

One can take $\theta = \frac{1}{8}\arccos\frac{175\pi}{972} + 2k\pi$ ($k \in \mathbb{Z}$) such that the condition (1.9) holds. Then, according to Theorem 1.1, we obtain the boundedness for the equation with $d = 1/3$, $c_\pm = 3$.

In fact, the conclusion of Theorem 1.1 is also true, if the conditions (1.2) and (1.9) are replaced respectively by

- $\dfrac{1}{\sqrt{a}} + \dfrac{1}{\sqrt{b}} = \dfrac{2m}{n}$, where m and n are relatively prime positive integers

and

- $\displaystyle\int_0^{2\pi} p(mt)\mathbf{C}(mt-\theta)\,dt = \frac{2n\sqrt{a}}{m}\left(\frac{\varphi(+\infty)}{a} - \frac{\varphi(-\infty)}{b}\right)$ for some $\theta \in \mathbb{R}$.

This paper is organized as follows. In Section 2, we give some technical lemmas. In Section 3, we introduce a rotation transformation and make a series of canonical transformations such that the new Hamiltonian system is closed to a nearly integrable one. In Section 4, we first give a twist condition in some weak way, then prove the boundedness of solutions of (1.4) by Moser's twist theorem.

Throughout this paper, we denote by

$$[I](\cdot) := \frac{1}{2\pi}\int_0^{2\pi} I(\,\cdot\,,\theta)\,d\theta$$

the average value of $I(\,\cdot\,,\theta)$ over $\mathbb{R}/2\pi\mathbb{Z}$. We denote by $C > 1$, $c < 1$ two positive constants without concerning their quantity.

2. Preliminaries

In this section, some technical lemmas will be given.

Introduce a new variable y as $x' = -y$, then the equation (1.4) is equivalent to a planar non-autonomous Hamiltonian system

$$x' = -\frac{\partial H}{\partial y}(x,y,t), \quad y' = \frac{\partial H}{\partial x}(x,y,t), \tag{2.1}$$

where $H(x,y,t) = \frac{1}{2}y^2 + \frac{1}{2}a(x^+)^2 + \frac{1}{2}b(x^-)^2 + \Phi(x) - xp(t)$, $\Phi(x) = \int_0^x \varphi(s)\,ds$.

Define $\mathbf{S}(t) = -\mathbf{C}'(t)$. Then $(\mathbf{C}(t),\mathbf{S}(t))$ is the solution of the following system

$$x' = -y, \quad y' = ax^+ - bx^-$$

with the initial condition $(\mathbf{C}(0),\mathbf{S}(0)) = (1,0)$ from (1.3). Hence

- $\mathbf{C}(-t) = \mathbf{C}(t)$, $\mathbf{S}(-t) = -\mathbf{S}(t)$;
- $\mathbf{C}(t)$ and $\mathbf{S}(t)$ are $\frac{2\pi}{n}$-periodic functions;
- $\mathbf{S}^2(t) + a(\mathbf{C}^+(t))^2 + b(\mathbf{C}^-(t))^2 \equiv a$;
- $\mathbf{C}(t)$ can be given by

$$\mathbf{C}(t) = \begin{cases} \cos\sqrt{a}t, & 0 \le |t| \le \frac{\pi}{2\sqrt{a}}, \\ -\sqrt{\frac{a}{b}}\sin\sqrt{b}\left(t - \frac{\pi}{2\sqrt{a}}\right), & \frac{\pi}{2\sqrt{a}} < |t| \le \frac{\pi}{n}. \end{cases}$$

For $r > 0$, we make the transformation $(r, \theta) \to (x, y)$:

$$x = a^{-1/2} r^{1/2} \mathbf{C}(\theta), \quad y = a^{-1/2} r^{1/2} \mathbf{S}(\theta),$$

then the Hamiltonian system (2.1) is changed into

$$\frac{dr}{dt} = -\frac{\partial h}{\partial \theta}(r, \theta, t), \quad \frac{d\theta}{dt} = \frac{\partial h}{\partial r}(r, \theta, t), \tag{2.2}$$

where

$$h(r, \theta, t) = r + I_1(r, \theta) + I_2(r, \theta, t) \tag{2.3}$$

with $I_1(r, \theta) = 2\Phi(a^{-1/2} r^{1/2} \mathbf{C}(\theta))$, $I_2(r, \theta, t) = -2a^{-1/2} r^{1/2} \mathbf{C}(\theta) p(t)$.

Similar to [9], we can obtain the following estimates on $I_1(r, \theta)$, $I_2(r, \theta, t)$ by direct calculations. We omit the proof here.

Lemma 2.1. *For $r \gg 1$, it holds that*

$$|I_1(r, \theta)| \le C r^{1/2}, \quad |\partial_r^i \partial_\theta^j I_1(r, \theta)| \le C r^{-i + \frac{1}{2} + \frac{1}{2}(\max(1, j) - 1)}, \quad i + j \le 11.$$

Lemma 2.2. *For $r \gg 1$, it holds that*

$$|\partial_r^i \partial_\theta^j \partial_t^k I_2(r, \theta, t)| \le C r^{-i + 1/2}, \quad i + j \le 11, \ k \le 6.$$

Lemma 2.3. *Let*

$$\alpha(r) = [I_1](r) - \frac{2n}{\pi} r^{1/2} \cdot \left(\frac{\varphi(+\infty)}{a} - \frac{\varphi(-\infty)}{b} \right).$$

Then $\alpha(r)$ satisfies

$$\begin{aligned}
& c r^{(1-d)/2 - k} \le \alpha^{(k)}(r) \le C r^{(1-d)/2 - k}, && k = 0, 1, \ 0 < d < 1, \\
& |\alpha^{(2)}(r)| \ge c r^{(1-d)/2 - 2}, \quad |\alpha^{(k)}(r)| \le c r^{(1-d)/2 - k}, && k \le 11, \ 0 < d < 1.
\end{aligned}$$

Proof. By definition of $[I_1](r)$, one has

$$\alpha(r) = \frac{1}{2\pi} \int_0^{2\pi} 2\Phi(a^{-1/2} r^{1/2} \mathbf{C}(\theta))\, d\theta - \frac{2n}{\pi} r^{1/2} \cdot \left(\frac{\varphi(+\infty)}{a} - \frac{\varphi(-\infty)}{b} \right).$$

By direct calculations, one has

$$\begin{aligned}
\alpha'(r) &= \frac{a^{-1/2} r^{-1/2}}{2\pi} \left(\int_0^{2\pi} \varphi(a^{-1/2} r^{1/2} \mathbf{C}(\theta)) \mathbf{C}(\theta)\, d\theta - 2n a^{1/2} \cdot \left(\frac{\varphi(+\infty)}{a} - \frac{\varphi(-\infty)}{b} \right) \right) \\
&= \frac{a^{-1/2} r^{-1/2}}{2\pi} \sum_{k=1}^{n} (\alpha_{k+}(r) + \alpha_{k-}(r)),
\end{aligned} \tag{2.4}$$

where

$$
\begin{aligned}
\alpha_{k+}(r) &= \int_{\frac{2k\pi}{n}-\frac{\pi}{2\sqrt{a}}}^{\frac{2k\pi}{n}+\frac{\pi}{2\sqrt{a}}} (\varphi(a^{-1/2}r^{1/2}\mathbf{C}(\theta)) - \varphi(+\infty))\mathbf{C}(\theta)\,d\theta, \\
\alpha_{k-}(r) &= \int_{\frac{2k\pi}{n}+\frac{\pi}{2\sqrt{a}}}^{\frac{2(k+1)\pi}{n}-\frac{\pi}{2\sqrt{a}}} (\varphi(a^{-1/2}r^{1/2}\mathbf{C}(\theta)) - \varphi(-\infty))\mathbf{C}(\theta)\,d\theta.
\end{aligned}
\tag{2.5}
$$

By (1.10), as $r \to \infty$, there exists a positive constant $C_1(d)$ such that

$$
\begin{aligned}
(a^{-1}r)^{d/2}\alpha_{k+}(r) &= \int_{\frac{2k\pi}{n}-\frac{\pi}{2\sqrt{a}}}^{\frac{2k\pi}{n}+\frac{\pi}{2\sqrt{a}}} (\varphi(a^{-1/2}r^{1/2}\mathbf{C}(\theta)) - \varphi(+\infty))(a^{-1/2}r^{1/2}\mathbf{C}(\theta))^d\mathbf{C}^{1-d}(\theta)\,d\theta \\
&\to C_1(d)c_+, \\
(a^{-1}r)^{d/2}\alpha_{k-}(r) &= \int_{\frac{2k\pi}{n}+\frac{\pi}{2\sqrt{a}}}^{\frac{(2k+1)\pi}{n}-\frac{\pi}{2\sqrt{a}}} (\varphi(a^{-1/2}r^{1/2}\mathbf{C}(\theta)) - \varphi(-\infty))(a^{-1/2}r^{1/2}\mathbf{C}(\theta))^d\mathbf{C}^{1-d}(\theta)\,d\theta \\
&\to C_1(d)c_-.
\end{aligned}
$$

Then we have, as $r \to \infty$,

$$
(a^{-1}r)^{(d+1)/2}\alpha'(r) \to \frac{1}{2\pi a}C_1(d)(c_+ + c_-),
$$

which implies that

$$
cr^{(1-d)/2-1} \le \alpha'(r) \le Ch_3^{(1-d)/2-1}. \tag{2.6}
$$

The conclusion

$$
cr^{(1-d)/2} \le \alpha(r) \le Cr^{(1-d)/2}
$$

is a consequence of (2.6) by the rule of L'Hospital.

By (1.10) and (2.5), as $r \to \infty$, there exists a positive constant $C_2(d)$ such that

$$
\begin{aligned}
2(a^{-1}r)^{d/2}r\alpha'_{k+}(r) &= \int_{\frac{2k\pi}{n}-\frac{\pi}{2\sqrt{a}}}^{\frac{2k\pi}{n}+\frac{\pi}{2\sqrt{a}}} \varphi'(a^{-1/2}r^{1/2}\mathbf{C}(\theta))(a^{-1/2}r^{1/2}\mathbf{C}(\theta))^{1+d}\mathbf{C}^{1-d}(\theta)\,d\theta \\
&\to -C_2(d)c_+d(1-d), \\
2(a^{-1}r)^{d/2}r\alpha'_{k-}(r) &= \int_{\frac{2k\pi}{n}+\frac{\pi}{2\sqrt{a}}}^{\frac{(2k+1)\pi}{n}-\frac{\pi}{2\sqrt{a}}} \varphi'(a^{-1/2}r^{1/2}\mathbf{C}(\theta))(a^{-1/2}r^{1/2}\mathbf{C}(\theta))^{1+d}\mathbf{C}^{1-d}(\theta)\,d\theta \\
&\to -C_2(d)c_-d(1-d).
\end{aligned}
$$

Thus

$$
(\alpha'_{k+}(r) + \alpha'_{k-}(r)) \to -C_2(d)(c_+ + c_-)d(1-d)\frac{1}{2}a^{d/2}r^{-1-d/2}. \tag{2.7}
$$

By (2.4), one has

$$\alpha''(r) = -\frac{a^{-1/2}r^{-3/2}}{4\pi}\sum_{k=1}^{n}(\alpha_{k+}(r)+\alpha_{k-}(r)) + \frac{a^{-1/2}r^{-1/2}}{2\pi}\sum_{k=1}^{n}(\alpha'_{k+}(r)+\alpha'_{k-}(r)),$$

which together with (2.6) and (2.7) implies $|\alpha''(r)| \geq cr^{(1-d)/2-2}$.

By direct calculations, one has for $m \leq 11$ that

$$r^{m+d/2}\alpha_{k+}^{(m)}(r) = \int_{\frac{2k\pi}{n}-\frac{\pi}{2\sqrt{a}}}^{\frac{2k\pi}{n}+\frac{\pi}{2\sqrt{a}}}\sum_{i=1}^{m} c\varphi^{(i)}(a^{-1/2}r^{1/2}\mathbf{C}(\theta))(a^{-1/2}r^{1/2}\mathbf{C}(\theta))^{i+d}\mathbf{C}^{1-d}(\theta)\,d\theta,$$

$$r^{m+d/2}\alpha_{k-}^{(m)}(r) = \int_{\frac{2k\pi}{n}+\frac{\pi}{2\sqrt{a}}}^{\frac{2(k+1)\pi}{n}-\frac{\pi}{2\sqrt{a}}}\sum_{i=1}^{m} c\varphi^{(i)}(a^{-1/2}r^{1/2}\mathbf{C}(\theta))(a^{-1/2}r^{1/2}\mathbf{C}(\theta))^{i+d}\mathbf{C}^{1-d}(\theta)\,d\theta.$$

As $r \to \infty$, we have

$$|\alpha_{k\pm}^{(m)}(r)| \leq Cr^{-m-d/2}. \tag{2.8}$$

Thus we can obtain the last estimate from (2.4) and (2.8). □

By (2.3), Lemmas 2.1 and 2.2, one has $\partial_r h > 0$ for $r \gg 1$. Then by the implicit function theorem, there exists a function $R(h,t,\theta)$ such that

$$r(h,t,\theta) = h - R(h,t,\theta),$$

where

$$R(h,t,\theta) = I_1(h-R,\theta) + I_2(h-R,\theta,t) = I_1(h,\theta) + I_2(h,\theta,t) - R_0(h,t,\theta)$$

with $R_0(h,t,\theta) = \int_0^1 \partial_r I_1(h-\mu R,\theta)R\,d\mu + \int_0^1 \partial_r I_2(h-\mu R,\theta,t)R\,d\mu$.

Now h, t and θ are the new action, angle and time variables respectively. Moreover $R(h,t,\theta)$ and $R_0(h,t,\theta)$ satisfy the following estimates.

Lemma 2.4. *For $h \gg 1$, it holds that*

$$\begin{aligned} &|\partial_h^i\partial_t^j\partial_\theta^k R| \leq Ch^{-i+\frac{1}{2}+\frac{1}{2}\{\max\{1,k\}-1\}}, && i+k \leq 11,\ j \leq 6,\\ &|\partial_h^i\partial_t^j\partial_\theta^k R_0| \leq Ch^{-i+k/2}, && i+k \leq 10,\ j \leq 6. \end{aligned} \tag{2.9}$$

The proof can be obtained by direct calculations similar to that of Lemma 3.1 in [14]. Thus we omit it here.

Now the Hamiltonian system (2.2) can be written

$$\frac{dh}{d\theta} = -\frac{\partial r}{\partial t}(h,t,\theta), \quad \frac{dt}{d\theta} = \frac{\partial r}{\partial h}(h,t,\theta),$$

where

$$r(h,t,\theta) = h - I_1(h,\theta) - I_2(h,\theta,t) + R_0(h,t,\theta). \tag{2.10}$$

3. Canonical transformations

In this section, we will make some canonical transformations such that the perturbation satisfies desirable estimates. First, we eliminate the linear part of the Hamiltonian by a rotation transformation.

Lemma 3.1. *There exists a rotation transformation Ψ_1 of the form*

$$\Psi_1: \quad h = h_1, \quad t = t_1 + \theta$$

such that the system with the Hamiltonian (2.10) *is transformed into a sublinear system with the following Hamiltonian*

$$r_1(h_1, t_1, \theta) = -I_1(h_1, \theta) - I_2(h_1, \theta, t_1 + \theta) + R_1(h_1, t_1, \theta), \tag{3.1}$$

where $R_1(h_1, t_1, \theta) = R_0(h, t_1 + \theta, \theta)$ satisfies

$$|\partial_{h_1}^i \partial_{t_1}^j \partial_\theta^k R_1(h_1, t_1, \theta)| \leq C h_1^{-i+k/2}, \quad i + k \leq 10,\ j \leq 6. \tag{3.2}$$

Proof. It is not difficult to obtain (3.1) and (3.2) from (2.10) and (2.9). □

Lemma 3.2. *There exists a canonical transformation Ψ_2 given by*

$$\Psi_2: \quad h_1 = h_2, \quad t_1 = t_2 - \partial_{h_2} S_2(h_2, \theta) \tag{3.3}$$

with

$$S_2(h_2, \theta) = \int_0^\theta (I_1(h_2, \theta) - [I_1](h_2))\, ds \tag{3.4}$$

such that the Hamiltonian (3.1) *is transformed into the following Hamiltonian*

$$r_2(h_2, t_2, \theta) = -[I_1](h_2) - I_2(h_2, \theta, t_2 + \theta) + R_2(h_2, t_2, \theta), \tag{3.5}$$

where $R_2(h_2, t_2, \theta)$ satisfies

$$|\partial_{h_2}^i \partial_{t_2}^j \partial_\theta^k R_2(h_2, t_2, \theta)| \leq C h_2^{-i+k/2}, \quad i + k \leq 10,\ j \leq 5. \tag{3.6}$$

Proof. Under the transformation Ψ_2, the transformed Hamiltonian is

$$\begin{aligned} r_2(h_2, t_2, \theta) &= -I_1(h_2, \theta) - I_2(h_2, \theta, t_2 - \partial_{h_2} S_2 + \theta) + R_1(h_2, t_2 - \partial_{h_2} S_2, \theta) + \partial_\theta S_2 \\ &= -[I_1](h_2) - I_2(h_2, \theta, t_2 + \theta) + [I_1](h_2) - I_1(h_2, \theta) + R_2(h_2, t_2, \theta) + \partial_\theta S_2, \end{aligned}$$

where

$$R_2(h_2, t_2, \theta) = R_1(h_2, t_2 - \partial_{h_2} S_2, \theta) + \int_0^1 \partial_{t_1} I_2(h_2, \theta, t_2 - \mu \partial_{h_2} S_2 + \theta) \partial_{h_2} S_2\, d\mu.$$

By (3.4), one has

$$[I_1](h_2) - I_1(h_2,\theta) + \partial_\theta S_2 = 0.$$

Thus we obtain the Hamiltonian (3.5). By Lemma 2.1, one has

$$|\partial_{h_2}^i \partial_\theta^j S_2(h_2,\theta)| \le C h_2^{-i+\frac{1}{2}+\frac{1}{2}(\max\{1,j\}-1)}, \quad i+j \le 11. \tag{3.7}$$

By (3.3), we have

$$|\partial_{h_2} t_1| \le C h_2^{-3/2}, \quad \frac{1}{2} \le |\partial_{t_2} t_1| \le 2, \quad |\partial_\theta t_1| \le C h_2^{-1/2}, \tag{3.8}$$

$$|\partial_{h_2}^i \partial_\theta^j \partial_{t_2}^k t_1| \le C h_2^{-i-\frac{1}{2}+\frac{1}{2}(\max\{1,j\}-1)}, \quad i+j+k \ge 2,\ i+j \le 10,\ k \le 6. \tag{3.9}$$

By Leibniz's rule, $\partial_{h_2}^i \partial_{t_2}^j \partial_\theta^k R_1(h_2, t_2 - \partial_{h_2} S_2, \theta)$ is the summation of terms

$$\partial_{h_1}^m \partial_{t_1}^s \partial_\theta^q R_1 \cdot \prod_{l=1}^{s} \partial_{h_2}^{i_l} \partial_{t_2}^{j_l} \partial_\theta^{k_l} t_1$$

with $1 \le m+s+q \le i+j+k$, $m + \sum_{l=1}^s i_l = i$, $\sum_{l=1}^s j_l = j$ and $q + \sum_{l=1}^s k_l = k$.

Combining (3.2), (3.8) with (3.9), we obtain

$$|\partial_{h_2}^i \partial_{t_2}^j \partial_\theta^k R_1(h_2, t_2 - \partial_{h_2} S_2, \theta)| \le C h_2^{-i+k/2}, \quad i+k \le 10,\ j \le 6. \tag{3.10}$$

Similarly, we obtain

$$\begin{aligned} &|\partial_{h_2}^i \partial_{t_2}^j \partial_\theta^k (\partial_{t_1} I_2(h_2, \theta, t_2 - \mu\partial_{h_2} S_2 + \theta))| \\ &\le C h_2^{-i+\frac{1}{2}+\frac{1}{2}(\max\{1,k\}-1)}, \quad i+k \le 10,\ j \le 5. \end{aligned} \tag{3.11}$$

By (3.7), (3.11) and Leibniz's rule, one has

$$\begin{aligned} &|\partial_{h_2}^i \partial_{t_2}^j \partial_\theta^k (\partial_{t_1} I_2(h_2, \theta, t_2 - \mu\partial_{h_2} S_2 + \theta) \cdot \partial_{h_2} S_2)| \\ &\le C h_2^{-i-\frac{1}{2}+\frac{1}{2}(\max\{1,k\}-1)}, \quad i+k \le 10,\ j \le 5. \end{aligned} \tag{3.12}$$

Then the estimate (3.6) follows from (3.10) and (3.12). □

Without causing confusion, denote

$$[I_2](h,t) = \frac{1}{2\pi} \int_0^{2\pi} I_2(h,\theta,t+\theta)\, d\theta.$$

Lemma 3.3. *There exists a canonical transformation* Ψ_3 *given by*

$$\Psi_3: \quad h_2 = h_3 + \partial_{t_2} S_3(h_3, t_2, \theta), \quad t_3 = t_2 + \partial_{h_3} S_3(h_3, t_2, \theta)$$

with

$$S_3(h_3, t_2, \theta) = \int_0^\theta (I_2(h_3, s, t_2 + s) - [I_2](h_3, t_2))\, ds \tag{3.13}$$

such that the Hamiltonian (3.5) *is transformed into the following Hamiltonian*

$$r_3(h_3,t_3,\theta)=I(h_3,t_3)+R_3(h_3,t_3,\theta), \tag{3.14}$$

where $R_3(h_3,t_3,\theta)$ *and* $I(h_3,t_3)$ *satisfy*

$$|\partial_{h_3}^i\partial_{t_3}^j\partial_\theta^k R_3(h_3,t_3,\theta)|\le Ch_3^{-i+k/2},\qquad i+k\le 10,\ j\le 5, \tag{3.15}$$

$$|\partial_{h_3}^k I(h_3,t_3)|\ge ch_3^{(1-d)/2-k},\qquad k=0,1,2, \tag{3.16}$$

$$|\partial_{h_3}^k\partial_{t_3}^l I(h_3,t_3)|\le Ch_3^{-k+1/2},\qquad k\le 11,\ l\le 6. \tag{3.17}$$

Proof. Under the transformation Ψ_3, the transformed Hamiltonian is

$$\begin{aligned}r_3(h_3,t_3,\theta)&=-[I_1](h_3+\partial_{t_2}S_3)-I_2(h_3+\partial_{t_2}S_3,\theta,t_2+\theta)\\&\quad+R_2(h_3+\partial_{t_2}S_3,t_3-\partial_{h_3}S_3,\theta)+\partial_\theta S_3\\&=-[I_1](h_3)-[I_2](h_3,t_3)+[I_2](h_3,t_2)-I_2(h_3,\theta,t_2+\theta)\\&\quad+\partial_\theta S_3+R_3(h_3,t_3,\theta),\end{aligned}$$

where

$$\begin{aligned}R_3(h_3,t_3,\theta)&=R_2(h_3+\partial_{t_2}S_3,t_3-\partial_{h_3}S_3,\theta)-\int_0^1\partial_{h_2}I_2(h_3+\mu\partial_{t_2}S_3,\theta,t_2+\theta)\partial_{t_2}S_3\,d\mu\\&\quad+\int_0^1\partial_{t_2}[I_2](h_3,t_3-\mu\partial_{h_3}S_3)\partial_{h_3}S_3\,d\mu-\int_0^1[I_1]'(h_3+\mu\partial_{t_2}S_3)\partial_{t_2}S_3\,d\mu.\end{aligned}$$

By (3.13), one has

$$[I_2](h_3,t_2)-I_2(h_3,\theta,t_2+\theta)+\partial_\theta S_3=0.$$

Denote $I(h_3,t_3)=-[I_1](h_3)-[I_2](h_3,t_3)$, then we obtain the Hamiltonian (3.14). By Lemma 2.2, one has

$$|\partial_{h_3}^i\partial_{t_2}^j\partial_\theta^k S_3(h_3,t_2,\theta)|\le Ch_2^{-i+1/2},\quad i+k\le 11,\ j\le 6.$$

From (3.3), we have

$$\begin{gathered}|\partial_{h_3}t_2|\le Ch_3^{-3/2},\quad \frac12\le|\partial_{t_3}t_2|\le 2,\quad |\partial_\theta t_2|\le Ch_2^{-1/2},\\|\partial_{h_3}^i\partial_\theta^j\partial_{t_3}^k t_2|\le Ch_2^{-i-1/2},\quad i+j+k\ge 2,\ i+j\le 10,\ k\le 6,\\\frac12\le|\partial_{h_3}h_2|\le 2,\quad |\partial_{t_3}h_2|\le Ch_3^{1/2},\quad |\partial_\theta h_2|\le Ch_3^{1/2},\\|\partial_{h_3}^i\partial_\theta^j\partial_{t_3}^k h_2|\le Ch_3^{-i+1/2},\quad i+j+k\ge 2,\ i+j\le 10,\ k\le 5.\end{gathered}$$

Similar to the proof of (3.6), we can obtain (3.15).

Notice that

$$
\begin{aligned}
-I(h_3,t_3) &= [I_1](h_3) + [I_2](h_3,t_3) \\
&= \alpha(h_3) + \frac{2h_3^{1/2}}{\pi}\cdot\left(\frac{n\varphi(+\infty)}{a} - \frac{n\varphi(-\infty)}{b} - \frac{1}{2\sqrt{a}}\int_0^{2\pi}\mathbf{C}(t-\theta)p(t)\,dt\right),
\end{aligned}
$$

which together with (1.9) and Lemma 2.3 implies (3.16) and (3.17). □

Lemma 3.4. [14] *Consider the system with Hamiltonian*

$$\widetilde{r} = \widetilde{I}(h,t) + \widetilde{R}(h,t,\theta), \tag{3.18}$$

where $\widetilde{I}(h,t)$ *satisfies* (3.16) *and* (3.17) *for* $k \le m$, $l \le n$, *and* $R(h,t,\theta)$ *satisfies*

$$|\partial_h^j\partial_t^k\partial_\theta^l R| \le Ch^{-j-i/2+\max\{0,(l-i)/2\}}$$

for h *large enough,* $j+l \le m_1$, $k \le n_1$ $(m_1 \le m,\ n_1 \le n)$. *Then there exists a transformation* Ψ_+ *of the form*

$$h = h_+ + \partial_t S_+(h_+,t,\theta), \quad t_+ = t + \partial_{h_+} S_+(h_+,t,\theta)$$

with

$$S_+(h_+,t,\theta) = -\int_0^\theta (\widetilde{R}(h_+,t,\theta) - [\widetilde{R}](h_+,t,))\,d\theta$$

such that the Hamiltonian (3.18) *is transformed to*

$$\widetilde{r}_+(h_+,t_+,\theta) = \widetilde{I}_+(h_+,t_+) + \widetilde{R}_+(h_+,t_+,\theta),$$

where $\widetilde{I}_+(h_+,t_+) = \widetilde{I}(h_+,t_+) + [\widetilde{R}](h_+,t_+)$ *satisfies* (3.16) *and* (3.17) *for* $k \le m_1$, $l \le n_1$. *Moreover for* $h_+ \gg 1$, $l \le n_1 - 1$, $k + j \le m_1 - 1$, *it holds that*

$$|\partial_{h_+}^j\partial_{t_+}^l\partial_\theta^k\widetilde{R}_+| \le Ch_+^{-j-(i+1)/2+\max\{0,(k-i-1)/2\}}.$$

Lemma 3.5. *There exists a canonical transformation* Ψ_4 *given by*

$$\Psi_4: \quad h_3 = h_4 + U(h_4,t_4,\theta), \quad t_3 = t_4 + V(h_4,t_4,\theta)$$

such that the Hamiltonian (3.14) *is transformed into the following Hamiltonian*

$$r_4(h_4,t_4,\theta) = I(h_4,t_4) + R_4(h_4,t_4,\theta) \tag{3.19}$$

with $I(h_4,t_4)$ *and* $R_4(h_4,t_4,\theta)$ *satisfying*

$$|\partial_{h_4}^k I(h_4,t_4)| \ge ch_4^{(1-d)/2-k}, \qquad k = 0,1,2, \tag{3.20}$$

$$|\partial_{h_4}^k\partial_{t_4}^l I(h_4,t_4)| \le Ch_4^{-k+1/2}, \qquad k \le 5,\ l \le 1, \tag{3.21}$$

$$|\partial_{h_4}^i\partial_{t_4}^j\partial_\theta^k R_4(h_4,t_4,\theta)| \le Ch_4^{-i-5/2}, \qquad i + k \le 5. \tag{3.22}$$

Proof. We can prove this lemma by Lemma 3.3 and repeated applications of canonical transformations given in Lemma 3.4. $\square$

Consider the Hamiltonian (3.19). Noting that $\partial_{h_4}^k I(h_4, t_4) \geq c h_4^{(1-d)/2-1} > 0$ as $h_4 \to \infty$, we can solve (3.19) for large h_4 as follows:

$$h_4(r_4, \theta, t_4) = N(r_4, t_4) + P(r_4, \theta, t_4), \tag{3.23}$$

where $h_4 = N(r_4, t_4)$ is the inverse function of $r_4 = I(h_4, t_4)$. By (3.19) and (3.23), one has

$$\begin{aligned} r_4 &= I(N+P, t_4) + R_4(N+P, t_4, \theta) \\ &= I(N, t_4) + \int_0^1 \partial_{h_4} I(N + \mu P, t_4) P \, d\mu + R_4(N+P, t_4, \theta). \end{aligned}$$

Note that $r_4 = I(h_4, t_4)$, thus

$$0 = \int_0^1 \partial_{h_4} I(N + \mu P, t_4) P \, d\mu + R_4(N+P, t_4, \theta), \tag{3.24}$$

that is,

$$P = -\frac{R_4(N+P, t_4, \theta)}{\int_0^1 \partial_{h_4} I(N + \mu P, t_4) \, d\mu}.$$

Lemma 3.6. *For r_4 large enough, it holds that*

$$c r_4^2 \leq |N| \leq C r_4^{2/(1-d)}, \quad c N r_4^{-k} \leq |\partial_{r_4}^k N| \leq C N r_4^{-k}, \qquad k = 1, 2, \tag{3.25}$$

$$|\partial_{r_4}^k N| \leq C N r_4^{-k}, \qquad k \leq 6, \tag{3.26}$$

$$|\partial_{r_4}^k \partial_\theta^j P| \leq C r_4^{-k-1} N |R_4|, \qquad k + j \leq 5. \tag{3.27}$$

Proof. By $r_4 \equiv I(N(r_4, t_4), t_4)$, one has

$$c r_4^2 \leq |N| \leq C r_4^{2/(1-d)}, \quad \partial_{h_4} I \cdot \partial_{r_4} N = 1, \quad \partial_{h_4} I \cdot \partial_{t_4} N + \partial_{t_4} I = 0.$$

Then it follows that

$$\partial_{r_4} h_4 = \partial_{r_4} N = (\partial_{h_4} I)^{-1} \in [c h r_4^{-1}, C h r_4^{-1}].$$

From (3.20) and (3.21), one has

$$c r_4^{-1} N \leq |\partial_{r_4} N| \leq C r_4^{-1} N. \tag{3.28}$$

By direct computation, one has

$$\partial_{r_4}^2 N = -\frac{\partial_{h_4}^2 I \cdot \partial_{r_4} h_4}{(\partial_{h_4} I)^2} = -\frac{\partial_{h_4}^2 I \cdot \partial_{r_4} N}{(\partial_{h_4} I)^2}.$$

From (3.20), (3.21) and (3.28), one has

$$cr_4^{-2}N \le |\partial_{r_4}^2 N| \le Cr_4^{-2}N.$$

Using Leibniz's rule, for $2 \le k \le 6$, $\partial_{r_4}^k N$ is the summation of terms

$$\partial_{h_4}^m \left(\frac{\partial_{h_4}^2 I}{(\partial_{h_4} I)^2} \right) \prod_{i=1}^m \partial_{r_4}^{k_i} N$$

with $0 \le m \le k$, $\sum_{i=1}^m k_i = k-1$, and $k_i \ge 1$, $i = 1, 2, \ldots, m$. By induction, we can obtain (3.26) from (3.20), (3.21) and (3.25).

By (3.24), one has

$$|P| \le Cr_4^{-1}N|R_4| \le C\partial_{r_4}N|R_4|.$$

Differentiating both sides of (3.24) on r_4, one has

$$\begin{aligned} |\partial_{r_4}P| &= \left| -\frac{\left(\partial_{h_4}R_4 + P\int_0^1 \partial_{h_4}^2 I(N+\mu P, t_4)\, d\mu\right) \cdot \partial_{r_4}N}{\int_0^1 \partial_{h_4} I(N+\mu P, t_4)\, d\mu + P\int_0^1 \partial_{h_4}^2 I(N+\mu P, t_4)\mu\, d\mu + \partial_{h_4}R_4} \right| \\ &\le Cr_4^{-1}|R_4| \cdot |\partial_{r_4}N|. \end{aligned}$$

Differentiating both sides of (3.24) on θ, one has

$$\begin{aligned} |\partial_\theta P| &= \left| \frac{-\partial_\theta R_4}{\int_0^1 \partial_{h_4} I(N+\mu P, t_4)\, d\mu + P\int_0^1 \partial_{h_4}^2 I(N+\mu P, t_4)\mu\, d\mu + \partial_{h_4}R_4} \right| \\ &\le C|R_4| \cdot |\partial_{r_4}N|. \end{aligned}$$

Similar to the proof of (3.26), by induction, we can prove (3.27) for $2 \le k+j \le 5$ using Leibniz's rule. □

4. Proof of Theorem 1.1

In this section, we will prove Theorem 1.1 by Moser's small twist theorem. Without leading to confusion, we denote (r_4, h_4, t_4) by (ρ, H, τ) in (3.23). The system with Hamiltonian (3.23) is

$$\frac{d\theta}{d\tau} = \partial_\rho N(\rho, \tau) + \partial_\rho P(\rho, \theta, \tau), \quad \frac{d\rho}{d\tau} = -\partial_\theta P(\rho, \theta, \tau). \tag{4.1}$$

Integrate the system (4.1) by τ from 0 to 2π, the Poincaré map P of system (4.1) is of the form

$$\theta_1 = \theta + \Lambda(\rho) + F_1(\rho, \theta), \quad \rho_1 = \rho + F_2(\rho, \theta) \tag{4.2}$$

with $(\rho,\theta)=(\rho(0),\theta(0))$, and

$$\begin{aligned}
\Lambda(\rho) &= \int_0^{2\pi} \partial_\rho N(\rho,\tau)\,d\tau,\\
F_1(\rho,\theta) &= \int_0^{2\pi} \partial_\rho P(\rho(\tau),\theta(\tau),\tau)\,d\tau + \int_0^{2\pi} \partial_\rho N(\rho(\tau),\tau)\,d\tau - \int_0^{2\pi} \partial_\rho N(\rho,\tau)\,d\tau,\\
F_2(\rho,\theta) &= -\int_0^{2\pi} \partial_\theta P(\rho(\tau),\theta(\tau),\tau)\,d\tau.
\end{aligned}$$

By Lemma 3.6, for $k+j\le 4$, $F_1(\rho,\theta)$ and $F_2(\rho,\theta)$ satisfy the following estimates

$$\begin{aligned}
&|\partial_\rho^k\partial_\theta^j F_1(\rho,\theta)| \le C(N\cdot\rho^{-2}+1)\cdot\rho^{-k-1}N\cdot|R_4|,\\
&|\partial_\rho^k\partial_\theta^j F_2(\rho,\theta)| \le C\rho^{-k-1}N\cdot|R_4|.
\end{aligned}$$

Moreover, the following estimates hold true for $\Lambda(\rho)$:

$$\begin{aligned}
&c\rho \le |\Lambda(\rho)| \le C\rho^{(1+d)/(1-d)}, \quad c\le|\Lambda'(\rho)|\le C\rho^{2d/(1-d)},\\
&|\Lambda^{(k)}(\rho)| \le C\rho^{(1+d)/(1-d)-k}, \quad k\le 5.
\end{aligned} \tag{4.3}$$

Denote by $\rho(\Lambda)$ the inverse function of $\Lambda(\rho)$. By (4.3), one has

$$c\Lambda^{(1+d)/(1-d)} \le \rho \le C\Lambda, \quad |\rho^{(k)}(\Lambda)| \le C\Lambda^{-k}|\rho|, \quad k\le 5. \tag{4.4}$$

The Poincaré map (4.2) can be rewritten in the following map

$$\theta_1 = \theta+\Lambda+\widehat{F}_1(\Lambda,\theta), \quad \Lambda_1 = \Lambda+\widehat{F}_2(\Lambda,\theta), \tag{4.5}$$

where

$$\widehat{F}_1(\Lambda,\theta) = F_1(\rho(r),\theta), \quad \widehat{F}_2(\Lambda,\theta) = \int_0^1 \Lambda'(\rho+\lambda F_2(\rho,\theta))F_2(\rho,\theta)\,d\lambda.$$

By Leibniz's rule, (3.22) and (4.4), one has

$$\begin{aligned}
|\partial_\Lambda^k\partial_\theta^j\widehat{F}_1| &\le \sum_{i=1}^k |\partial_\rho^i\partial_\theta^j F_1|\cdot|\rho^{(k_1)}(r)\cdots\rho^{(k_i)}(r)|\\
&\le CN^2\cdot\rho^{-i-3}|R_4|\cdot\rho^i\cdot\Lambda^{-k}\\
&\le C\rho^{-3}\cdot\Lambda^{-k}\cdot|R_4|\cdot N^2\\
&\le C\rho^{-3}, \quad k+j\le 4
\end{aligned} \tag{4.6}$$

with $\sum_{l=1}^i k_l = k$. Similarly, one has

$$|\partial_\Lambda^k\partial_\theta^j\widehat{F}_2| \le C\rho^{-3}, \quad k+j\le 4. \tag{4.7}$$

By (3.5), we have the estimate

$$|\partial_{h_4}^k\partial_\theta^j\partial_{t_4}^l R| \le Ch_4^{-\upsilon/2-k}, \quad k\le 5,\ k+j\le 5,\ l\le 4.$$

Since $N=h_4$, thus for $j\le 5$, $k+j\le 4$, (4.6) and (4.7) yield that

$$|\partial_\Lambda^k\partial_\theta^j\widehat{F}_i| \le C\rho^{-3}, \quad i=1,2.$$

Proof of Theorem 1.1. The map (4.5) satisfies all the conditions of Moser's small twist theorem [10]. Thus we obtain the boundedness result of Theorem 1.1. □

Acknowledgments

This work was supported by the NSFC (grant no. 11801295). The authors are grateful to the referees for their valuable suggestions and comments.

References

[1] J. M. Alonso and R. Ortega, *Unbounded solutions of semilinear equations at resonance*, Nonlinearity **9** (1996), no. 5, 1099–1111.

[2] ______, *Roots of unity and unbounded motions of an asymmetric oscillator*, J. Differential Equations **143** (1998), no. 1, 201–220.

[3] E. N. Dancer, *On the Dirichlet problem for weakly non-linear elliptic partial differential equations*, Proc. Roy. Soc. Edinburgh Sect. A **76** (1977), no. 4, 283–300.

[4] C. Fabry and J. Mawhin, *Oscillations of a forced asymmetric oscillator at resonance*, Nonlinearity **13** (2000), no. 3, 493–505.

[5] L. Jiao, D. Piao and Y. Wang, *Boundedness for the general semilinear Duffing equations via the twist theorem*, J. Differential Equations **252** (2012), no. 1, 91–113.

[6] A. C. Lazer and D. E. Leach, *Bounded perturbations of forced harmonic oscillators at resonance*, Ann. Mat. Pura Appl. (4) **82** (1969), 49–68.

[7] A. C. Lazer and P. J. McKenna, *Large-amplitude periodic oscillations in suspension bridges: Some new connections with nonlinear analysis*, SIAM Rev. **32** (1990), no. 4, 537–578.

[8] B. Liu, *Boundedness in asymmetric oscillations*, J. Math. Anal. Appl. **231** (1999), no. 2, 355–373.

[9] ______, *Boundedness in nonlinear oscillations at resonance*, J. Differential Equations **153** (1999), no. 1, 142–174.

[10] J. Moser, *On invariant curves of area-preserving mappings of an annulus*, Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II **1962** (1962), 1–20.

[11] R. Ortega, *Asymmetric oscillators and twist mappings*, J. London Math. Soc. (2) **53** (1996), no. 2, 325–342.

[12] X. P. Wang, *Invariant tori and boundedness in asymmetric oscillations*, Acta Math. Sin. (Engl. Ser.) **19** (2003), no. 4, 765–782.

[13] Z. Wang, Y. Wang and D. Piao, *A new method for the boundedness of semilinear Duffing equations at resonance*, Discrete Contin. Dyn. Syst. Ser. **36** (2016), no. 7, 3961–3991.

[14] X. Xing, J. Wang and Y. Wang, *Boundedness of semilinear Duffing equations at resonance in a critical situation*, J. Differential Equations **266** (2019), no. 4, 2294–2326.

[15] X. Zhang, Y. Peng and D. Piao, *Quasi-periodic solutions for the general semilinear Duffing equations with asymmetric nonlinearity and oscillating potential*, Sci. China Math. **64** (2021), no. 5, 931–946.

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