On the Exponential Diophantine Equation $F_{n+1}^x - F_{n-1}^x = F_m^y$

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Abstract. In this paper, we find all the solutions of Diophantine equation written in our title, where n, m, x and y are positive integers and F_k denotes the kth term of the Fibonacci sequence.

1. Introduction

Let $(F_n)_{n \geq 0}$ be the classical *Fibonacci sequence* given by

$$(1.1) \qquad F_{n+2} = F_{n+1} + F_n \quad \text{for all } n \geq 0$$

and initial values $F_0 = 0$, $F_1 = 1$.

The Fibonacci numbers satisfy numerous identities that have been discovered over centuries. For instance, in 1876, E. Lucas [12] showed that for all $n \geq 1$,

$$(1.2) \qquad F_n^2 + F_{n+1}^2 = F_{2n+1},$$

$$(1.3) \qquad F_{n+1}^2 - F_{n-1}^2 = F_{2n}.$$

In the last decade, equation (1.2) has been revisited in the context of exponential Diophantine equation $A^x + B^y = C^z$ for A, B, C positive integers which are members of the Fibonacci sequence or generalizations. Indeed, Marques and Togbé [13], investigated analogues of Diophantine equation (1.2) in higher powers. They proved that if $x \geq 1$ is an integer such that $F_n^x + F_{n+1}^x$ is a Fibonacci number for all sufficiently large n, then $x \in \{1, 2\}$. Later, Luca and Oyono [10] extended the above result on the nonexistence of positive integer solutions (n, m, x) to the Diophantine equation $F_n^x + F_{n+1}^x = F_m$ with

$n \geq 2$, $m \geq 1$ and $x \geq 3$. Subsequently, Chaves and Marques [3] proved that the analogues of the Diophantine equation (1.2), in k-generalized Fibonacci numbers[1] instead of Fibonacci numbers, has no positive integer solution (k, n, m) with $k \geq 3$ and $n \geq 1$. Then, Gómez and Luca [8] solved completely the Diophantine equation $(F_n^{(k)})^x + (F_{n+1}^{(k)})^x = F_m^{(k)}$ in k-generalized Fibonacci numbers, showing that it has no solutions. In the same direction, in [2] it was proved that the equation $(F_n^{(k)})^2 + (F_{n+1}^{(k)})^2 = F_m^{(l)}$ has no solutions in positive integers with $2 \leq k < l$ and $n \geq 2$. Similar problems with different recurrent sequences have been considered in [4, 6, 19–21].

On the other hand, identity (1.2) was studied from the point of view of Diophantine equations: $F_n^x + F_{n+1}^x = F_m^y$ by Luca and Oyono in [9]; $F_n^x + F_{n+1}^y = F_m^x$ and $F_n^y + F_{n+1}^x = F_m^x$ by Hirata-Kohno and Luca in [11]; and $F_n^x + F_{n+1}^y = F_{2n+1}^z$ by Miyazaki in [16], all in positive integers n, m, x, y and z.

Recently, identity (1.3) was investigated in the form

$$F_{n+1}^x - F_n^x = F_m$$

by Patel and Chaves in [18]. Here, we consider adding an extra exponent y on the right-hand side, so we study the equation

(1.4) $$F_{n+1}^x - F_{n-1}^x = F_m^y$$

in positive integers n, m, x and y.

Following the argument used in [9], we establish our main theorem.

Theorem 1.1. *The only positive integers solutions (n, m, x, y) of Diophantine equation (1.4) correspond to the trivial ones in Section 3.1.*

The proof requires mainly the use of lower bounds for nonzero linear forms in logarithms of algebraic numbers as well as reduction methods based on the algorithms of Baker and Davenport [1] and LLL [5, Section 2.3.5].

2. Preliminary results

2.1. The Fibonacci sequence

Recall that the characteristic equation of the Fibonacci sequence is $x^2 - x - 1 = 0$. Let α and β to denote its roots, with $\alpha := (1 + \sqrt{5})/2$. We have the Binet formula

(2.1) $$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}} \quad \text{for all } n \geq 0.$$

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[1]The k-generalized Fibonacci sequence $F^{(k)}$, for an integer $k \geq 2$, satisfies that its first k terms are $0, \dots, 0, 1$ and each term afterwards is the sum of the preceding k terms. For $k = 2$, this reduces to the familiar *Fibonacci* numbers, while for $k = 3$ these are the *Tribonacci* numbers. They are followed by the *Tetranacci* numbers for $k = 4$, and so on.

The Fibonacci companion sequence $(L_n)_{n \geq 0}$ known as the *Lucas sequence*, given by $L_0 = 2$, $L_1 = 1$ and $L_{n+2} = L_{n+1} + L_n$ for all $n \geq 0$, has the Binet formula

$$L_n = \alpha^n + \beta^n \quad \text{for all } n \geq 0.$$

We will make use of some of the several properties and relations that this two sequences fulfill.

Lemma 2.1. (i) $\alpha^{n-2} \leq F_n \leq \alpha^{n-1}$ *for all* $n \geq 1$.

(ii) $F_{n-1}/F_{n+1} \leq 2/5$ *for all* $n \geq 3$.

(iii) F_k *is divisible by a prime which is at least as large as* $k - 1$ *for all* $k \geq 13$.

(iv) $L_n^2 - 5F_n^2 = 4(-1)^n$ *for all* $n \geq 0$.

2.2. Linear forms in logarithms

For any nonzero algebraic number γ of degree d over $\mathbb{Q}$, with minimal polynomial over $\mathbb{Z}$ given by $a \prod_{j=1}^{d} \left(X - \gamma^{(j)} \right)$ where $a \geq 1$, we denote by

$$h(\gamma) = \frac{1}{d} \left(\log a + \sum_{j=1}^{d} \log \max \left\{ 1, \left| \gamma^{(j)} \right| \right\} \right)$$

the usual absolute logarithmic height of γ. With the previous notation, the following theorem is a result of Matveev from [14].

Theorem 2.2. *Let* $s \geq 1$, $\gamma_1, \ldots, \gamma_s$ *be real algebraic numbers and let* $b_1, \ldots, b_s$ *be nonzero integers. Let* D *be the degree of the number field* $\mathbb{Q}(\gamma_1, \ldots, \gamma_s)$ *over* $\mathbb{Q}$ *and let* A_j *be a positive real number satisfying*

$$A_j \geq \max\{Dh(\gamma_j), |\log \gamma_j|, 0.16\} \quad \text{for } j = 1, \ldots, s.$$

Assume that

$$B \geq \max\{|b_1|, \ldots, |b_s|\}.$$

If $\gamma_1^{b_1} \cdots \gamma_s^{b_s} - 1 \neq 0$, *then*

$$|\gamma_1^{b_1} \cdots \gamma_s^{b_s} - 1| \geq \exp(-1.4 \cdot 30^{s+3} \cdot s^{4.5} \cdot D^2 (1 + \log D)(1 + \log B) A_1 \cdots A_s).$$

2.3. Reduction methods

The first result is Lemma 5(a) in [7].

Lemma 2.3. *Let T be a positive integer, p/q be a convergent of the continued fraction of the irrational γ such that $q > 6T$ and A, B, μ be some real numbers with $A > 0$ and $B > 1$. Let*

$$\varepsilon := \|\mu q\| - T\|\gamma q\|,$$

where $\|\cdot\|$ denotes the distance from the nearest integer. If $\varepsilon > 0$, then there is no solution of the inequality

$$0 < |m\gamma - n + \mu| < AB^{-k}$$

in positive integers m, n and k with

$$m \leq T \quad and \quad k \geq \frac{\log(Aq/\varepsilon)}{\log B}.$$

Note that the above lemma cannot be applied when $\mu = 0$. In this case, we use the following classical results in the theory of Diophantine approximation, where the item (i) is the wellknown Legendre criterion (see Theorem 8.2.4 in [17]), while the item (ii) follows from inequality (3.5) in De Weger's doctoral dissertation [22].

Lemma 2.4. *Let M be a positive integer, $p_1/q_1, p_2/q_2, \ldots$ the convergents of an irrational γ, and $[a_0, a_1, \ldots]$ its continued fraction.*

(i) *Let x and y be integers such that*

$$\left|\gamma - \frac{x}{y}\right| < \frac{1}{2y^2}.$$

Then $x/y = p_k/q_k$ for some positive integer k.

(ii) *If $a_M := \max\{a_t : 0 \leq t \leq K+1\}$, where K is a positive integer such that $q_{K+1} > M$. Then*

$$\left|\gamma - \frac{x}{y}\right| > \frac{1}{(a_M + 2)y^2}$$

for every rational number x/y with $1 \leq y < M$.

Since there are no methods based on continued fractions to find a lower bound for linear forms in three variables with bounded integer coefficients, we need to use a method based on the LLL-algorithm (see Proposition 2.3.20 in [5, Section 2.3.5]).

Let $\gamma_1, \ldots, \gamma_s \in \mathbb{R}$ and the linear form

$$x_1\gamma_1 + x_2\gamma_2 + \cdots + x_s\gamma_s \quad \text{with } |x_i| \leq X_i.$$

We set $X := \max\{X_i\}$ and $C > (sX)^s$. Let Ω be the integer lattice generated by

$$\mathbf{b}_j := \mathbf{e}_j + \lfloor C\gamma_j \rfloor \mathbf{e}_s \text{ for } 1 \le j \le s - 1 \quad \text{and} \quad \mathbf{b}_s := \lfloor C\gamma_s \rfloor \mathbf{e}_s$$

with C a sufficiently large positive constant.

Lemma 2.5. *With the above notation, we consider $\{\mathbf{b}_i\}$ as a reduced basis of Ω and $\{\mathbf{b}_i^*\}$ as its associated Gram–Schmidt basis. We set*

$$c_1 := \max_{1 \le i \le s} \frac{\|\mathbf{b}_1\|}{\|\mathbf{b}_i^*\|}, \quad \mathbf{m}_\Omega := \frac{\|\mathbf{b}_1\|}{c_1}, \quad Q := \sum_{i=1}^{s-1} X_i^2 \quad and \quad T := \frac{1}{2}\left(1 + \sum_{i=1}^{s} X_i\right).$$

If the integers x_i satisfy that $|x_i| \le X_i$ for $i = 1, \ldots, t$ and $\mathbf{m}_\Omega^2 \ge T^2 + Q$, then we have

$$\left|\sum_{i=1}^{s} x_i \gamma_i\right| \ge \frac{\sqrt{\mathbf{m}_\Omega^2 - Q} - T}{C}.$$

3. The trivial solutions, nonzero linear forms and relations among indeterminates

3.1. The trivial solutions

Since $y = 1$ was solved by [18], we may assume that $y \ge 2$. Let us take $n = 1$. Then, we have

$$F_m^y = F_2^x - F_0^x = 1,$$

which implies $m \in \{1, 2\}$ for every $x \ge 1$ and $y \ge 2$. If we take $n = 2$, then

$$F_m^y = F_3^x - F_1^x = 2^x - 1.$$

For $x = 1$, we get $m \in \{1, 2\}$ and $y \ge 2$. However, there are no solutions for $x \ge 2$ by the solution to Catalan's conjecture [15]. Now, let us take $n = 3$. So, we have

$$F_m^y = F_4^x - F_2^x = 3^x - 1,$$

which, according to Catalan conjecture, implies $F_m = 2$; i.e., $m = 3$, and, therefore, $x = 2$ and $y = 3$. From now on we assume that $n \ge 4$.

On the other hand, if $x \in \{1, 2\}$, then

$$F_m^y \in \{F_{n+1} - F_{n-1}, F_{n+1}^2 - F_{n-1}^2\} = \{F_n, F_{2n}\}.$$

Thus, we are looking for Fibonacci numbers which are perfect powers and the only such are 1, 8 and 144. So, we have $F_m^y \in \{1, 8, 144\}$. Therefore, we get $m = 1$ and $y \ge 2$, or, $m = 3$ and $y = 3$, respectively. Thus, we may assume that $x \ge 3$. By Wiles' solution of Fermat's last theorem [23] it follows that $x \ne y$.

Finally, if we assume that $m \in \{1, 2\}$, equation (1.4) implies

$$1 = F_{n+1}^x - F_{n-1}^x = (F_n + F_{n-1})^x - F_{n-1}^x \geq F_n^x.$$

Therefore, we have $n = 1$ with $x \geq 1$ or $n = 2$ with $x = 1$, that we had already found. Thus, we may assume that $m \geq 3$.

To sum up, the trivial positive integers solutions (n, m, x, y) of equation (1.4) correspond to those in the set:

$$\{(1, 1, x, y), (1, 2, x, y), (2, 1, 1, y), (2, 2, 1, y) : x, y \geq 1\} \cup \{(3, 3, 2, 3), (6, 3, 1, 2)\},$$

along with the parametric ones in $\{(n, n, 1, 1), (n, 2n, 2, 1) : n \geq 1\}$ that come from the identities (1.1) and (1.3).

From now on, we work with $n \geq 4$, $m \geq 3$, $x \geq 3$ and $y \geq 2$ with $x \neq y$.

3.2. Nonzero linear forms

In our work, we apply Theorem 2.2, Matveev's result, with $\mathbb{K} := \mathbb{Q}$ or $\mathbb{Q}(\sqrt{5})$ (so $D := 1$ or 2) to the following linear forms in logarithms of algebraic numbers (written in exponential form):

$$\Lambda_1 := 1 - F_{n+1}^{-x} F_m^y,$$
$$\Lambda_2 := F_M^w \alpha^{-Nz} 5^{z/2} - 1,$$
$$\Lambda_3 := F_m^y \alpha^{-nx} ((\alpha^x - \alpha^{-x}) / 5^{x/2})^{-1} - 1,$$
$$\Lambda_4 := \alpha^{my - (n+1)x} 5^{(x-2)/2} - 1,$$
$$\Lambda_5 := \alpha^{my - nx} 5^{(x-y)/2} (\alpha^x - \alpha^{-x})^{-1} - 1$$

with

(3.1) $$M := \min\{n + 1, m\} \quad \text{and} \quad N := \max\{n + 1, m\}.$$

Besides, we denote $w, z \in \{x, y\}$ such that (w, M), (z, N) are the two pairs $(x, n + 1)$, (y, m). Clearly, $M \geq 3$. The notation (3.1) will be used in the rest of this work.

Below, we show that the above linear forms in logarithms are nonzero.

Lemma 3.1. $\Lambda_j \neq 0$ for all $j = 1, 2, 3, 4, 5$.

Proof. By equation (1.4), it is clear that $\Lambda_1 = (F_{n-1}/F_{n+1})^x > 0$. For the rest of the linear forms we proceed by contradiction. If $\Lambda_2 = 0$, then, after squaring, we get $\alpha^{2Nz} = F_M^{2w} 5^z \in \mathbb{Z}$. However, no power of positive integer exponent of α is an integer. Next, if $\Lambda_3 = 0$, then $\alpha^{2(n-1)x}(\alpha^{2x} - 1)^2 = F_m^{2y} 5^x \in \mathbb{Z}$, and by conjugation in $\mathbb{Q}(\sqrt{5})$ we get

$\alpha^{2(n-1)x}(\alpha^{2x}-1)^2 = \beta^{2(n-1)x}(\beta^{2x}-1)^2$, which, due to the fact that $x \geq 3$ and $n \geq 4$, is not possible (the right-hand side is at most 4 and the left-hand side is larger). If $\Lambda_4 = 0$, then we get $\alpha^{2my-2(n+1)x} = 5^{x-y} \in \mathbb{Z}$, which implies $my = (n+1)x$ and $x = y$, but the last equality is not allowed. Finally, if $\Lambda_5 = 0$, then we get that $\alpha^x - \alpha^{-x} = 5^{(x-y)/2}\alpha^{mx-ny}$. Conjugating the above relation in $\mathbb{Q}(\sqrt{2})$, we get $\beta^x - \beta^{-x} = 5^{(x-y)/2}\beta^{mx-ny}$. Hence, multiplying the two previous relations, we get

$$2(-1)^x - (-1)^x\alpha^{2x} - (-1)^x\beta^{2x}$$
$$= (\alpha^x - \alpha^{-x})(\beta^x - \beta^{-x}) = (\alpha\beta)^{my-nx}5^{x-y} = (-1)^{my-nx}5^{x-y},$$

which implies

(3.2) $$(-1)^x(2 - L_{2x}) = (-1)^{my-nx}5^{x-y}.$$

Since the left-hand side of equation (3.2) is an integer and $x \neq y$, we get that $x > y$. Taking absolute values we obtain

$$L_{2x} = 2 + 5^{x-y}.$$

Reducing the above equation modulo 5 we get $L_{2x} \equiv 2 \pmod 5$, so $x = 2x_1$ is even (the Lucas sequence is periodic modulo 5 with period 4). Thus,

$$5^{x-y} = L_{4x_1} - 2 = 5F_{2x_1}^2.$$

It follows that F_{2x_1} is a power of 5. By the Primitive Divisor Theorem (see Lemma 2.1(iii)), the only possibilities are $x_1 = 1$ (for which $F_{2x_1} = F_2 = 1 = 5^0$) which is not convenient since for us $x \geq 3$, or $2x_1 = 5$, which is impossible since it does not lead to an integer solution x_1. $\qquad\square$

3.3. A relation between the indeterminates n, m, x, y

Lemma 3.2. *If (n, m, x, y) is a solution of (1.4) with $n \geq 4$, $m \geq 3$, $x \geq 3$ and $y \geq 2$, then the inequality*

(3.3) $$|nx - my| < 2\max\{x, y\}$$

holds.

Proof. By (1.4) and Lemma 2.1(i), we have

$$\alpha^{(m-2)y} < F_m^y = F_{n+1}^x - F_{n-1}^x < F_{n+1}^x < \alpha^{xn},$$

and

$$\alpha^{(n-2)x} < F_n^x < F_{n+1}^x - F_{n-1}^x = F_m^y < \alpha^{(m-1)y},$$

which imply

$$(m-2)y < nx \quad \text{and} \quad (n-2)x < (m-1)y.$$

Therefore

$$-2y < nx - my \quad \text{and} \quad nx - my < 2x - y < 2x,$$

obtaining inequality (3.3).   □

3.4. Bounds on x and y in terms of M and N

In this section, we will prove the following lemma.

Lemma 3.3. *If (n, m, x, y) is a solution in positive integers of equation (1.4), with $n \geq 4$, $m \geq 2$, $x \geq 3$ and $y \geq 2$, then both inequalities*

$$x < 1.2 \times 10^{11} MN \log N \quad \text{and} \quad y < 1.2 \times 10^{11} MN^2 \log N$$

hold.

Proof. To start, by (1.4) and Lemma 2.1(ii), we get

$$(3.4) \qquad\qquad \Lambda_1 \leq \frac{1}{2.5^x}$$

for Λ_1 given in Section 3.2. To get a lower bound for Λ_1, we apply Matveev's Theorem 2.2 with the data

$$s := 2, \quad \gamma_1 := F_m, \quad \gamma_2 := F_{n+1}, \quad b_1 = y, \quad b_2 = -x.$$

Here, we have $D = 1$, since γ_1 and γ_2 are integers. Also, we can take $A_1 := m \log \alpha$ and $A_2 := n \log \alpha$. Now, by Matveev's result, we have

$$(3.5) \qquad \begin{aligned} |\Lambda_1| &> \exp(-1.4 \times 30^5 \times 2^{4.5}(m \log \alpha)(n \log \alpha)(1 + \log \max\{x, y\})) \\ &> \exp(-1.2 \times 10^9 \times m \times n \times \log(\max\{x, y\})), \end{aligned}$$

where we have used the fact that $1 + \log \max\{x, y\} < 2 \log \max\{x, y\}$ which holds because $\max\{x, y\} \geq x \geq 3$. Comparing inequalities (3.4) and (3.5), we obtain

$$(3.6) \qquad\qquad x < 1.31 \times 10^9 mn \log \max\{x, y\}.$$

If $x > y$, the above inequality gives

$$(3.7) \qquad\qquad x < 1.31 \times 10^9 mn \log x.$$

If instead we consider $y > x$, by Lemma 3.2, we get $|my - nx| < 2y$. Now, since $m \geq 3$, we obtain

$$(3.8) \qquad\qquad y \leq (m-2)y < nx < Nx.$$

Since $x \geq 3$ the case $x = y$ is not possible according to Fermat's Last Theorem. So, in this case the inequality (3.6) implies

$$(3.9) \qquad x < 1.31 \times 10^9 mn \log(Nx).$$

Now, if we assume

$$(3.10) \qquad x \leq N,$$

then we have an even better upper bound on x. Therefore, we proceed under the assumption that $x > N$ and inequality (3.9) gives us

$$(3.11) \qquad x < 1.31 \times 10^9 mn \log(Nx) < 2.65 \times 10^9 mn \log x.$$

Thus, comparing inequalities (3.7), (3.10) and (3.11), we can conclude that (3.11) holds in all cases. So, we have

$$\frac{x}{\log x} < 2.65 \times 10^9 mn.$$

Due to the fact that, for all $x > 2$, $x/\log x < A$ implies $x < 2A \log A$, whenever $A \geq 3$, the previous inequality gives us

$$x < 5.30 \times 10^9 mn \log(2.65 \times 10^9 N^2) < 1.2 \times 10^{11} mn \log N,$$

where we have used the fact that $\log(2.65 \times 10^9 N^2) < 22 \log N$ for all $N \geq 3$. Finally, by estimate (3.8), we obtain

$$y < Nx < 1.2 \times 10^{11} MN^2 \log N,$$

as we wanted to prove. $\qquad \square$

Due to Section 3.1, to complete the proof of Theorem 1.1, it remains to show that there are no solutions for equation (1.4) when $n \geq 4$, $m \geq 3$, $x \geq 3$ and $y \geq 2$. In order to do it, we proceed by cases over N.

4. The case N big: $N > 1200$

4.1. Bounds for the variables in terms of M

First, we need to prove the following lemma which give us some bounds for x, y and N in terms of M.

Lemma 4.1. *If (n, m, x, y) is a solution to (1.4) with $n \geq 4$, $x \geq 3$, $y \geq 2$ and $N > 1200$, then*

$$N < 2.4 \times 10^{30} M^2 \log^2 M,$$
$$x < 3.9 \times 10^{27} M^2 \log^2 M,$$
$$y < 6.9 \times 10^{45} M^2 \log^3 M,$$
$$\max\{Mw, Nz\} < 6.9 \times 10^{45} M^3 \log^3 M.$$

Proof. From Lemma 3.3, we have

(4.1) $$\max\{x, y\} < 1.2 \times 10^{11} N^3 \log N < \alpha^N,$$

where the right-hand inequality holds for all $N \geq 84$. Clearly, inequality (4.1) implies that

(4.2) $$\frac{z}{\alpha^{2N}} < \frac{1}{\alpha^N}.$$

Now, since $\beta = -\alpha^{-1}$, by the Binet formula (2.1) we get

$$F_N^z = \frac{\alpha^{Nz}}{5^{z/2}} \left(1 - \frac{(-1)^N}{\alpha^{2N}}\right)^z = \frac{\alpha^{Nz}}{5^{z/2}} \exp\left(z \log\left(1 - \frac{(-1)^N}{\alpha^{2N}}\right)\right).$$

If we take

$$\varepsilon_{N,z} := \left(1 - \frac{(-1)^N}{\alpha^{2N}}\right)^z - 1,$$

by the argument in [9], we deduce that

(4.3) $$F_N^z = \frac{\alpha^{Nz}}{5^{z/2}}(1 + \varepsilon_{N,z}) \quad \text{with } |\varepsilon_{N,z}| < \frac{2}{\alpha^N}.$$

Since $x \geq 3$ and $N > 1200$, it is clear from (3.4) and (4.3) that

(4.4) $$\frac{F_m^y}{F_{n+1}^x}, \frac{F_N^z 5^{z/2}}{\alpha^{Nz}} \in (1/2, \sqrt{5}).$$

Let us suppose that $N = n + 1$. Thus, $z = x$ and

$$F_m^y = F_{n+1}^x - F_{n-1}^x = \frac{\alpha^{(n+1)x}}{5^{x/2}} + \left(\frac{\alpha^{(n+1)x}}{5^{x/2}}\right) \varepsilon_{n+1,x} - F_{n-1}^x,$$

so

(4.5)
$$\left|F_m^y \alpha^{-(n+1)x} 5^{x/2} - 1\right| = \left|\varepsilon_{n+1,x} - \frac{F_{n-1}^x 5^{x/2}}{\alpha^{(n+1)x}}\right| < |\varepsilon_{n+1,x}| + \left(\frac{F_{n-1}}{F_{n+1}}\right)^x \left(\frac{F_{n+1}^x 5^{x/2}}{\alpha^{(n+1)x}}\right)$$
$$< \frac{2}{\alpha^{n+1}} + \frac{\sqrt{5}}{2.5^x} \leq \frac{5}{\alpha^\lambda},$$

where we have used the equation (4.4) and the notation

$$(4.6) \qquad\qquad \lambda := \min\{x, N\}.$$

We also used the fact that $\alpha < 2.5$. Let us show that we get a similar inequality when $N = m$. In this case we have $z = y$ and

$$F_{n+1}^x = F_m^y + F_{n-1}^x = \frac{\alpha^{my}}{5^{y/2}} + \left(\frac{\alpha^{my}}{5^{y/2}}\right)\varepsilon_{m,y} + F_{n-1}^x,$$

therefore,

$$\left|F_{n+1}^x \alpha^{-my} 5^{y/2} - 1\right| = \left|\varepsilon_{m,y} + \frac{F_{n-1}^x 5^{y/2}}{\alpha^{my}}\right|$$

$$(4.7) \qquad\qquad < |\varepsilon_{m,y}| + \left(\frac{F_{n-1}}{F_{n+1}}\right)^x \left(\frac{F_{n+1}^x}{F_m^y}\right)\left(\frac{F_m^y 5^{y/2}}{\alpha^{my}}\right)$$

$$< \frac{2}{\alpha^m} + \frac{2\sqrt{5}}{2.5^x} \leq \frac{7}{\alpha^\lambda}$$

with λ given by (4.6). To summarize, by (4.5) and (4.7), we get that

$$(4.8) \qquad\qquad \left|F_M^w \alpha^{-Nz} 5^{z/2} - 1\right| < \frac{7}{\alpha^\lambda},$$

where the expression inside the absolute value corresponds to Λ_2 in Section 3.2. Now, we use Matveev's result to get a lower bound on $|\Lambda_2|$. In order to do this, let us take

$$s := 3, \quad \gamma_1 := F_M, \quad \gamma_2 := \alpha, \quad \gamma_3 := \sqrt{5}, \quad b_1 := w, \quad b_2 := -Nz, \quad b_3 := z.$$

Since γ_1, γ_2, γ_3 belong to the field $\mathbb{K} = \mathbb{Q}(\sqrt{5})$, we can take $D = 2$. Besides, since $F_M < \alpha^M$, we can take

$$A_1 = 2M \log \alpha > D \log F_M = Dh(\gamma_1).$$

Note that $h(\gamma_2) = (\log \alpha)/2$ and $h(\gamma_3) = (\log 5)/2$, so we can take $A_2 = \log \alpha$ and $A_3 = \log 5$. Even more, by Lemma 3.3, we have

$$\max\{Nz, z, w\} < (1.2 \times 10^{11} MN^2 \log N) \times N$$

$$= (1.2 \times 10^{11} \times \log N) \times MN^3$$

$$< N^4 \times N^4 = N^8,$$

where we have used the fact that $1.2 \times 10^{11} \log N < N^4$ for all $N > 1200$. Thus, we can take $B := N^8$. Finally, by Matveev's result, we have

$$(4.9) \qquad |\Lambda_2| > \exp\left(-5.7 \times 10^{11}(1 + \log 2)(1 + 8\log N)(2M \log \alpha)(\log \alpha)(\log 5)\right)$$

$$> \exp(-6.5 \times 10^{12} \times M \log N),$$

where we used the fact that $1 + 8 \log N < 9 \log N$ for all $N \geq 3$. Now, we compare (4.8) and (4.9) to get

$$(4.10) \qquad \lambda < 1.4 \times 10^{13} M \log N.$$

Let us use inequality (3.3) to get some bounds on Mw in terms of Nz. To start, if $z > w$, then we have

$$Mw \leq (N + 2)z \leq 2Nz,$$

while if $z < w$, then, due to the fact that $M \geq 3$, we have

$$\frac{Mw}{3} \leq (M - 2)w \leq Nz.$$

From the above inequalities we conclude that $Mw \leq 3Nz$ always holds. A similar argument and the fact that $N \geq 4$, allows us to show that

$$Nz \leq 2Mw,$$

therefore we have that

$$(4.11) \qquad \frac{Nz}{Mw} \in (1/3, 2).$$

At this point we need to proceed by cases according to the different options that we are making along the way.

Case $\lambda = N$. By inequality (4.10), we get

$$N < 1.4 \times 10^{13} M \log N.$$

Thus, we get

$$
\begin{aligned}
(4.12) \qquad N &< 2 \times 1.4 \times 10^{13} M \log(1.4 \times 10^{13} M) \\
&< 2.8 \times 10^{13} M (30 \log M) \\
&< 8.4 \times 10^{14} M \log M.
\end{aligned}
$$

Therefore, by Lemma 3.3, we get

$$
\begin{aligned}
(4.13) \qquad x &< 1.2 \times 10^{11} MN \log N \\
&< 1.2 \times 10^{11} M (8.4 \times 10^{14} M \log M) \log(8.4 \times 10^{14} M \log M) \\
&< 1.1 \times 10^{26} M^2 \log M (35 \log M) \\
&< 3.9 \times 10^{27} M^2 \log^2 M,
\end{aligned}
$$

where we have used the fact that $\log M < M$ and $8.4 \times 10^{14} M^2 < M^{35}$ for all $M \geq 3$. So, if $w = x$, we have

$$(4.14) \qquad Mw = Mx < 3.9 \times 10^{27} M^3 \log^2 M,$$

or, if $w = y$, then $z = x$ and

$$(4.15) \qquad \begin{aligned} Nz = Nx &< (8.4 \times 10^{14} M \log M)(3.9 \times 10^{27} M^2 \log^2 M) \\ &< 3.3 \times 10^{42} M^3 \log^3 M. \end{aligned}$$

Thus, by (4.11) and estimates (4.14) and (4.15), we deduce that

$$(4.16) \qquad \max\{Nz, Mw\} < 10^{43} M^3 \log^3 M.$$

Since $My \leq \max\{Mw, Nz\}$, it is clear from the previous inequality that

$$(4.17) \qquad y < 10^{43} M^2 \log^3 M.$$

Case $\lambda = x$. In this case, from inequality (4.10), we get

$$(4.18) \qquad x < 1.4 \times 10^{13} M \log N,$$

and we have to consider the following two subcases:

• When $N = m$. Therefore, we have $M = n + 1$. If we assume that $x > y$, then, by inequality (3.3) and the fact that $y \geq 2$, we have

$$m \leq \frac{my}{2} < \frac{(n+3)x}{2} < (n+1)x = Mx,$$

which, together with inequality (4.18), implies $N < 1.4 \times 10^{13} M^2 \log N$. If instead we assume that $x < y$, then by inequality (3.3) and the fact that, in this case, $y \geq 3$, we have

$$m \leq my - 2y < (n+1)x = Mx,$$

so, again by inequality (4.18), we get $N < 1.4 \times 10^{13} M^2 \log N$. Either way, we can conclude that

$$(4.19) \qquad N < 1.4 \times 10^{13} M^2 \log N.$$

• When $N = n + 1$. Let us start by pointing out that, either $y > x$ or $x > y$ holds. By an argument similar to the one used in the analysis of the previous subcase around inequality (3.3), we get that inequality

$$(4.20) \qquad y < Nx$$

holds in this case. Further, observe that $z = x$. Now, we have

$$F_{n-1}^x = \frac{\alpha^{(n-1)x}}{5^{x/2}} \left(1 - \frac{(-1)^{n-1}}{\alpha^{2(n-1)}}\right)^x,$$

and, by (4.2), we also have

$$\frac{x}{\alpha^{2n-1}} = \frac{x}{\alpha^{2N-4}} < \frac{\alpha^4}{\alpha^N}.$$

Following the argument to get (4.3), in this case we have

$$F_{n-1}^x = \frac{\alpha^{(n-1)x}}{5^{x/2}}(1 + \varepsilon_{n-1,x}), \quad \text{where } |\varepsilon_{n-1,x}| < \frac{2\alpha^4}{\alpha^N}.$$

Next, we rewrite equation (1.4) as follows

$$F_m^y = F_{n+1}^x - F_{n-1}^x = \frac{\alpha^{nx}(\alpha^x - \alpha^{-x})}{5^{x/2}} + \left(\frac{\alpha^{(n+1)x}}{5^{x/2}}\right)\varepsilon_{n+1,x} - \left(\frac{\alpha^{(n-1)x}}{5^{x/2}}\right)\varepsilon_{n-1,x},$$

so, we can get

$$\left|F_m^y \alpha^{-nx}\left(\frac{5^{x/2}}{\alpha^x - \alpha^{-x}}\right) - 1\right| < |\varepsilon_{n+1,x}|\left(\frac{\alpha^x}{\alpha^x - \alpha^{-x}}\right) + |\varepsilon_{n-1,x}|\left(\frac{\alpha^{-x}}{\alpha^x - \alpha^{-x}}\right),$$

which, due to the fact that

$$(4.21) \qquad |\varepsilon_{n+1,x}| < \frac{2}{\alpha^N}, \quad |\varepsilon_{n-1,x}| < \frac{2\alpha^4}{\alpha^N}, \quad \frac{\alpha^x}{\alpha^x - \alpha^{-x}} < \alpha, \quad \frac{\alpha^{-x}}{\alpha^x - \alpha^{-x}} < 1,$$

with $x \geq 3$, implies

$$(4.22) \qquad \left|F_m^y \alpha^{-nx}\left(\frac{5^{x/2}}{\alpha^x - \alpha^{-x}}\right) - 1\right| < \frac{2\alpha + 2\alpha^4}{\alpha^N} < \frac{17}{\alpha^N}.$$

Note that, the expression inside the absolute value corresponds to Λ_3 defined in Section 3.2. We now use Matveev's result to get a lower bound to the expression on the left-hand side of previous inequality. In order to do this, we take

$$s := 3, \quad \gamma_1 := F_m, \quad \gamma_2 := \alpha, \quad \gamma_3 := (\alpha^x - \alpha^{-x})/5^{x/2}, \quad b_1 := y, \quad b_2 := -nx, \quad b_3 := -1.$$

Since $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{Q}(\sqrt{5})$, we can take $D = 2$ and as in the previous application we can take

$$A_1 := 2M\log\alpha \quad \text{and} \quad A_2 := \log\alpha.$$

Concerning γ_3, since its conjugate in $\mathbb{Q}(\sqrt{5})$ is $(-1)^x(\beta^x - \beta^{-x})/5^{x/2}$, it's minimal polynomial over the integers is a divisor of

$$a_0 X^2 + a_1 X + a_2 = 5^x\left(X - \frac{\alpha^x - \alpha^{-x}}{5^{x/2}}\right)\left(X - (-1)^x\frac{\beta^x - \beta^{-x}}{5^{x/2}}\right)$$
$$= 5^x X^2 - 5^{x/2}(\alpha^x + (-1)^x\beta^x - \alpha^{-x} - (-1)^x\beta^{-x})X + (2 - \alpha^{2x} - \beta^{2x}).$$

Thus, we have $a_0 \leq 5^x$,

$$|\gamma_3^{(1)}| = |\gamma_3| = \frac{\alpha^x - \alpha^{-x}}{5^{x/2}} < \left(\frac{\alpha}{\sqrt{5}}\right)^x < 1$$

and

$$|\gamma_3^{(2)}| = \frac{|\beta^x - \beta^{-x}|}{5^{x/2}} < \frac{\alpha^x + 1}{5^{x/2}} < 1.1 \left(\frac{\alpha}{\sqrt{5}}\right)^x < 1.$$

Therefore, we can take

$$A_3 = \log 5^x = \frac{D \log 5^x}{2} \geq \frac{D \log a_0}{2} \geq Dh(\gamma_3).$$

Now, by inequalities (4.18) and (4.20), and the fact that $1.4 \times 10^{13} \log N < N^5$ holds for all $N > 1200$, we get

$$\max\{|b_1|, |b_2|, |b_3|\} = \max\{y, nx, 1\} < Nx < N \times 1.4 \times 10^{13} \times M \log N$$
$$= (1.4 \times 10^{13} \log N) \times MN < N^5 \times N^2 = N^7.$$

Finally, by Matveev's result, we have that

$$\log |\Lambda_3| > -5.73 \times 10^{11} \times (1 + \log 2)(1 + 7 \log N)(2M \log \alpha)(\log \alpha)(x \log 5)$$
$$> -5.8 \times 10^{12} Mx \log N,$$

where we have used the fact that $1 + 7 \log N < 8 \log N$ holds for all $N > 1200$. Now, we use (4.18) to get

$$\text{(4.23)} \qquad \begin{aligned} \log |\Lambda_3| &> -5.8 \times 10^{12} M(1.4 \times 10^{13} M \log N) \log N \\ &> -8.2 \times 10^{25} M^2 \log^2 N. \end{aligned}$$

By inequalities (4.22) and (4.23), we get that

$$\text{(4.24)} \qquad N < 1.8 \times 10^{26} M^2 \log^2 N.$$

Clearly, between inequalities (4.19) and (4.24), we can conclude that

$$N < 1.8 \times 10^{26} M^2 \log^2 N.$$

Now, we use the fact that $t < A \log^2 t$ implies $t < 4A(\log A)^2$, when $A > 100$. Since $A = 2 \times 10^{26} M^2$, we get that

$$N < 4 \times 1.8 \times 10^{26} M^2 (\log(1.8 \times 10^{26} M^2))^2$$

$$\text{(4.25)} \qquad < 7.2 \times 10^{26} M^2 (2 \log M)^2 \left(\frac{\log(1.8 \times 10^{26})}{2 \log 3} + 1\right)^2$$

$$< 2.4 \times 10^{30} M^2 \log^2 M.$$

Also, by inequality (4.18), we get

$$x < 1.4 \times 10^{13} M \log N$$

$$< 1.4 \times 10^{13} M \log(2.4 \times 10^{30} M^2 \log^2 M)$$

$$\text{(4.26)} \qquad < 1.4 \times 10^{13} M(4 \log M) \left(\frac{\log(2.4 \times 10^{30})}{4 \log 3} + 1\right)$$

$$< 9.5 \times 10^{14} M \log M.$$

So, if $w = x$, then, by the previous inequality

$$(4.27) \qquad Mw = Mx < 9.5 \times 10^{14} M^2 \log M,$$

while if $w = y$, then $z = x$ and, by (4.25), we get

$$(4.28) \qquad \begin{aligned} Nz = Nx &< 2.4 \times 10^{30} M^2 \log^2 M (9.5 \times 10^{14} M \log M) \\ &< 2.3 \times 10^{45} M^3 \log^3 M. \end{aligned}$$

Now, taking into account (4.11), we deduce from estimates (4.27) and (4.28),

$$(4.29) \qquad \max\{Nz, Mw\} < 6.9 \times 10^{45} M^3 \log^3 M.$$

In particular, due to the fact $My \leq \max\{Mz, Mw\}$, we have that

$$(4.30) \qquad y < 6.9 \times 10^{45} M^2 \log^3 M.$$

Finally, between (4.12) and (4.25), (4.13) and (4.26), (4.16) and (4.29), and (4.17) and (4.30), we conclude that

$$N < 2.4 \times 10^{30} M^2 \log^2 M, \quad x < 3.9 \times 10^{27} M^2 \log^2 M,$$

$$\max\{Mw, Nz\} < 6.9 \times 10^{45} M^3 \log^3 M \quad \text{and} \quad y < 6.9 \times 10^{45} M^2 \log^3 M,$$

respectively. $\qquad \square$

4.2. The case M small

Since Lemma 4.1 gave us some bounds for the variables of our interest in terms of M, allow us to assume $M \leq 750$ and to show that there are no solutions of equation (1.4).

Lemma 4.2. *If (n, m, x, y) is a solution of equation (1.4) with $n \geq 4$, $x \geq 3$ and $y \geq 2$, then $M > 750$.*

Proof. Proceeding by contradiction, let us assume $M \leq 750$. By Lemma 4.1, we have that $X := \max\{w, Nz, z\} < 10^{57}$. Now, let us take

$$\Gamma_2 := w \log F_M - Nz \log \alpha + z \log(\sqrt{5}).$$

Since $|\Gamma_2| \leq e^{|\Gamma_2|} |e^{\Gamma_2} - 1|$, if we assume that $\lambda \geq 6$, then the right-hand side of (4.8) is at most $1/2$, and we get

$$(4.31) \qquad \left| w \log F_M - Nz \log \alpha + z \log(\sqrt{5}) \right| < \frac{14}{\alpha^\lambda}.$$

We need to look for a lower bound for $|\Gamma_2|$. Since $F_5 = 5$, we start by assuming that $M \neq 5$ to apply LLL-algorithm. In order to do it, we take $C := (30X)^3$ and consider the lattice Ω spanned by

$$\gamma_1 := \big(1, 0, \lfloor C \log F_M \rfloor\big), \quad \gamma_2 := \big(0, 1, \lfloor C \log \alpha \rfloor\big), \quad \gamma_3 := \big(0, 0, \lfloor C \log \sqrt{5} \rfloor\big).$$

Now, for each $M \in [3, 750] \setminus \{5\}$, we use Mathematica to calculate the parameters Q, T, c_1 and $\mathbf{m}_\Omega = \|\mathbf{b}_1\|/c_1$ from Lemma 2.5. Thus, we obtain

$$8 \times 10^{-119} < \big|w \log F_M - Nz \log \alpha + z \log \sqrt{5}\big|,$$

which, comparing with inequality (4.31), implies

(4.32) $$\lambda \leq 570.$$

Next, we take $M = 5$. Thus, inequality (4.31) is rewritten as

$$\big|(2w + z) \log 5 - 2Nz \log \alpha\big| < \frac{28}{\alpha^\lambda}.$$

Dividing the previous estimate by $Nz \log 5$, we get

(4.33) $$\left|\frac{2w + z}{Nz} - \frac{2 \log \alpha}{\log 5}\right| < \frac{18}{\alpha^\lambda Nz}.$$

By Lemma 4.1 and our assumption that $M \leq 750$, we have

(4.34) $$Nz < 10^{46} M^3 \log^3 M < 10^{57}.$$

Due to the fact that $F_{275} > 10^{57}$, we can use item (ii) of Lemma 2.4 with $K = 274$. In this case we get $a_M = 330$. Thus, we conclude

(4.35) $$\frac{1}{332(Nz)^2} < \left|\frac{2w + z}{Nz} - \frac{2 \log \alpha}{\log 5}\right|.$$

Comparing (4.33) with (4.35), and using (4.34), we get $\alpha^\lambda < 5976Nz < 6 \times 10^{60}$, which implies

(4.36) $$\lambda \leq 290.$$

It is clear that, from estimates (4.32) and (4.36), the inequality

$$\lambda \leq 570$$

holds for all $M \in [3, 750]$. If we suppose that $\lambda = N$, then, according to what we established, we have $N \leq 570$, a contradiction with our assumption that $N > 1200$. Thus, we can conclude that $\lambda = x$ and

(4.37) $$x \leq 570.$$

Next, we proceed to consider whether $(N, z) = (n + 1, x)$ or $(N, z) = (m, y)$.

Case $(N, z) = (n + 1, x)$. So $(M, w) = (m, y)$. By inequality (4.22) and the fact that $N > 1200$, we have

$$\left| w \log F_M - (N - 1)z \log \alpha + \log(5^{x/2}/(\alpha^x - \alpha^{-x})) \right| < \frac{34}{\alpha^N}.$$

Here, we have the expression

$$|u\gamma_M - v + \mu_x| < AB^{-N}$$

with $(u, v) = (w, (N - 1)z)$ and

$$\gamma_M := \frac{\log F_M}{\log \alpha}, \quad \mu_x := \frac{\log(5^{x/2}/(\alpha^x - \alpha^{-x}))}{\log \alpha}, \quad A := 71 \left(> \frac{34}{\log \alpha} \right), \quad B := \alpha.$$

We apply the Baker–Davenport reduction method for each $x \in [3, 570]$. Since $x \leq 570$ and $M \leq 750$, by estimate (4.20) and Lemma 4.1, we get that

$$u = w < Nx < 3.3 \times 10^{40} =: T.$$

Hence, for each γ_M with $M \in [3, 750]$, we look for the smallest q such that $q > 6T$ and, in all the cases, we get

$$\varepsilon > 4.7 \times 10^{-102} \quad \text{and} \quad \log(Aq/\varepsilon)/\log B < 1157,$$

the latter being an upper bound for N and, therefore, a contradiction with our assumption that $N > 1200$.

Case $(N, z) = (m, y)$. So, $(M, w) = (n + 1, x)$. Then, by inequalities (3.3), (4.37) and the fact that $M \leq 750$, we have

$$\max\{N, y\} < (n + 1)x = Mx < 427500.$$

Here, due to the previous inequality, we can take

$$X := \max\{w, Nz, z\} < 10^{12}.$$

So, we use inequality (4.31) to apply LLL-algorithm for each $M \in [3, 750]$. We get that $x = \lambda \leq 144$. We repeat this process once more and we obtained $x \leq 134$, which allow us to conclude that $(x, n) \in [3, 134] \times [4, 749]$. Then, due to the fact that $m = N > 1200$, by inequality (3.3), we get

$$1198y < (m - 2)y < (n + 1)x < 100500,$$

which implies that $(y, m) \in [3, 83] \times [1201, 2 + \lfloor 100500/y \rfloor]$. Next, we check the congruence $F_{n+1}^x - F_{n-1}^x \equiv F_m^y \pmod{10^{20}}$, where (n, m, x, y) is a quadruple of integers in the above ranges. Again we use the feature $\texttt{PowerMod}[A, u, B]$ in Mathematica to create both lists

$$L = \{F_{n+1}^x - F_{n-1}^x \pmod{10^{20}} : (x, n) \in [3, 134] \times [4, 749]\}$$

and

$$R_y = \{F_m^y \pmod{10^{20}} : m \in [1201, 2 + \lfloor 100500/y \rfloor]\}.$$

Finally, we were able to verify that $L \cap R_y = \emptyset$ for each $y \in [3, 83]$.

We conclude that equation (1.4) have no solutions when $M \leq 750$. $\qquad\square$

4.3. The case M big

Due to Lemma 4.2, here we study the case $M > 750$ looking for some absolutes bounds on the variables n, m, x and y.

Lemma 4.3. *If (n, m, x, y) is a solution of equation (1.4) with $n \geq 2$, $x \geq 3$ and $y \geq 2$, then*

$$\max\{x, y\} < 3 \times 10^{94}.$$

Proof. Since $M > 750$, by Lemma 4.1 we have that

$$\max\{x, y\} < 6.9 \times 10^{45} M^2 \log^3 M < \alpha^{M-2} \leq \min\{\alpha^{n-1}, \alpha^m\}.$$

In fact, the middle inequality holds for all $M \geq 256$. Hence, all three inequalities

$$\frac{x}{\alpha^{2n-2}} \leq \frac{1}{\alpha^{n-1}}, \quad \frac{x}{\alpha^{2n+2}} \leq \frac{1}{\alpha^{n+3}}, \quad \frac{y}{\alpha^{2m}} \leq \frac{1}{\alpha^m}$$

hold, so we may write

$$(4.38) \quad F_{n-1}^x = \frac{\alpha^{(n-1)x}}{5^{x/2}}(1 + \zeta_{n-1,x}), \quad F_{n+1}^x = \frac{\alpha^{(n+1)x}}{5^{x/2}}(1 + \zeta_{n+1,x}), \quad F_m^y = \frac{\alpha^{my}}{5^{y/2}}(1 + \zeta_{m,y}),$$

where

$$(4.39) \qquad \max\{|\zeta_{n-1,x}|, |\zeta_{n+1,x}|\} \leq \frac{2}{\alpha^{n-1}} \quad \text{and} \quad |\zeta_{m,x}| \leq \frac{2}{\alpha^m}.$$

We also have some additional conditions to (4.4), namely that

$$(4.40) \qquad \frac{F_{n-1}^x}{\alpha^{(n-1)x}/5^{x/2}}, \quad \frac{F_{n+1}^x}{\alpha^{(n+1)x}/5^{x/2}}, \quad \frac{F_m^y}{\alpha^{my}/5^{y/2}} \quad \text{belong to } (1/2, \sqrt{5}).$$

Using approximations (4.38) to rewrite equation (1.4) and reordering some terms, we get

$$(4.41) \quad \frac{\alpha^{my}}{5^{y/2}} - \frac{\alpha^{(n+1)x}}{5^{x/2}} + \frac{\alpha^{(n-1)x}}{5^{x/2}} = \left(\frac{\alpha^{(n+1)x}}{5^{x/2}}\right)\zeta_{n+1,x} - \left(\frac{\alpha^{(n-1)x}}{5^{x/2}}\right)\zeta_{n-1,x} - \left(\frac{\alpha^{my}}{5^{y/2}}\right)\zeta_{m,y},$$

which, due to the fact that

$$(4.42) \qquad \left(\frac{\alpha^{my}/5^{y/2}}{\alpha^{(n+1)x}/5^{x/2}} \right) = \left(\frac{\alpha^{my}/5^{y/2}}{F_m^y} \right) \left(\frac{F_m^y}{F_{n+1}^x} \right) \left(\frac{F_{n+1}^x}{\alpha^{(n+1)x}/5^{x/2}} \right) < 20,$$

and together with (4.40), implies

$$\left| \alpha^{my-(n+1)x} 5^{(x-y)/2} - 1 \right| < |\zeta_{n+1,x}| + \frac{1}{\alpha^{2x}} + \frac{|\zeta_{n-1,x}|}{\alpha^{2x}} + \left(\frac{\alpha^{my}/5^{y/2}}{\alpha^{(n+1)x}/5^{x/2}} \right) |\zeta_{m,y}|$$

$$(4.43) \qquad\qquad\qquad < \frac{2}{\alpha^{n-1}} + \frac{1}{\alpha^{2x}} + \frac{|\zeta_{n-1,x}|}{2} + \frac{20}{\alpha^m}$$

$$\qquad\qquad\qquad < \frac{8}{\alpha^{n+1}} + \frac{1}{\alpha^x} + \frac{20}{\alpha^m} < \frac{29}{\alpha^{\lambda_1}},$$

where $\lambda_1 := \min\{x, M\}$, as the λ that we had defined in (4.6). Note also that the expression inside the absolute value corresponds to Λ_4 defined in Section 3.2.

We need to establish the parameters to apply Matveev's result. So, we take

$$s := 2, \quad \gamma_1 := \alpha, \quad \gamma_2 := \sqrt{5}, \quad b_1 := my - (n+1)x, \quad b_2 := x - y.$$

Then, we have $D = 2$. Here, we take $A_1 := \log \alpha$ and $A_2 := \log 5$. Besides, since $x \geq 3$, the right-hand side in (4.43) is at most $29/\alpha^3 < 7$, therefore,

$$\frac{\alpha^{|my-(n+1)x|}}{5^{|x-y|/2}} < 8,$$

which implies

$$(4.44) \qquad \begin{aligned} |b_1| = |my - (n+1)x| &< \frac{\log(8 \times 5^{|y-x|/2})}{\log \alpha} \\ &= \left(\frac{\log 5}{2 \log \alpha} \right) |y - x| + \frac{\log 8}{\log \alpha} < 2|y - x| + 5 < 8|y - x|. \end{aligned}$$

Thus, by Lemmas 4.1 and 4.2, we can take

$$(4.45) \qquad \begin{aligned} \max\{|b_1|, |b_2|\} &< 8 \max\{x, y\} < 8 \times 6.9 \times 10^{45} M^2 \log^3 M \\ &< (5.6 \times 10^{46} \log^3 M) M^2 < M^{18} \times M^2 = M^{20} =: B. \end{aligned}$$

Matveev's theorem tells us that

$$(4.46) \qquad \begin{aligned} \log |\Lambda_4| &> -3.1 \times 10^9 (1 + \log 2)(1 + 20 \log M)(\log \alpha)(\log 5) \\ &> -8.6 \times 10^{10} \log M. \end{aligned}$$

Comparing estimates (4.43) and (4.46), we get that

$$(4.47) \qquad\qquad\qquad\qquad \lambda_1 < 1.8 \times 10^{11} \log M.$$

Hence, we have to proceed by cases according to various scenarios concerning λ_1:

Case $\lambda_1 = M$. In this case, by (4.47), we get $M < 1.8 \times 10^{11} \log M$, so,

$$(4.48) \qquad M < 5.6 \times 10^{12}.$$

Case $\lambda_1 = x$. In this case, by (4.47), we get

$$(4.49) \qquad x < 1.8 \times 10^{11} \log M.$$

Next, we rewrite (4.41) as

$$\frac{\alpha^{my}}{5^{y/2}} - \frac{\alpha^{nx}(\alpha^x - \alpha^{-x})}{5^{x/2}} = \left(\frac{\alpha^{(n+1)x}}{5^{x/2}}\right) \zeta_{n+1,x} - \left(\frac{\alpha^{(n-1)x}}{5^{x/2}}\right) \zeta_{n-1,x} - \left(\frac{\alpha^{my}}{5^{y/2}}\right) \zeta_{m,y},$$

which implies

$$\left| \alpha^{my-nx} 5^{(x-y)/2} (\alpha^x - \alpha^{-x})^{-1} - 1 \right|$$

$$(4.50) \qquad < \frac{\alpha^x}{\alpha^x - \alpha^{-x}} |\zeta_{n+1,x}| + \frac{\alpha^{-x}}{\alpha^x - \alpha^{-x}} |\zeta_{n-1,x}| + \left(\frac{\alpha^x}{\alpha^x - \alpha^{-x}}\right) \left(\frac{\alpha^{my}/5^{y/2}}{\alpha^{(n+1)x}/5^{x/2}}\right) |\zeta_{m,y}|$$

$$< \frac{2}{\alpha^{n-2}} + \frac{2}{\alpha^{n-1}} + \frac{40}{\alpha^{m-1}} < \frac{6}{\alpha^n} + \frac{4}{\alpha^n} + \frac{65}{\alpha^m} < \frac{75}{\alpha^M},$$

where we have used estimates (4.21), (4.39) and (4.42). Note that the expression inside the absolute value corresponds to Λ_5 as in Section 3.2.

Now, we indicate the parameters to apply Matveev's result to the left-hand side of the inequality (4.50). We have $s := 3$ and we take

$$\gamma_1 := \alpha, \quad \gamma_2 := 2\sqrt{2}, \quad \gamma_3 := \alpha^x - \alpha^{-x}, \quad b_1 := my - nx, \quad b_2 := x - y, \quad b_3 := -1.$$

Thus, it is clear that $D := 2$. As in prior application of Matveev's theorem, we take $A_1 := \log \alpha$, $A_2 := \log 5$. As for $\gamma_3 := \alpha^x - \alpha^{-x}$, we have

$$h(\gamma_3) \leq h(\alpha^x) + h(\alpha^{-x}) + 2 = 2x \log \alpha + 2 < 1.7x,$$

since $x \geq 3$, which allow us to take $A_3 := 4x$. By estimate (4.44), we have

$$|b_1| = |my - nx| \leq |my - (n+1)x| + x < 8|y - x| + x < 9 \max\{x, y\}.$$

Hence, using Lemmas 4.1 and 4.2, we conclude, as at estimate (4.45), that we can take

$$\max\{|b_1|, |b_2|\} < 9 \max\{x, y\} < 9 \times 6.9 \times 10^{45} M^2 \log^3 M$$
$$< (6.3 \times 10^{46} \log^3 M) M^2 < M^{20} =: B,$$

where we have used the fact that $6.3 \times 10^{46} \log^3 M < M^{18}$ for all $M > 750$. Thus, by Matveev's result, we have that

$$\begin{aligned}
(4.51) \qquad \log|\Lambda_5| &> -5.8 \times 10^{11}(1 + \log 2)(1 + 20 \log M)(\log \alpha)(\log 5)(4x) \\
&> -6.4 \times 10^{13} x \log M.
\end{aligned}$$

From estimates (4.50) and (4.51), we get that

$$(4.52) \qquad\qquad\qquad M < 1.4 \times 10^{14} x \log M.$$

Inserting estimate (4.49) into (4.52), we get

$$M < 1.4 \times 10^{14}(1.8 \times 10^{11} \log M) \log M < 2.6 \times 10^{25} \log^2 M.$$

Therefore, we have

$$(4.53) \qquad\qquad\qquad M < 1.2 \times 10^{29}.$$

Comparing (4.48) and (4.53), we conclude that the inequality (4.53) always holds. Hence, by Lemma 4.1, we get

$$N < 2 \times 10^{92}, \quad x < 3 \times 10^{89} \quad \text{and} \quad y < 3 \times 10^{94}. \qquad\qquad \square$$

4.4. Reducing the bound

We need to work some more on inequality (4.43) under the assumption that $\lambda_1 > 470$. Then $29/\alpha^{\lambda_1} < 1/2$, so by a classic argument we get that

$$\left|(my - (n+1)x) \log \alpha - (y - x) \log \sqrt{5}\right| < \frac{58}{\alpha^{\lambda_1}}.$$

Thus, since $x \neq y$, we get

$$(4.54) \qquad\qquad \left|\frac{my - (n+1)x}{x - y} - \frac{\log \sqrt{5}}{\log \alpha}\right| < \frac{121}{|x - y|\alpha^{\lambda_1}}.$$

By Lemma 4.3, we have that

$$250|x - y| < 250 \max\{x, y\} < 10^{97} < \alpha^{470} < \alpha^{\lambda_1}$$

which implies that the expression on the right-hand side of (4.54) is smaller than $1/(2|x - y|^2)$. By Legendre's criterion, $(my - (n+1)x)/(x - y)$ is a convergent p_k/q_k of $\gamma := \log \sqrt{5}/\log \alpha$ for some nonnegative integer k.

Case $k < 100$. Since $q_{99} < 4.1 \times 10^{46}$ and $\max\{a_k : 0 \leq k \leq 99\} = 29$, then

$$\frac{1}{10^{95}} < \frac{1}{31 q_{99}^2} < \min\left\{\left|\gamma - \frac{p_k}{q_k}\right| : k \in \{0, 1, \ldots, 100\}\right\} \leq \left|\gamma - \frac{my - (n+1)x}{x - y}\right|.$$

Therefore, comparing with (4.54) and using the fact that $1 \leq |x - y|$,

$$\lambda_1 < \frac{\log(121 \times 10^{95})}{\log \alpha} < 465,$$

a contradiction with our assumption that $\lambda_1 > 470$. Thus, $k \geq 100$.

Case $k \geq 100$. Since $q_{189} > 3 \times 10^{94} > |x - y|$, we conclude that $k \in [100, 189]$. Besides, we have that

$$\frac{1}{10^{192}} < \frac{1}{332 \times q_{189}^2} < \left| \gamma - \frac{p_{189}}{q_{189}} \right|.$$

Thus, comparing with (4.54), we get that

$$\frac{1}{10^{192}} < \left| \gamma - \frac{p_k}{q_k} \right| < \frac{121}{|x - y|\alpha^{\lambda_1}} \leq \frac{121}{q_{100}\alpha^{\lambda_1}} < \frac{121}{10^{48}\alpha^{\lambda_1}},$$

which implies

$$\lambda_1 < \frac{\log(121 \times 10^{144})}{\log \alpha} < 700.$$

If $M \leq x$, then we obtain $M = \lambda_1 < 700$, a contradiction with the fact that $M > 750$. Thus, $x = \lambda_1$, therefore we have $x < 700$. Now, by estimate (4.52) we get

$$M < 1.4 \times 700 \times 10^{14} \log M < 9.8 \times 10^{16} \log M,$$

which gives $M < 4.3 \times 10^{18}$. Comparing it with estimate (4.48), we conclude that, in any case, this last inequality always holds. Hence, Lemma 4.1 yields

(4.55) $$N < 9 \times 10^{70}, \quad x < 2 \times 10^{68} \quad \text{and} \quad y < 2 \times 10^{88}.$$

Now, we get that $q_{177} > 2 \times 10^{88} > |x - y|$ and

$$\left| \gamma - \frac{p_{177}}{q_{177}} \right| > \frac{1}{332 q_{177}^2} > \frac{1}{10^{180}},$$

therefore, as we proceeded before, we have

$$\lambda_1 < \frac{\log(121 \times 10^{132})}{\log \alpha} < 642.$$

Again, since $M > 750$ and $\lambda_1 := \min\{x, M\}$, the previous inequality implies that $\lambda = x$, with $x \in [3, 641]$.

Now, due to the fact that $M > 750$, inequality (4.50) implies

(4.56) $$\left| (x - y) \log \sqrt{5} - (nx - my) \log \alpha - \log(\alpha^x - \alpha^{-x}) \right| < \frac{150}{\alpha^M}.$$

Here, we fix x and apply again the Baker–Davenport reduction method. We have

$$|u\gamma - v + \mu_x| < AB^{-M},$$

where $(u, v) = (x - y, nx - my)$ and

$$\gamma := \frac{\log \sqrt{5}}{\log \alpha}, \quad \mu_x := -\frac{\log(\alpha^x - \alpha^{-x})}{\log \alpha}, \quad A := 312 \left(> \frac{150}{\log \alpha} \right), \quad B := \alpha.$$

By estimate (4.55), we can take $T = 10^{89}$ as the bound on $|u|$. We loop over all values $x \in [3, 641]$. In all cases we choose q_{180}, i.e., the denominator of the convergent of index 180 of γ, since $q_{180} > 6T$, and we obtained

$$\varepsilon > 1.3 \times 10^{-90} \quad \text{and} \quad \log(Aq/\varepsilon)/\log B < 1293.$$

Hence, as before, the latter inequality implies $M < 1293$. We repeat this reduction process. By Lemma 4.1, we get

(4.57) $$N < 3 \times 10^{38}, \quad x < 4 \times 10^{35} \quad \text{and} \quad y < 5 \times 10^{54}.$$

This time, we get that $q_{115} > 5 \times 10^{54} > |x - y|$ and

$$\left| \gamma - \frac{p_{115}}{q_{115}} \right| > \frac{1}{10^{114}},$$

which implies $\lambda_1 < 326$. Thus, we can conclude that $x \in [3, 325]$. We apply again the Baker–Davenport reduction method to (4.56) taking into account that, by estimate (4.57), we can take $T = 10^{55}$. This time, in all the cases, we get

$$\varepsilon > 1.9 \times 10^{-41} \quad \text{and} \quad \log(Aq/\varepsilon)/\log B < 662.$$

Therefore, we have $M < 662$, a contradiction with our assumption that $M > 750$.

What we have done until now allows us to conclude that, no matter whether $M > 750$ or $M \leq 750$, equation (1.4) has no solution when $N > 1200$.

5. The case N small: $N \leq 1200$

5.1. The final computations

Due to the previous section, it remains to study the case $N \leq 1200$. Recall that we are working under the assumption that $n \geq 4$, $m \geq 3$, $x \geq 3$ and $y \geq 2$.

Lemma 5.1. *Under the previous assumptions over the variables and $N \leq 1200$, there are no non-trivial solutions to the Diophantine equation* (1.4).

Proof. By Lemma 3.3 and the fact that $M \leq N$, we have

$$x < 1.3 \times 10^{18} \quad \text{and} \quad y < 1.5 \times 10^{21}.$$

Now, let us take

$$\Gamma_1 := y \log F_m - x \log F_{n+1},$$

and observe that $\Gamma_1 < 0$, due to the fact that $\Lambda_1 = 1 - e^{\Gamma_1} > 0$. Hence, by (3.4), we get

(5.1) $$0 < 1 - e^{-|\Gamma_1|} \le 2.5^{-x},$$

which implies

$$0 < \left(e^{|\Gamma_1|} - 1\right) e^{-|\Gamma_1|} \le 2.5^{-x}.$$

Thus, we get

$$0 < |\Gamma_1| < e^{|\Gamma_1|} - 1 \le 2.5^{-x} e^{|\Gamma_1|}.$$

However, since $x \ge 3$, by (5.1) we get $e^{|\Gamma_1|} \le 1.2$. Thus, the previous inequality implies

$$0 < \left|y \log F_m - x \log F_{n+1}\right| < \frac{1.2}{2.5^x}.$$

Now, dividing both sides of the last inequality above by $x \log F_m$ and taking into account that $\Gamma_1 < 0$, we get

(5.2) $$0 < \frac{\log F_{n+1}}{\log F_m} - \frac{y}{x} < \frac{1.2}{x 2.5^x \log F_m}.$$

Due to the fact that

$$2.5^x \log F_m \ge 2.5^x \log 2 > 3x \quad \text{for all } x \ge 3 \text{ and } m \ge 3,$$

by inequality (5.2), we get

$$0 < \frac{\log F_{n+1}}{\log F_m} - \frac{y}{x} < \frac{1}{2x^2}.$$

By Legendre's criterion, item (i) in Lemma 2.4, we infer that y/x is a convergent to the continued fraction of $\log F_{n+1}/\log F_m$. Let $d := \gcd(x, y)$. Due to Fermat's last theorem once again, it follows that $d \in \{1, 2\}$, otherwise the triple

$$(X, Y, Z) = \left(F_{n-1}^{x/d}, F_m^{y/d}, F_{n+1}^{x/d}\right)$$

would be a positive integer solution to the Fermat equation $X^d + Y^d = Z^d$ with integer exponent $d \ge 3$ which does not exist. Therefore, since the convergent p_k/q_k of any irrational number γ satisfies $q_k \ge F_k$, where F_k is the kth Fibonacci number, and since $F_{86} > 3 \times 10^{17}$, it follows that $(x, y) = (q_k, p_k)$ or $(2q_k, 2p_k)$ for some $k \le 86$. Thus, we have that p_k/q_k is the kth convergent to $\log F_{n+1}/\log F_m$ for some $m \ge 3$, $n \ge 3$, such that $\gcd(m, n^2 - 1) \le 4$, and $N \le 1200$. The only issue to justify is that $\gcd(m, n^2 - 1) \le 4$. We assume that $r = \gcd(m, n - 1)$. Then r divides both m and $n - 1$, so F_r divides both F_m^y and F_{n-1}^x. In particular, F_r divides F_{n+1}^x. Thus,

$$F_r \mid \gcd(F_{n-1}^x, F_{n+1}^x) = \gcd(F_{n-1}, F_{n+1})^x = F_{\gcd(n-1,n+1)}^x \mid F_2^x = 1.$$

So, $r \in \{1, 2\}$. The same argument shows that $\gcd(m, n+1) \leq 2$, therefore $\gcd(m, n^2-1) \leq 4$. Now, we generated the first 86 convergents of $\log F_{n+1}/\log F_m$, with m and n satisfying all of the above conditions, and checked for each pair $(x, y) \in \{(q_k, p_k), (2q_k, 2p_k)\}$ with $x \geq 3$, if the congruence

$$F_{n+1}^x - F_{n-1}^x \equiv F_m^y \pmod{10^{20}}$$

holds. The computations which took almost three hours used the Mathematica feature `PowerMod[A, u, B]` and found no new solutions.   □

This ends the proof of our Theorem 1.1.

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