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Finite Morse Index Solutions of the Fractional Henon–Lane–Emden Equation with Hardy Potential

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Abstract. In this paper, we study the fractional Henon–Lane–Emden equation associated with Hardy potential

$$(-\Delta)^s u - \gamma |x|^{-2s} u = |x|^a |u|^{p-1} u \quad \text{in } \mathbb{R}^n.$$

Extending the celebrated result of [14], we obtain a classification result on finite Morse index solutions to the fractional elliptic equation above with Hardy potential. In particular, a critical exponent p of Joseph–Lundgren type is derived in the supercritical case studying a Liouville type result for the s-harmonic extension problem.

1. Introduction

For given constants $0 < s < 1$, $n > 2s$, $a > -2s$ and $p > 1$, we consider the following fractional Henon–Lane–Emden equation associated with the Hardy potential

$$(-\Delta)^s u - \gamma |x|^{-2s} u = |x|^a |u|^{p-1} u \quad \text{in } \mathbb{R}^n. \tag{1.1}$$

The fractional Laplacian $(-\Delta)^s$ is defined by

$$(-\Delta)^s u(x) = \mathcal{A}_{n,s} \, \mathrm{P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}} \, dy \quad \text{for } x \in \mathbb{R}^n,$$

which is well-defined in the principal-value sense for $u \in C^{2\sigma}_{\mathrm{loc}}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n; (1+|x|)^{-n-2s})$; refer [17] for instance. In this paper, we are concerned with solutions to (1.1) which have finite Morse index assuming $\gamma < \gamma_{n,s,a}(p)$ with a critical constant $\gamma_{n,s,a}(p)$ defined by (1.11).

In recent years, nonlocal diffusion operators such as the fractional Laplacians $(-\Delta)^s$ have drawn a great attention of many mathematicians. Integro-differential operators including the fractional Laplacians appear naturally in the study of stochastic processes with jumps, which allow long-distance interactions and have numerous applications to physics

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and finance. As the order s of the fractional Laplacian tends to 1, the Henon–Lane–Emden equation with the Hardy potential

$$-\Delta u-\gamma|x|^{-2}u=|x|^a|u|^{p-1}u \quad \text{in } \mathbb{R}^n \tag{1.2}$$

can be seen as the limit of the equation (1.1) (see [17] for instance). In the local case when $s=1$ with $a=0$ and $\gamma=0$, the equation (1.2) becomes the Lane–Emden equation

$$-\Delta u=|u|^{p-1}u \quad \text{in } \mathbb{R}^n \tag{1.3}$$

which arises in the study of stellar structure in astrophysics [6,9], and the prescribed scalar curvature problem in conformal geometry [6,47]. During the last few decades, there have been extensive literatures on the equation (1.3). Among them, Gidas and Spruck in the pioneering work [33] proved no existence of positive solutions to the equation (1.3) for $1<p<p_S(n,1,0)$, where $p_S(n,1,0)$ is the so-called classical Sobolev exponent given by

$$p_S(n,1,0)=\begin{cases}+\infty & \text{if } n\le 2,\\ \frac{n+2}{n-2} & \text{if } n>2.\end{cases}$$

Moreover, in the case when $p=p_S(n,1,0)$, it was proved by Caffarelli, Gidas and Spruck in the remarkable paper [6] that there exists a unique positive solution of (1.3) up to translation and rescaling, which is radial and explicit. Regarding finite Morse index solutions (not necessarily positive solutions), Farina in the seminal paper [25] completely classified finite Morse index solutions with the Joseph–Lundgren exponent $p_c(n)$ which is given by

$$p_c(n)=\begin{cases}+\infty & \text{if } n\le 10,\\ \frac{(n-2)^2-4n+8\sqrt{n-1}}{(n-2)(n-10)} & \text{if } n\ge 11,\end{cases} \tag{1.4}$$

see also [37]. Farina's result has been extended to the equation involving the Henon term $|x|^a|u|^{p-1}u$ and the Hardy term $\gamma|x|^{-2}u$; for instance, we refer to [1, 2, 11, 19, 20, 36, 48] and the references therein. Moreover, stable and finite Morse index solutions of Gelfand–Liouville problem $-\Delta u=e^u$ has been also studied in [12, 26], and extended to non-local operators in [27, 30, 31, 35].

This paper concerns the classification of finite Morse index solutions to the fractional Henon–Lane–Emden equation (1.1) with the Hardy potential. Throughout this paper, we always assume that $0<s<1$, $n>2s$, $a>-2s$ and $p>1$ unless otherwise stated. We first recall some definitions and notations regarding fractional Laplacians. The fractional Laplacian $(-\Delta)^s$ on the Schwartz space is defined as a pseudo-differential operator with the symbol $|\xi|^{2s}$ by the Fourier transform. Associated to the fractional Laplacian $(-\Delta)^s$, we denote by $H^s(\mathbb{R}^n)$ the usual L^2-based fractional Sobolev spaces, and by $\dot{H}^s(\mathbb{R}^n)$ its homogeneous version defined via the Fourier transform as the completion of $C_c^\infty(\mathbb{R}^n)$ under the norm

$$\|u\|^2_{\dot{H}^s(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\xi|^{2s} |\widehat{u}(\xi)|^2 \, d\xi;$$

see [17,43] for the details. Here $\widehat{u}$ stands for the Fourier transform of u. Then the fractional Laplacian $(-\Delta)^{s/2}$ is defined as a bounded linear operator $(-\Delta)^{s/2}\colon \dot{H}^s(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, and

$$\|u\|^2_{\dot{H}^s(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\xi|^{2s} |\widehat{u}(\xi)|^2 \, d\xi = \|(-\Delta)^{s/2} u\|^2_{L^2(\mathbb{R}^n)}, \quad \forall\, u \in \dot{H}^s(\mathbb{R}^n)$$

in view of Plancherel's Theorem. For $0 < s < 1$ and $u \in \dot{H}^s(\mathbb{R}^n)$, the following equivalence of the norms holds (see [17, Propositions 3.4 and 3.6]):

$$\|u\|^2_{\dot{H}^s(\mathbb{R}^n)} = \frac{\mathcal{A}_{n,s}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} \, dxdy. \tag{1.5}$$

Here a constant $\mathcal{A}_{n,s}$ is given by

$$\mathcal{A}_{n,s} := \frac{2^{2s}}{\pi^{n/2}} \frac{\Gamma\left(\frac{n+2s}{2}\right)}{|\Gamma(-s)|} \tag{1.6}$$

and is of order $s(1-s)$ as $s \in (0,1)$ tends to 0 or 1.

For $0 < s < \sigma < 1$, and $u \in C^{2\sigma}_{\mathrm{loc}}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n; (1+|x|)^{-n-2s})$, the following integral representation for the fractional Laplacian

$$\begin{aligned}(-\Delta)^s u(x) &= -\mathcal{A}_{n,s}\, \mathrm{P.V.} \int_{\mathbb{R}^n} \frac{u(y) - u(x)}{|y-x|^{n+2s}} \, dy \\ &= -\frac{\mathcal{A}_{n,s}}{2} \int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}} \, dy, \quad \forall\, x \in \mathbb{R}^n\end{aligned} \tag{1.7}$$

is well-defined. Moreover, if $u \in C^{2\sigma}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and $u \in \dot{H}^s(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, then it holds that

$$\int_{\mathbb{R}^n} (-\Delta)^s u \, u \, dx = \frac{\mathcal{A}_{n,s}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} \, dxdy = \|u\|^2_{\dot{H}^s(\mathbb{R}^n)} \tag{1.8}$$

in light of (1.5) and (1.7).

It should be noted that any finite Morse index solution u is stable outside some compact set. Here we say that a solution u to (1.1) is stable on a set Ω if

$$\int_{\mathbb{R}^n} \left(p|x|^a |u|^{p-1} \phi^2 + \gamma |x|^{-2s} \phi^2 \right) dx \le \|\phi\|^2_{\dot{H}^s(\mathbb{R}^n)} \tag{1.9}$$

for any $\phi \in C_c^\infty(\Omega)$. With regard to stability results on the fractional Laplacian, the corresponding results of Gidas and Spruck [33] and Caffarelli, Gidas and Spruck [6] have been established by Li [38], and Chen, Li and Ou [10] respectively. Indeed, the following fractional Lane–Emden equation

$$(1.10) \qquad (-\Delta)^s u = |u|^{p-1}u \quad \text{in } \mathbb{R}^n$$

was studied with the use of the fractional critical Sobolev exponent $p_S(n,s,0)$ given by

$$p_S(n,s,0) = \begin{cases} +\infty & \text{if } n \le 2s, \\ \frac{n+2s}{n-2s} & \text{if } n > 2s. \end{cases}$$

Recently, Davila, Dupaigne and Wei in their remarkable paper [14] provided a complete classification of finite Morse index solutions for the fractional Lane–Emden equation (1.10). With the use of the harmonic extension method for the fractional Laplacian developed by Caffarelli and Silvestre [8], one of main ingredients in studying the supercritical cases $p > p_S(n,s,0)$ in [14] is a monotonicity formula for the extension problem of (1.10), which enables us to employ a blow-down analysis. A more discussion on various monotonicity formulas for fractional Laplacian operators can be found in [5, 7, 8, 24, 32]. Moreover, Fazly and Wei in [28, 29] extended the result of [14] to the fractional Henon–Lane–Emden equation, and the fractional Lane–Emden equations of higher order $s \in (1,2)$, respectively.

It would be interesting to study stable solutions to the p-fractional Laplace equation which has attracted increasing attention in recent years. In the nonlinear case, the s-harmonic extension approach might not be applicable and it seems crucial to work with the integral definition of the fractional operator; we refer to [15, 16, 42] for some recent results of the nonlocal tail for the p-fractional Laplacian. We hope to consider the stability problem of the p-fractional Laplace equation in future works.

Before stating our main result, we introduce some constants which will play crucial roles in the classification of finite Morse index solutions to the fractional Henon–Lane–Emden equation (1.1) with Hardy potential. We define

$$p_S(n,s,a) := \begin{cases} +\infty & \text{if } n \le 2s, \\ \frac{n+2s+2a}{n-2s} & \text{if } n > 2s, \end{cases}$$

and

$$(1.11) \qquad \gamma_{n,s,a}(p) := \begin{cases} \lambda(0) =: \Lambda_{n,s} & \text{if } 1 < p \le p_S(n,s,a), \\ \lambda\big(\frac{n-2s}{2} - \frac{2s+a}{p-1}\big) & \text{if } p > p_S(n,s,a). \end{cases}$$

Here a function $\lambda\colon [0,(n-2s)/2) \to \mathbb{R}$ is defined by

$$(1.12) \qquad \lambda(\alpha) = 2^{2s} \frac{\Gamma\big(\frac{n+2s+2\alpha}{4}\big)\Gamma\big(\frac{n+2s-2\alpha}{4}\big)}{\Gamma\big(\frac{n-2s-2\alpha}{4}\big)\Gamma\big(\frac{n-2s+2\alpha}{4}\big)}$$

(see [23, Lemma 3.1]). The function $\lambda|_{[0,(n-2s)/2)}$ is continuous and monotone decreasing with respect to α, and has an asymptotic behavior such that $\lambda(\alpha) \to 0$ as $\alpha \to (n-2s)/2$. In particular, we note that $0 < \gamma_{n,s,a}(p) \le \Lambda_{n,s}$.

Now we state our main result in this paper.

Theorem 1.1. *Let $n > 2s$, $0 < s < \sigma < 1$, $a > -2s$ and $\gamma < \gamma_{n,s,a}(p)$. Let $u \in C^{2\sigma}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ be a solution to* (1.1) *which is stable outside a compact set, i.e., there exists a constant $R_0 \ge 0$ such that the inequality* (1.9) *holds for any $\phi \in C_c^\infty(\mathbb{R}^n \setminus \overline{B_{R_0}})$.*

(a) *If $1 < p < p_S(n,s,a)$ and $u \in L^1(\mathbb{R}^n)$, then $u \equiv 0$.*

(b) *If $p = p_S(n,s,a)$ and if $u \in L^1(\mathbb{R}^n)$, then u has finite energy, that is,*

$$\|u\|^2_{\dot{H}^s(\mathbb{R}^n)} < +\infty.$$

In this case, if u is stable in $\mathbb{R}^n$, then $u \equiv 0$.

(c) *If $p > p_S(n,s,a)$ and*

$$\text{(P)} \qquad p > \frac{\Lambda_{n,s} - \gamma}{\gamma_{n,s,a}(p) - \gamma},$$

then $u \equiv 0$.

We remark that the condition (P) is exactly the inequality (1.6) of [14] when $\gamma = a = 0$. In order to prove Theorem 1.1, we employ the approach used in [14] which is a nonlocal counterpart of the results [13, 49] for the local operators: the classical Laplacian and the biharmonic operator. Following [14], we introduce the s-harmonic extension $\overline{u}$ of u on the upper half space related to the fractional Laplacian of order $s \in (0,1)$ in Theorem 2.1. Based on the extension technique, the problem for the solution u of the equation (1.1) may be reduced to the classification problem for the extension $\overline{u}$ which satisfies the following local equation on the upper half space with a Neumann boundary condition

$$\begin{cases} -\nabla \cdot (t^{1-2s} \nabla \overline{u}) = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ -\lim_{t\to 0} t^{1-2s} \partial_t \overline{u} = \kappa_s \big(\gamma |x|^{-2s} u + |x|^a |u|^{p-1} u\big) & \text{on } \partial\mathbb{R}^{n+1}_+, \end{cases} \tag{1.13}$$

and is stable outside some set in the sense of Lemma 2.4. Then we first obtain suitable energy estimates for the solution u and its s-harmonic extension $\overline{u}$ utilizing the following Hardy inequality (see [34, 50]): if $n > 2s$, then

$$\Lambda_{n,s} \int_{\mathbb{R}^n} |x|^{-2s} \phi^2(x)\, dx \le \|\phi\|^2_{\dot{H}^s(\mathbb{R}^n)}, \quad \forall\, \phi \in \dot{H}^s(\mathbb{R}^n). \tag{1.14}$$

Here the constant $\Lambda_{n,s} = \lambda(0) = 2^{2s} \frac{\Gamma\left(\frac{n+2s}{4}\right)^2}{\Gamma\left(\frac{n-2s}{4}\right)^2}$ is optimal. The proof for the subcritical case follows by applying the Pohozaev identity based on energy estimates. Here for the

Pohozaev identity, we give a different proof from [14] in order to address the regularity issue due to the Hardy term. When dealing with the supercritical case, we derive the monotonicity formula for the extension problem (1.13) in Theorem 5.1 which plays a key role in the blow-down analysis in Section 7. In fact, in light of the monotonicity formula together with energy estimates, we show that the blow-down limit of the harmonic extension $\overline{u}$ is a homogeneous solution to the extension problem (1.13) which is stable except the origin. A Liouville type result on such stable, homogeneous solutions is established in Theorem 6.1, where the assumptions $\gamma < \gamma_{n,s,a}(p)$ and (P) are used. Then it is proved that the extension $\overline{u}$ is trivial thanks to the monotonicity formula, and in turn, the solution u of the original problem is zero. Finally, we notice that the result in Theorem 1.1 would be optimal taking into account the following remark.

Remark 1.2. As seen in [14, 23], there is an explicit singular solution to (1.1), provided that $\gamma < \gamma_{n,s,a}(p) = \lambda\big(\frac{n-2s}{2} - \frac{2s+a}{p-1}\big)$. For $p > p_S(n,s,a)$, let

$$u_s(x) := A|x|^{-\frac{2s+a}{p-1}},$$

with a constant A satisfying

$$|A|^{p-1} = \lambda\left(\frac{n-2s}{2} - \frac{2s+a}{p-1}\right) - \gamma = \gamma_{n,s,a}(p) - \gamma.$$

Then it can be easily checked that u_s is a singular solution to (1.1). In fact,

$$(-\Delta)^s u_s(x) = \gamma_{n,s,a}(p)|x|^{-2s}u_s \quad \text{in } \mathbb{R}^n \setminus \{0\}.$$

In light of the Hardy inequality (1.14), we see that u_s is unstable if and only if $p|A|^{p-1}+\gamma > \Lambda_{n,s}$, i.e., the condition (P) holds. Here we used the fact that $\Lambda_{n,s}$ is the sharp constant in the Hardy inequality.

The rest of the paper is organized as follows. In Section 2, we prepare some preliminary results. In Section 3, we derive various energy estimates for a finite Mores index solution u to (1.1) and its s-harmonic extension. In Section 4, we prove Theorem 1.1 for the subcritical and critical case. In Section 5, we obtain the monotonicity formula for the extension problem. In Section 6, we obtain a Liouville type theorem for stable homogeneous solutions to the extension problem in the supercritical case. Section 7 is devoted to the proof of Theorem 1.1 for the supercritical case. Lastly, in Section 8, we analyze the asymptotic behavior of our assumption (P) of Joseph–Lundgren type as the order $s \in (0,1)$ tends to 1, the local case.

Notations.

(a) $\mathbb{R}^{n+1}_+ := \big\{(x,t) \in \mathbb{R}^{n+1} : t > 0\big\}$.

(b) $B_R := B_R^{(n)}(0) \subset \mathbb{R}^n$ is a ball of radius R centered at the origin in the n-dimensional space.

(c) $B_R^+ := \mathbb{R}_+^{n+1} \cap B_R^{(n+1)}(0) = \big\{(x,t) \in \mathbb{R}^{n+1} : t > 0, |(x,t)| < R\big\}$.

(d) For $0 < \sigma < 1$ and a domain Ω in $\mathbb{R}^n$, a seminorm $[u]_{C^{2\sigma}(\Omega)}$ denotes

$$\begin{cases} [u]_{C^{0,2\sigma}(\Omega)} & \text{if } 2\sigma \le 1, \\ [u]_{C^{1,2\sigma-1}(\Omega)} & \text{if } 2\sigma > 1. \end{cases}$$

A function space $C^{2\sigma}(\Omega)$ consists of functions u such that $[u]_{C^{2\sigma}(\Omega)}$ is finite. $C^{2\sigma}_{\mathrm{loc}}(\mathbb{R}^n)$ stands for a space of functions which belong to $C^{2\sigma}(K)$ for any compact subset K in $\mathbb{R}^n$.

(e) $L^1(\mathbb{R}^n; (1+|x|)^{-n-2s})$ denotes the L^1-space over $\mathbb{R}^n$ with measure $(1+|x|)^{-n-2s}\,dx$. Others are similar.

(f) We may extend a function $\overline{u}$ defined on $\overline{\mathbb{R}_+^{n+1}}$ to the function on the whole space $\mathbb{R}^{n+1}$, still denoted by $\overline{u}$, by setting

$$\overline{u}(x,t) = \begin{cases} \overline{u}(x,t), & \forall\, x \in \mathbb{R}^n,\ t \ge 0, \\ \overline{u}(x,-t), & \forall\, x \in \mathbb{R}^n,\ t < 0. \end{cases} \tag{1.15}$$

$\overline{u} \in H^1_{\mathrm{loc}}(\overline{\mathbb{R}_+^{n+1}}; t^{1-2s})$ means that the even extension of $\overline{u}$ given by (1.15) belongs to $H^1_{\mathrm{loc}}(\mathbb{R}^{n+1}; |t|^{1-2s})$. The spaces $L^2_{\mathrm{loc}}(\overline{\mathbb{R}_+^{n+1}}; t^{1-2s})$ and $H^1_{\mathrm{loc}}(\overline{\mathbb{R}_+^{n+1}} \setminus \{0\}; t^{1-2s})$ can be understood similarly.

2. Preliminaries

In this section, we collect some known results on the fractional Laplacian operators used in the paper. First of all, we recall the s-harmonic extension due to Caffarelli and Silvestre [8] from which the fractional Laplacian can be considered the Dirichlet-to-Neumann map; see also [39, 46].

Theorem 2.1. [8, 39, 46] *Let $0 < s < \sigma < 1$ and $u \in C^{2\sigma}_{\mathrm{loc}}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n; (1+|x|)^{-(n+2s)})$. Let*

$$\overline{u}(x,t) = \int_{\mathbb{R}^n} P_{n,s}(x-\xi, t) u(\xi)\, d\xi \quad \textit{for } (x,t) \in \mathbb{R}_+^{n+1}.$$

Here the fractional Poisson kernel $P_{n,s}$ is defined by

$$P_{n,s}(x,t) = p_{n,s} t^{2s} |(x,t)|^{-(n+2s)}$$

with the positive constant $p_{n,s}$ *satisfying*

$$\int_{\mathbb{R}^n} P_{n,s}(x-\xi,t)\,d\xi = 1 \quad \text{for any } (x,t)\in\mathbb{R}^{n+1}_+.$$

Then $\overline{u}$ *belongs to* $C^2(\mathbb{R}^{n+1}_+)\cap C(\overline{\mathbb{R}^{n+1}_+})$ *with* $t^{1-2s}\partial_t\overline{u}\in C(\overline{\mathbb{R}^{n+1}_+})$, *and* $\overline{u}$ *satisfies*

$$\begin{cases} -\nabla\cdot(t^{1-2s}\nabla\overline{u}) = 0 & \text{in } \mathbb{R}^{n+1}_+,\\ \overline{u} = u & \text{on } \partial\mathbb{R}^{n+1}_+, \end{cases}$$

and

$$-\lim_{t\to 0} t^{1-2s}\partial_t\overline{u} = \kappa_s(-\Delta)^s u \quad \text{on } \partial\mathbb{R}^{n+1}_+,$$

where the constant κ_s *is given by*

$$\kappa_s = \frac{\Gamma(1-s)}{2^{2s-1}\Gamma(s)}.$$

In the paper, unless specifically stated, $\overline{u}$ denotes the s-harmonic extension of u given by Theorem 2.1. Applying Theorem 2.1 to a solution u of the fractional Henon–Lane–Emden equation (1.1) with the Hardy potential, the equation for the extension $\overline{u}$ can be written as follows:

$$\text{(2.1)} \qquad \begin{cases} -\nabla\cdot(t^{1-2s}\nabla\overline{u}) = 0 & \text{in } \mathbb{R}^{n+1}_+,\\ -\lim_{t\to 0} t^{1-2s}\partial_t\overline{u} = \kappa_s\big(\gamma|x|^{-2s}u + |x|^a|u|^{p-1}u\big) & \text{on } \partial\mathbb{R}^{n+1}_+, \end{cases}$$

which will be used in the paper.

In [8], it was shown that if $u\in\dot{H}^s(\mathbb{R}^n)$, then

$$\text{(2.2)} \qquad \|u\|^2_{\dot{H}^s(\mathbb{R}^n)} = \frac{1}{\kappa_s}\int_{\mathbb{R}^{n+1}_+} t^{1-2s}|\nabla\overline{u}|^2\,dxdt.$$

The next lemma concerns some condition on u, which guarantees that $\overline{u}\in H^1_{\text{loc}}(\overline{\mathbb{R}^{n+1}_+};t^{1-2s})$.

Lemma 2.2. *Let* $u\in C^{2\sigma}(\mathbb{R}^n)\cap L^\infty(\Omega)$. *For any constant* $R>0$, *there is a constant* $C_R>0$ *such that*

$$\text{(2.3)} \qquad \int_{B_R^+} t^{1-2s}|\nabla\overline{u}|^2\,dxdt \le C_R.$$

Proof. In view of Theorem 2.1, we have

$$\overline{u}(x,t) = p_{n,s}\int_{\mathbb{R}^n}\frac{t^{2s}\{u(\xi)-u(x)\}}{(|x-\xi|^2+t^2)^{(n+2s)/2}}\,d\xi + u(x).$$

Moreover direct computations show that for $(x,t)\in\mathbb{R}^{n+1}_+$,

$$\partial_{x_i}\overline{u}(x,t) = -(n+2s)p_{n,s}\int_{\mathbb{R}^n}\frac{t^{2s}\{u(\xi)-u(x)\}(x-\xi)_i}{(|x-\xi|^2+t^2)^{(n+2s)/2+1}}\,d\xi$$

and

$$\partial_t \overline{u}(x,t) = 2sp_{n,s} \int_{\mathbb{R}^n} \frac{t^{2s-1}\{u(\xi) - u(x)\}}{(|x-\xi|^2 + t^2)^{(n+2s)/2}}\, d\xi \\ - (n+2s)p_{n,s} \int_{\mathbb{R}^n} \frac{t^{2s+1}\{u(\xi) - u(x)\}}{(|x-\xi|^2 + t^2)^{(n+2s)/2+1}}\, d\xi.$$

Then we have

$$\begin{aligned} t^{1-2s}|\nabla \overline{u}(x,t)|^2 &\le C\Big\{ t^{1+2s}\Big[\int_{\mathbb{R}^n} \frac{|u(\xi) - u(x)||x-\xi|}{(|x-\xi|^2+t^2)^{(n+2s)/2}(|x-\xi|^2+t^2)}\, d\xi \Big]^2 \\ &\quad + t^{2s-1}\Big[\int_{\mathbb{R}^n} \frac{|u(\xi) - u(x)|}{(|x-\xi|^2+t^2)^{(n+2s)/2}}\, d\xi \Big]^2 \\ &\quad + t^{2s+3}\Big[\int_{\mathbb{R}^n} \frac{|u(\xi) - u(x)|}{(|x-\xi|^2+t^2)^{(n+2s)/2}(|x-\xi|^2+t^2)}\, d\xi \Big]^2 \Big\} \\ &\le Ct^{2s-1}\Big[\int_{\mathbb{R}^n} \frac{|u(\xi) - u(x)|}{(|x-\xi|^2+t^2)^{(n+2s)/2}}\, d\xi \Big]^2, \end{aligned}$$

where a positive constant C may vary from line to line. Using a change of variables, it follows that

$$t^{1-2s}|\nabla \overline{u}(x,t)|^2 \le Ct^{-2s-1} \left[\int_{\mathbb{R}^n} \frac{|u(tz) - u(x)|}{(|\frac{x}{t} - z|^2 + 1)^{(n+2s)/2}}\, dz \right]^2,$$

and hence

$$\int_0^R \int_{B_R} t^{1-2s}|\nabla \overline{u}|^2\, dxdt \le C \int_0^R \int_{B_{R/t}} t^{n-2s-1} \left[\int_{\mathbb{R}^n} \frac{|u(tz) - u(ty)|}{(|y-z|^2+1)^{(n+2s)/2}}\, dz \right]^2 dydt.$$

In order to compute the inner integral above, we divide the space $\mathbb{R}^n$ into two regions $D_1 := \{z \in \mathbb{R}^n : t|y-z| \le R\}$ and $D_2 := \{z \in \mathbb{R}^n : t|y-z| > R\}$. Firstly, we assume that $2\sigma \le 1$. By applying the condition $u \in C^{2\sigma}(\mathbb{R}^n)$ to the region D_1 and the condition $u \in L^\infty(\mathbb{R}^n)$ to the region D_2, respectively, we obtain that

$$\begin{aligned} &\int_0^R \int_{B_R} t^{1-2s}|\nabla \overline{u}|^2\, dxdt \\ &\le C \int_0^R \int_{B_{R/t}} t^{n-2s-1} \Big[\int_{|z-y|\le 1} \frac{t^{2\sigma}|z-y|^{2\sigma}}{(|y-z|^2+1)^{(n+2s)/2}}\, dz + \int_{1\le |z-y| \le R/t} \frac{t^{2\sigma}|z-y|^{2\sigma}}{|y-z|^{n+2s}}\, dz \\ &\qquad + \int_{|z-y|>R/t} \frac{\|u\|_{L^\infty(\mathbb{R}^n)}}{|y-z|^{n+2s}}\, dz \Big]^2 dydt \\ &\le C \int_0^R \int_{B_{R/t}} t^{n-2s-1}\big(t^{4\sigma} + t^{4s}\big)\, dydt \le C\big(R^{n+4\sigma-2s} + R^{n+2s}\big). \end{aligned}$$

Here we note that $\sigma > s > 0$, and a positive constant C may vary from line to line and depend on $R > 0$. When $2\sigma > 1$, one can prove the boundedness (2.3) similarly by utilizing the Lipschitz (or Hölder) continuity of u and the fact that $0 < s < 1$. Here we refer to [45, Proposition 2.9] for the regularity regarding the fractional Laplacian operators. □

Employing the even extension of $\overline{u}$ as (1.15), some results on the weighted Sobolev spaces with weight $|t|^\mu$ (for a constant $-1 < \mu < 1$) on the whole space $\mathbb{R}^{n+1}$ can be used to analyze the s-harmonic extension $\overline{u}$ (and a solution u in the fractional Sobolev space; see (2.2)). Here the weight function $|t|^\mu$ (for $|\mu| < 1$) belongs to the class of the Muckenhoupt weight of order 2, denoted by A_2. The Muckenhoupt weights have been extensively studied in the theory of harmonic analysis and partial differential equations; we refer to [21, 22, 40, 41] for instance.

The following compactness of the weighted Sobolev spaces is a local version of [18, Lemma 3.1.2], which will be used in the blow-down analysis. The proof involves the results of the weighted Sobolev spaces in the whole space $\mathbb{R}^{n+1}$ for the even extension given as (1.15).

Lemma 2.3 (Compactness)**.** *Let R and μ be constants with $R > 0$ and $|\mu| < 1$, and let $\{v_k\}_{k=1}^\infty$ be a sequence of functions in $H^1(B_{2R}^+; t^\mu\, dxdt)$ such that*

$$\sup_{k\in\mathbb{N}} \int_{B_{2R}^+} t^\mu \left(|\nabla v_k|^2 + R^{-2}|v_k|^2\right) dxdt < C$$

for a constant $C > 0$. Then there is a convergent subsequence of $\{v_k\}_{k=1}^\infty$ in $L^2(B_R^+; t^\mu\, dxdt)$.

In the next, we will explain that the s-harmonic extension $\overline{u}$ of a finite Morse index solution to the original problem (1.1) satisfies the stability in the following sense.

Lemma 2.4 (Stability for the extension problem)**.** *Let u be a solution to* (1.1) *which is stable on a set $\Omega \subset \mathbb{R}^n$. Then the s-harmonic extension $\overline{u}$ is stable (on Ω) in the following sense: for any $\phi \in C_c^1(\overline{\mathbb{R}_+^{n+1}})$ satisfying* $\operatorname{supp} \phi(\,\cdot\,, 0) \Subset \Omega$*,*

$$\int_{\mathbb{R}^n} \left\{p|x|^a|\overline{u}|^{p-1}\phi^2(x,0) + \gamma|x|^{-2s}\phi^2(x,0)\right\} dx \le \frac{1}{\kappa_s}\int_{\mathbb{R}_+^{n+1}} t^{1-2s}|\nabla\phi(x,t)|^2\, dxdt. \tag{2.4}$$

Proof. We first recall the following trace inequality. Letting X^s be the completion of $C_c^\infty(\overline{\mathbb{R}_+^{n+1}})$ under the norm

$$\|\phi\|_{X^s}^2 = \int_{\mathbb{R}_+^{n+1}} t^{1-2s}|\nabla\phi(x,t)|^2\, dxdt,$$

it holds from [3, Lemma 2.4] that for any $\phi \in X^s$,

$$\|\phi\|_{X^s}^2 = \|\overline{\phi(\,\cdot\,,0)}\|_{X^s}^2 + \|\phi - \overline{\phi(\,\cdot\,,0)}\|_{X^s}^2 \ge \|\overline{\phi(\,\cdot\,,0)}\|_{X^s}^2.$$

Here $\overline{\phi(\,\cdot\,,0)}$ is the s-harmonic extension of $\phi(\,\cdot\,,0)$ given by Theorem 2.1. Then in light of (2.2), we see that for $\phi \in C_c^\infty(\overline{\mathbb{R}_+^{n+1}})$,

$$\kappa_s\|\phi(\,\cdot\,,0)\|_{\dot{H}^s(\mathbb{R}^n)}^2 = \|\overline{\phi(\,\cdot\,,0)}\|_{X^s}^2 \le \|\phi\|_{X^s}^2. \tag{2.5}$$

By the stability of u on Ω (see (1.9)) and (2.5), we have that for $\phi \in C_c^\infty(\overline{\mathbb{R}^{n+1}_+})$ satisfying $\operatorname{supp}\phi(\,\cdot\,,0) \Subset \Omega$,

$$\int_{\mathbb{R}^n} \left\{p|x|^a|u|^{p-1}\phi^2(x,0) + \gamma|x|^{-2s}\phi^2(x,0)\right\} dx \le \|\phi(\,\cdot\,,0)\|^2_{\dot{H}^s(\mathbb{R}^n)} \le \frac{1}{\kappa_s}\|\phi\|^2_{X^s},$$

which yields (2.4). □

3. Energy estimates

In this section, we give energy estimates following proofs of estimates in Section 2 of [14].

Lemma 3.1. *Let $0 < s < \sigma < 1$, $n > 2s$, $a > -2s$, $p > 1$, and $\gamma < \Lambda_{n,s}$. Fix a constant $R_0 \ge 1$ and let $u \in C^{2\sigma}_{\mathrm{loc}}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ be a solution to* (1.1)*, which is stable outside a ball $B_{R_0} \subset \mathbb{R}^n$. For a function $\eta \in C_c^\infty(\mathbb{R}^n \setminus \overline{B_{R_0}})$, define*

$$\rho(x) = \int_{\mathbb{R}^n} \frac{\{\eta(x) - \eta(y)\}^2}{|x-y|^{n+2s}}\, dy, \quad \forall\, x \in \mathbb{R}^n. \tag{3.1}$$

Then

$$\frac{1}{p}\left\{1 - \frac{\max(\gamma, 0)}{\Lambda_{n,s}}\right\}\|u\eta\|^2_{\dot{H}^s(\mathbb{R}^n)} + \int_{\mathbb{R}^n} |x|^a|u|^{p+1}\eta^2\, dx \le \frac{\mathcal{A}_{n,s}}{p-1}\int_{\mathbb{R}^n} u^2\rho\, dx, \tag{3.2}$$

where a constant $\mathcal{A}_{n,s} > 0$ is given by (1.6).

Proof. Multiplying (1.1) by $u\eta^2$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left(|x|^a|u|^{p+1}\eta^2 + \gamma|x|^{-2s}u^2\eta^2\right) dx = \int_{\mathbb{R}^n} (-\Delta)^s u \cdot u\eta^2\, dx \\
&= \mathcal{A}_{n,s}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}} \cdot u(x)\eta^2(x)\, dxdy \\
&= \frac{\mathcal{A}_{n,s}}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\{u(x)-u(y)\}\cdot\{u(x)\eta^2(x) - u(y)\eta^2(y)\}}{|x-y|^{n+2s}}\, dxdy \\
&= \frac{\mathcal{A}_{n,s}}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\{u(x)\eta(x) - u(y)\eta(y)\}^2 - \{\eta(x)-\eta(y)\}^2u(x)u(y)}{|x-y|^{n+2s}}\, dxdy.
\end{aligned}$$

Here we used that $u \in C^{2\sigma}_{\mathrm{loc}}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Then it follows from (1.5) that

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left(|x|^a|u|^{p+1}\eta^2 + \gamma|x|^{-2s}u^2\eta^2\right) dx \\
&= \|u\eta\|^2_{\dot{H}^s(\mathbb{R}^n)} - \frac{\mathcal{A}_{n,s}}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\{\eta(x)-\eta(y)\}^2u(x)u(y)}{|x-y|^{n+2s}}\, dxdy,
\end{aligned}$$

and hence using Young's inequality, we have

$$\|u\eta\|^2_{\dot{H}^s(\mathbb{R}^n)} - \int_{\mathbb{R}^n} \left(|x|^a|u|^{p+1}\eta^2 + \gamma|x|^{-2s}u^2\eta^2\right) dx \le \frac{\mathcal{A}_{n,s}}{2}\int_{\mathbb{R}^n} u^2(x)\rho(x)\, dx. \tag{3.3}$$

This combines with the stability of u outside B_{R_0} to obtain

$$(p-1)\int_{\mathbb{R}^n}|x|^a|u|^{p+1}\eta^2\,dx \le \frac{\mathcal{A}_{n,s}}{2}\int_{\mathbb{R}^n}u^2\rho\,dx \tag{3.4}$$

since the stability of u outside B_{R_0} shows that

$$\int_{\mathbb{R}^n}\left(p|x|^a|u|^{p-1}u^2\eta^2+\gamma|x|^{-2s}u^2\eta^2\right)dx \le \|u\eta\|^2_{\dot{H}^s(\mathbb{R}^n)}.$$

Thus, we deduce from (3.3) and (3.4) that

$$\|u\eta\|^2_{\dot{H}^s(\mathbb{R}^n)}-\int_{\mathbb{R}^n}\gamma|x|^{-2s}u^2\eta^2\,dx \le \frac{p}{p-1}\frac{\mathcal{A}_{n,s}}{2}\int_{\mathbb{R}^n}u^2\rho\,dx.$$

On the other hand, in view of the Hardy equality (1.14), it holds that

$$\int_{\mathbb{R}^n}\gamma|x|^{-2s}u^2\eta^2\,dx \le \frac{\max(\gamma,0)}{\Lambda_{n,s}}\|u\eta\|^2_{\dot{H}^s(\mathbb{R}^n)}.$$

Therefore, the two estimates above imply that

$$\left\{1-\frac{\max(\gamma,0)}{\Lambda_{n,s}}\right\}\|u\eta\|^2_{\dot{H}^s(\mathbb{R}^n)} \le \frac{p}{p-1}\frac{\mathcal{A}_{n,s}}{2}\int_{\mathbb{R}^n}u^2\rho\,dx,$$

which together with (3.4) yields (3.2). □

We recall from [14] the following estimates for ρ given by (3.1) with a particular choice of η.

Lemma 3.2. [14, Lemma 2.2] *For $m > n/2$, let*

$$\eta(x)=(1+|x|^2)^{-m/2} \quad \text{and} \quad \rho(x)=\int_{\mathbb{R}^n}\frac{\{\eta(x)-\eta(y)\}^2}{|x-y|^{n+2s}}\,dy, \quad \forall\, x\in\mathbb{R}^n. \tag{3.5}$$

Then there is a constant $C=C(n,s,m)>1$ such that

$$C^{-1}(1+|x|^2)^{-\frac{n+2s}{2}} \le \rho(x) \le C(1+|x|^2)^{-\frac{n+2s}{2}}, \quad \forall\, x\in\mathbb{R}^n.$$

Corollary 3.3. *Let $m > n/2$ and $R \ge R_0 \ge 1$. Let η be the function as in* (3.5)*, and $\psi \in C^\infty(\mathbb{R}^n)$ be a function such that $0\le\psi\le 1$, $\psi\equiv 0$ on B_1, and $\psi\equiv 1$ on $\mathbb{R}^n\setminus B_2$. Let*

$$\eta_R(x)=\eta\left(\frac{x}{R}\right)\psi\left(\frac{x}{R_0}\right) \quad \text{and} \quad \rho_R(x)=\int_{\mathbb{R}^n}\frac{\{\eta_R(x)-\eta_R(y)\}^2}{|x-y|^{n+2s}}\,dy, \quad \forall\, x\in\mathbb{R}^n.$$

Then there is a constant $C=C(n,s,m,R_0)>0$ such that

$$\rho_R(x) \le C\left\{\eta^2\left(\frac{x}{R}\right)|x|^{-(n+2s)}+R^{-2s}\rho\left(\frac{x}{R}\right)\right\}, \quad \forall\,|x|\ge 3R_0. \tag{3.6}$$

Moreover,

$$\rho_R(x) \ge cR^n|x|^{-(n+2s)}, \quad \forall\,|x|\ge R\ge 3R_0 \tag{3.7}$$

for some constant $c=c(n,s,m)>0$.

Proof. The estimate (3.6) follows from [14, Corollary 2.3]. For a lower bound estimate (3.7), direct computation shows that

$$\begin{aligned}\rho_R(x) &\geq \int_{2R/3\leq|y|\leq 5R/6}\frac{\left\{\eta(\frac{x}{R})-\eta(\frac{y}{R})\right\}^2}{|x-y|^{n+2s}}\,dy = \int_{2/3\leq|z|\leq 5/6} R^{-2s}\frac{\left\{\eta(\frac{x}{R})-\eta(z)\right\}^2}{|\frac{x}{R}-z|^{n+2s}}\,dz \\ &\geq c_0R^n|x|^{-(n+2s)}\int_{2/3\leq|z|\leq 5/6}\left\{\eta(z)-2^{-m/2}\right\}^2dz\end{aligned}$$

for some constant $c_0>0$ since $R\geq 3R_0$. This implies the estimation (3.7). □

Now we estimate the right-hand side of the energy estimate (3.2).

Lemma 3.4. *With the same assumptions as Lemma* 3.1, *let ρ_R be the function given as in Corollary* 3.3 *with $m\in\left(\frac{n}{2},\frac{n}{2}+\frac{s(p+1)+a}{2}\right)$. Then there is a constant $C=C(n,s,p,a,m,R_0)>0$ such that for any $R\geq 3R_0$,*

$$\begin{aligned}&\int_{\mathbb{R}^n}u^2\rho_R\,dx \\ &\leq C\begin{cases}\int_{B_{3R_0}}u^2\rho_R\,dx+R_0^{n-2s\frac{p+1}{p-1}-\frac{2a}{p-1}}+R^{n-2s\frac{p+1}{p-1}-\frac{2a}{p-1}} & \text{if } n\neq\frac{2a}{p-1},\\ \int_{B_{3R_0}}u^2\rho_R\,dx+R_0^{n-(n+2s)\frac{p+1}{p-1}-\frac{2a}{p-1}}+R^{-2s\frac{p+1}{p-1}}\left(\log\frac{R}{3R_0}+1\right) & \text{if } n=\frac{2a}{p-1}.\end{cases}\end{aligned}\tag{3.8}$$

Proof. The proof is similar to the one for [14, Lemma 2.4] and [28, Lemma 4.3]. For the reader's convenience, we will sketch the proof of the case when $n\neq\frac{2a}{p-1}$ since the other is similar. By using Hölder's inequality, we have

$$\begin{aligned}\int_{\mathbb{R}^n}u^2\rho_R\,dx &\leq \int_{B_{3R_0}}u^2\rho_R\,dx \\ &\quad+\left(\int_{\mathbb{R}^n\setminus B_{3R_0}}|x|^a|u|^{p+1}\eta_R^2\,dx\right)^{\frac{2}{p+1}}\left(\int_{\mathbb{R}^n\setminus B_{3R_0}}|x|^{-\frac{2a}{p-1}}\rho_R^{\frac{p+1}{p-1}}\eta_R^{-\frac{4}{p-1}}\,dx\right)^{\frac{p-1}{p+1}}.\end{aligned}$$

Utilizing Young's inequality and Lemma 3.1, we get that

$$\int_{\mathbb{R}^n}u^2\rho_R\,dx\leq C\left(\int_{B_{3R_0}}u^2\rho_R\,dx+\int_{\mathbb{R}^n\setminus B_{3R_0}}|x|^{-\frac{2a}{p-1}}\rho_R^{\frac{p+1}{p-1}}\eta_R^{-\frac{4}{p-1}}\,dx\right)\tag{3.9}$$

for a constant $C>0$ depending on n, s, and p. Here Lemma 3.1 holds true with $\eta=\eta_R$ by an approximation argument. By Lemma 3.2 and Corollary 3.3, it holds that $\rho_R(x)\leq C\left(|x|^{-(n+2s)}+R^{-2s}\right)$ for $3R_0\leq|x|\leq R$, and hence

$$\begin{aligned}&\int_{B_R\setminus B_{3R_0}}|x|^{-\frac{2a}{p-1}}\rho_R^{\frac{p+1}{p-1}}\eta_R^{-\frac{4}{p-1}}\,dx \\ &\leq C\int_{3R_0}^R r^{n-1-\frac{2a}{p-1}}r^{-(n+2s)\frac{p+1}{p-1}}\,dr+CR^{-2s\frac{p+1}{p-1}}\int_{3R_0}^R r^{n-1-\frac{2a}{p-1}}\,dr \\ &\leq C\left(R_0^{n-(n+2s)\frac{p+1}{p-1}-\frac{2a}{p-1}}+R_0^{n-2s\frac{p+1}{p-1}-\frac{2a}{p-1}}+R^{n-2s\frac{p+1}{p-1}-\frac{2a}{p-1}}\right)\end{aligned}\tag{3.10}$$

for a constant $C > 0$ which may depend on n, s, p, a, m and R_0, and vary from line to line. Here we note that $a > -2s$ and $n - \frac{2a}{p-1} \neq 0$. Similarly, by Lemma 3.2 and Corollary 3.3, if $|x| \geq R \geq 3R_0$, then

$$\rho_R(x) \leq c\left\{\left(1+\frac{|x|^2}{R^2}\right)^{-m}|x|^{-(n+2s)} + R^{-2s}\left(1+\frac{|x|^2}{R^2}\right)^{-\frac{n+2s}{2}}\right\},$$

which yields

$$\int_{|x|\geq R} |x|^{-\frac{2a}{p-1}}\rho_R^{\frac{p+1}{p-1}}\eta_R^{-\frac{4}{p-1}}\,dx \leq C\left(R^{n-(n+2s)\frac{p+1}{p-1}-\frac{2a}{p-1}} + R^{n-2s\frac{p+1}{p-1}-\frac{2a}{p-1}}\right). \tag{3.11}$$

Here we used that $m < \frac{n}{2} + \frac{s(p+1)+a}{2}$. From (3.9), (3.10) and (3.11), the estimate (3.8) follows. □

For the supercritical case $p > p_S(n,s,a)$, we derive energy estimates for the s-harmonic extension $\overline{u}$, which will lead to uniform estimates for scaled solutions in the blow-down analysis.

Lemma 3.5. *With the same assumption as Lemma* 3.1*, let* $p > p_S(n,s,a)$ *and* $\overline{u}$ *be the* s*-harmonic extension which satisfies* (2.1)*. Then there is a constant* $C = C(n,s,p,a,R_0,u) > 0$ *such that for any* $R \geq 3R_0$*,*

$$\int_{B_R^+} t^{1-2s}\overline{u}^2\,dxdt \leq CR^{n+2-2s\frac{p+1}{p-1}-\frac{2a}{p-1}}.$$

Proof. By Theorem 2.1, we have that for $(x,t) \in \mathbb{R}^{n+1}_+$,

$$\overline{u}(x,t) = p_{n,s}\int_{\mathbb{R}^n} u(z)\cdot\frac{t^{2s}}{(|x-z|^2+t^2)^{(n+2s)/2}}\,dz$$

and Hölder's inequality implies that

$$\overline{u}^2(x,t) \leq p_{n,s}\int_{\mathbb{R}^n} u^2(z)\cdot\frac{t^{2s}}{(|x-z|^2+t^2)^{(n+2s)/2}}\,dz.$$

Integrating over B_R^+, we get that

$$\begin{aligned}
&\int_{B_R^+} t^{1-2s}\overline{u}^2\,dxdt\\
&\leq p_{n,s}\int_{|x|\leq R, z\in\mathbb{R}^n} u^2(z)\left\{\int_0^R \frac{t}{(|x-z|^2+t^2)^{(n+2s)/2}}\,dt\right\}dzdx\\
&\leq C\int_{|x|\leq R, z\in\mathbb{R}^n} u^2(z)\cdot\left||x-z|^{-(n+2s-2)} - \left(|x-z|^2+R^2\right)^{-\frac{n+2s-2}{2}}\right|\,dzdx,
\end{aligned}$$

where we note that $n + 2s \neq 2$.

Now we split the above integral into integrals over $\{|x-z|<4R\}$ and $\{|x-z|\geq 4R\}$. For the region $\{|x-z|<4R\}$, we have

$$\begin{aligned}
&\int_{\{|x|\leq R,|x-z|<4R\}} u^2(z)\cdot\left||x-z|^{-(n+2s-2)}-\left(|x-z|^2+R^2\right)^{-\frac{n+2s-2}{2}}\right|dzdx \\
&\leq \int_{\{|x|\leq R,|x-z|<4R\}} u^2(z)\cdot\left\{|x-z|^{-(n+2s-2)}+\left(|x-z|^2+R^2\right)^{-\frac{n+2s-2}{2}}\right\}dzdx \\
&\leq CR^{2(1-s)}\int_{B_{5R}} u^2(z)\,dz
\end{aligned} \tag{3.12}$$

since $\{|x|\leq R,|x-z|<4R\}\subset\{|z|\leq 5R,|x-z|<4R\}$. Then Hölder's inequality and Lemmas 3.1, 3.2 and 3.4 yield that

$$\begin{aligned}
&\int_{B_{5R}} u^2(z)\,dz \\
&\leq \int_{B_{3R_0}} u^2(z)\,dz+\left(\int_{\mathbb{R}^n\setminus B_{3R_0}}|x|^a|u|^{p+1}\eta_R^2\right)^{\frac{2}{p+1}}\left(\int_{B_{5R}\setminus B_{3R_0}}|x|^{-\frac{2a}{p-1}}\eta_R^{-\frac{4}{p-1}}\right)^{\frac{p-1}{p+1}} \\
&\leq \int_{B_{3R_0}} u^2(z)\,dz+CR^{\left(n-\frac{2a}{p-1}\right)\frac{p-1}{p+1}}\left(\int_{\mathbb{R}^n} u^2(z)\rho_R(z)\,dz\right)^{\frac{2}{p+1}} \\
&\leq CR^{n-\frac{4s}{p-1}-\frac{2a}{p-1}},
\end{aligned}$$

where we used the assumption $p>p_S(n,s,a)$, and a constant $C=C(n,s,p,a,R_0,u)>0$ may vary from line to line. Thus this estimate combines with (3.12) to have

$$\begin{aligned}
&\int_{\{|x|\leq R,|x-z|<4R\}} u^2(z)\cdot\left||x-z|^{-(n+2s-2)}-\left(|x-z|^2+R^2\right)^{-\frac{n+2s-2}{2}}\right|dzdx \\
&\leq CR^{n+2-2s\frac{p+1}{p-1}-\frac{2a}{p-1}}.
\end{aligned} \tag{3.13}$$

For the region $\{|x-z|\geq 4R\}$, it follows by the mean-value theorem, Corollary 3.3 and Lemma 3.4 that

$$\begin{aligned}
&\int_{|x|\leq R,|x-z|\geq 4R} u^2(z)\cdot\left||x-z|^{-(n+2s-2)}-\left(|x-z|^2+R^2\right)^{-\frac{n+2s-2}{2}}\right|dzdx \\
&\leq CR^2\int_{\{|x|\leq R,|x-z|\geq 4R\}} u^2(z)|x-z|^{-(n+2s)}\,dzdx \\
&\leq CR^{n+2}\int_{\{|z|\geq 3R\}} u^2(z)|z|^{-(n+2s)}\,dz \\
&\leq CR^2\int_{\{|z|\geq R\}} u^2(z)\rho_R(z)\,dz\leq CR^{n+2-2s\frac{p+1}{p-1}-\frac{2a}{p-1}}.
\end{aligned}$$

This finishes the proof with the use of (3.13). □

Lemma 3.6. *With the same assumption as Lemma* 3.5*, there is a constant* $C = C(n, s, p, a, \gamma, R_0, u) > 0$ *such that for any* $R \geq 3R_0$*,*

$$\int_{B_R^+ \setminus B_{2R_0}^+} t^{1-2s} |\nabla \overline{u}|^2 \, dxdt + \int_{B_R} \left(|x|^a |u|^{p+1} + |x|^{-2s} u^2 \right) dx \leq C R^{n-2s\frac{p+1}{p-1} - \frac{2a}{p-1}}.$$

Proof. Let $\eta \in C_c^\infty(\overline{\mathbb{R}_+^{n+1}})$ be a cut-off function such that $\eta \equiv 1$ on $\overline{B_R^+ \setminus B_{2R_0}^+}$ and $\eta \equiv 0$ on $B_{R_0}^+ \cup (\mathbb{R}_+^{n+1} \setminus B_{2R}^+)$. Multiplying (2.1) by $\overline{u}\eta^2$, it holds that

$$\begin{aligned}
&\kappa_s \int_{\partial \mathbb{R}_+^{n+1}} \left\{ |x|^a |\overline{u}|^{p+1} \eta^2(x,0) + \gamma |x|^{-2s} \overline{u}^2 \eta^2(x,0) \right\} dx \\
&= \int_{\mathbb{R}_+^{n+1}} t^{1-2s} \nabla \overline{u} \cdot \nabla (\overline{u}\eta^2) \, dxdt \\
&= \int_{\mathbb{R}_+^{n+1}} t^{1-2s} \left\{ |\nabla (\overline{u}\eta)|^2 - \overline{u}^2 |\nabla \eta|^2 \right\} dxdt.
\end{aligned} \tag{3.14}$$

Since u is stable outside B_{R_0}, it follows from Lemma 2.4, the Hardy inequality (1.14) and the trace inequality (2.5) that

$$\begin{aligned}
&\kappa_s \int_{\partial \mathbb{R}_+^{n+1}} \left\{ |x|^a |\overline{u}|^{p+1} \eta^2(x,0) + \gamma |x|^{-2s} \overline{u}^2 \eta^2(x,0) \right\} dx \\
&\leq \frac{1}{p} \int_{\mathbb{R}_+^{n+1}} t^{1-2s} |\nabla (\overline{u}\eta)|^2 \, dxdt + \kappa_s \left(1 - \frac{1}{p} \right) \int_{\partial \mathbb{R}_+^{n+1}} \gamma |x|^{-2s} \overline{u}^2 \eta^2(x,0) \, dx \\
&\leq \frac{1}{p} \int_{\mathbb{R}_+^{n+1}} t^{1-2s} |\nabla (\overline{u}\eta)|^2 \, dxdt + \kappa_s \left(1 - \frac{1}{p} \right) \frac{\max(\gamma, 0)}{\Lambda_{n,s}} \| \overline{u}\eta(\,\cdot\,, 0) \|_{\dot{H}^s(\mathbb{R}^n)}^2 \\
&\leq \left\{ \frac{1}{p} + \left(1 - \frac{1}{p} \right) \frac{\max(\gamma, 0)}{\Lambda_{n,s}} \right\} \int_{\mathbb{R}_+^{n+1}} t^{1-2s} |\nabla (\overline{u}\eta)|^2 \, dxdt.
\end{aligned} \tag{3.15}$$

This combined with (3.14) implies

$$\left[1 - \left\{ \frac{1}{p} + \left(1 - \frac{1}{p} \right) \frac{\max(\gamma, 0)}{\Lambda_{n,s}} \right\} \right] \int_{\mathbb{R}_+^{n+1}} t^{1-2s} |\nabla (\overline{u}\eta)|^2 \, dxdt \leq \int_{\mathbb{R}_+^{n+1}} t^{1-2s} \overline{u}^2 |\nabla \eta|^2 \, dxdt. \tag{3.16}$$

Then we have that

$$\begin{aligned}
\int_{B_R^+ \setminus B_{2R_0}^+} t^{1-2s} |\nabla \overline{u}|^2 \, dxdt &\leq C \int_{\mathbb{R}_+^{n+1}} t^{1-2s} \overline{u}^2 |\nabla \eta|^2 \, dxdt \\
&\leq C \int_{B_{2R_0}^+} t^{1-2s} \overline{u}^2 \, dxdt + C R^{-2} \int_{B_{2R}^+ \setminus B_R^+} t^{1-2s} \overline{u}^2 \, dxdt.
\end{aligned} \tag{3.17}$$

By utilizing (1.14) and (2.5), we deduce

$$\kappa_s \int_{\partial \mathbb{R}_+^{n+1}} |x|^{-2s} \overline{u}^2 \eta^2(\,\cdot\,, 0) \, dx \leq \frac{\kappa_s}{\Lambda_{n,s}} \| \overline{u}\eta(\,\cdot\,, 0) \|_{\dot{H}^s(\mathbb{R}^n)}^2 \leq \frac{1}{\Lambda_{n,s}} \int_{\mathbb{R}_+^{n+1}} t^{1-2s} |\nabla (\overline{u}\eta)|^2 \, dxdt, \tag{3.18}$$

and in light of (3.15) and (3.18),

(3.19)
$$\kappa_s \int_{\partial\mathbb{R}^{n+1}_+} |x|^a |\overline{u}|^{p+1}\eta^2(x,0)\,dx \le \left\{\frac{1}{p} + \left(1-\frac{1}{p}\right)\frac{\max(\gamma,0)}{\Lambda_{n,s}}\right\}\int_{\mathbb{R}^{n+1}_+} t^{1-2s}|\nabla(\overline{u}\eta)|^2\,dxdt + \frac{\max(-\gamma,0)}{\Lambda_{n,s}}\int_{\mathbb{R}^{n+1}_+} t^{1-2s}|\nabla(\overline{u}\eta)|^2\,dxdt.$$

Therefore the result follows from (3.16)–(3.19) and Lemma 3.5. □

4. Subcritical and critical cases

In this section, we prove Theorem 1.1 in the case when $1 < p \le p_S(n,s,a)$.

Proof of Theorem 1.1 *in the subcritical and critical cases.* Firstly, we may assume that a solution u to (1.1) is stable outside B_{R_0} with a constant $R_0 \ge 1$. By letting $R \to +\infty$ in (3.8) and utilizing Lemma 3.1, we have

$$\limsup_{R\to\infty}\left\{\|u\eta_R\|^2_{\dot{H}^s(\mathbb{R}^n)} + \left\||x|^a|u|^{p+1}\eta_R^2\right\|_{L^1(\mathbb{R}^n)}\right\} < \infty$$

and hence it follows from (1.5) that

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u\psi_0(x)-u\psi_0(y)|^2}{|x-y|^{n+2s}}\,dxdy < \infty$$

and $u\psi_0$ belongs to $\dot{H}^s(\mathbb{R}^n)\cap L^{p+1}(\mathbb{R}^n;|x|^a\,dx)$, where $\psi_0 := \psi\left(\frac{\cdot}{R_0}\right)$ with ψ is given in Corollary 3.3. Then using the assumption that $u \in C^{2\sigma}_{\mathrm{loc}}(\mathbb{R}^n)$, we deduce that

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dxdy < \infty;$$

see the proof of Lemma 5.1 of [17] for the estimate of $u(1-\psi_0)$. Here we note that $\mathrm{supp}(1-\psi_0) \subset B_{2R_0}$. Thus in light of (1.5), we conclude that u belongs to $\dot{H}^s(\mathbb{R}^n)\cap L^{p+1}(\mathbb{R}^n;|x|^a\,dx)$, and the Hardy inequality (1.14) holds with $\phi = u$. By multiplying the equation (1.1) by u and integrating, it holds that

$$\gamma\int_{\mathbb{R}^n}|x|^{-2s}u^2\,dx + \int_{\mathbb{R}^n}|x|^a|u|^{p+1}\,dx = \|u\|^2_{\dot{H}^s(\mathbb{R}^n)} \tag{4.1}$$

in view of (1.8). Here we used the assumption that $u \in C^{2\sigma}(\mathbb{R}^n)\cap L^1(\mathbb{R}^n)$.

Direct computation shows that for $x \in \mathbb{R}^n\setminus\{0\}$,

$$\left(\gamma|x|^{-2s}u + |x|^a|u|^{p-1}u\right)\nabla u\cdot x = \mathrm{div}\left(\frac{\gamma|x|^{-2s}u^2}{2}x + \frac{|x|^a|u|^{p+1}}{p+1}x\right) - \left(\frac{n-2s}{2}\gamma|x|^{-2s}u^2 + \frac{n+a}{p+1}|x|^a|u|^{p+1}\right).$$

Here we note that $u \in C^1(\mathbb{R}^n \setminus \{0\})$ from the regularity theory since $u \in C^{2\sigma}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n; (1+|x|)^{-n-2s})$; we refer to [44] for instance. Thus we have

$$
\begin{aligned}
&\int_{B_R \setminus B_\varepsilon} \big(\gamma |x|^{-2s} u + |x|^a |u|^{p-1} u\big) \nabla u \cdot x \, dx \\
&+ \int_{B_R \setminus B_\varepsilon} \left(\frac{n-2s}{2} \gamma |x|^{-2s} u^2 + \frac{n+a}{p+1} |x|^a |u|^{p+1} \right) dx \qquad (4.2)\\
&= R \int_{\partial B_R} \left(\frac{\gamma |x|^{-2s} u^2}{2} + \frac{|x|^a |u|^{p+1}}{p+1} \right) dS_x - \varepsilon \int_{\partial B_\varepsilon} \left(\frac{\gamma |x|^{-2s} u^2}{2} + \frac{|x|^a |u|^{p+1}}{p+1} \right) dS_x
\end{aligned}
$$

for a small $\varepsilon > 0$.

On the other hand, let $X = (x,t)$. By an argument in the proof of [4, Lemma 3.1] with the use of the first equation of (2.1), we have

$$
\operatorname{div} \left\{ t^{1-2s} \left((X \cdot \nabla \overline{u}) \nabla \overline{u} - \frac{|\nabla \overline{u}|^2}{2} X \right) \right\} + \frac{n-2s}{2} t^{1-2s} |\nabla \overline{u}|^2 = 0 \quad \text{in } \mathbb{R}^{n+1}_+.
$$

Integrating on $B_R^+ \setminus B_\varepsilon^+$ and using (2.1) imply that

$$
\begin{aligned}
&\frac{n-2s}{2} \int_{B_R^+ \setminus B_\varepsilon^+} t^{1-2s} |\nabla \overline{u}|^2 \, dxdt + \kappa_s \int_{B_R \setminus B_\varepsilon} \big(\gamma |x|^{-2s} u + |x|^a |u|^{p-1} u\big) \nabla u \cdot x \, dx \\
&= -R \int_{\partial B_R^+ \cap \mathbb{R}^{n+1}_+} t^{1-2s} \big|\partial_r \overline{u}\big|^2 \, dS_{x,t} + \varepsilon \int_{\partial B_\varepsilon^+ \cap \mathbb{R}^{n+1}_+} t^{1-2s} \big|\partial_r \overline{u}\big|^2 \, dS_{x,t} \\
&\quad + \frac{R}{2} \int_{\partial B_R^+ \cap \mathbb{R}^{n+1}_+} t^{1-2s} |\nabla \overline{u}|^2 \, dS_{x,t} - \frac{\varepsilon}{2} \int_{\partial B_\varepsilon^+ \cap \mathbb{R}^{n+1}_+} t^{1-2s} |\nabla \overline{u}|^2 \, dS_{x,t}
\end{aligned}
$$

where $r = |X|$ for $X = (x,t) \in \mathbb{R}^{n+1}_+$. Together with (4.2), this yields that

$$
\begin{aligned}
&\frac{n-2s}{2} \int_{B_R^+ \setminus B_\varepsilon^+} t^{1-2s} |\nabla \overline{u}|^2 \, dxdt - \kappa_s \int_{B_R \setminus B_\varepsilon} \left(\frac{n-2s}{2} \gamma |x|^{-2s} u^2 + \frac{n+a}{p+1} |x|^a |u|^{p+1} \right) dx \\
&= -R \int_{\partial B_R^+ \cap \mathbb{R}^{n+1}_+} t^{1-2s} \big|\partial_r \overline{u}\big|^2 \, dS_{x,t} + \varepsilon \int_{\partial B_\varepsilon^+ \cap \mathbb{R}^{n+1}_+} t^{1-2s} \big|\partial_r \overline{u}\big|^2 \, dS_{x,t} \\
&\quad + \frac{R}{2} \int_{\partial B_R^+ \cap \mathbb{R}^{n+1}_+} t^{1-2s} |\nabla \overline{u}|^2 \, dS_{x,t} - \frac{\varepsilon}{2} \int_{\partial B_\varepsilon^+ \cap \mathbb{R}^{n+1}_+} t^{1-2s} |\nabla \overline{u}|^2 \, dS_{x,t} \\
&\quad - \kappa_s R \int_{\partial B_R} \left(\frac{\gamma |x|^{-2s} u^2}{2} + \frac{|x|^a |u|^{p+1}}{p+1} \right) dS_x + \kappa_s \varepsilon \int_{\partial B_\varepsilon} \left(\frac{\gamma |x|^{-2s} u^2}{2} + \frac{|x|^a |u|^{p+1}}{p+1} \right) dS_x.
\end{aligned}
$$

Since $u \in \dot{H}^s(\mathbb{R}^n)$, and $|x|^{-2s}u^2$ and $|x|^a|u|^{p+1}$ are integrable by (4.1) and the Hardy inequality (1.14) with $\phi = u$, we let $R \to \infty$ and $\varepsilon \to 0$ (with suitably chosen sequences using the coarea formula) in order to get

$$
\frac{n-2s}{2} \int_{\mathbb{R}^{n+1}_+} t^{1-2s} |\nabla \overline{u}|^2 \, dxdt = \kappa_s \int_{\mathbb{R}^n} \left(\frac{n-2s}{2} \gamma |x|^{-2s} u^2 + \frac{n+a}{p+1} |x|^a |u|^{p+1} \right) dx.
$$

Here we also used the equality (2.2). Then utilizing (2.2) yields the following Pohozaev identity

$$\frac{n-2s}{2}\left(\|u\|^2_{\dot{H}^s(\mathbb{R}^n)} - \gamma\int_{\mathbb{R}^n}|x|^{-2s}u^2\,dx\right) = \frac{n+a}{p+1}\int_{\mathbb{R}^n}|x|^a|u|^{p+1}\,dx.$$

This combined with (4.1) gives that

$$\left(\frac{n-2s}{2} - \frac{n+a}{p+1}\right)\int_{\mathbb{R}^n}|x|^a|u|^{p+1}\,dx = 0.$$

Therefore we conclude that $u \equiv 0$ when $1 < p < p_S(n,s,a)$.

In the case when $p = p_S(n,s,a)$, suppose that u is a stable solution in $\mathbb{R}^n$. Since $u \in \dot{H}^s(\mathbb{R}^n)$, it follows from the stability (1.9) with a test function $\phi = u$ and (4.1) that

$$p\int_{\mathbb{R}^n}|x|^a|u|^{p+1}\,dx \le \|u\|^2_{\dot{H}^s(\mathbb{R}^n)} - \gamma\int_{\mathbb{R}^n}|x|^{-2s}u^2\,dx = \int_{\mathbb{R}^n}|x|^a|u|^{p+1}\,dx$$

which yields $u \equiv 0$. □

5. Monotonicity formula

This section is devoted to the proof of the following monotonicity formula.

Theorem 5.1. *Let $\overline{u} \in C^2(\mathbb{R}^{n+1}_+) \cap C(\overline{\mathbb{R}^{n+1}_+})$ with $t^{1-2s}\partial_t\overline{u} \in C(\overline{\mathbb{R}^{n+1}_+})$ be a solution to (2.1). For $\lambda > 0$, let*

$$\begin{aligned} &E(\overline{u};\lambda)\\ &= \lambda^{2s\frac{p+1}{p-1}+\frac{2a}{p-1}-n}\left\{\frac{1}{2}\int_{B^+_\lambda} t^{1-2s}|\nabla\overline{u}|^2\,dxdt - \kappa_s\int_{B_\lambda\cap\partial\mathbb{R}^{n+1}_+}\left(\frac{\gamma|x|^{-2s}|\overline{u}|^2}{2} + \frac{|x|^a|\overline{u}|^{p+1}}{p+1}\right)dx\right\}\\ &\quad + \lambda^{2s\frac{p+1}{p-1}+\frac{2a}{p-1}-n-1}\cdot\frac{2s+a}{2(p-1)}\int_{\partial B^+_\lambda\cap\mathbb{R}^{n+1}_+} t^{1-2s}\overline{u}^2\,dS_{x,t}. \end{aligned}$$

Then, E is a nondecreasing function of λ, and

$$\frac{dE}{d\lambda} = \lambda^{2s\frac{p+1}{p-1}+\frac{2a}{p-1}-n-2}\int_{\partial B^+_\lambda\cap\mathbb{R}^{n+1}_+} t^{1-2s}\left(r\partial_r\overline{u} + \frac{2s+a}{p-1}\overline{u}\right)^2 dS_{x,t}, \tag{5.1}$$

where $r = |X|$ for $X = (x,t) \in \mathbb{R}^{n+1}_+$.

Proof. The proof is similar to the proof of [14, Theorem 1.4]. For the reader's convenience, we will briefly sketch it. Let

$$\begin{aligned} E_1(\overline{u};\lambda) &:= \lambda^{2s\frac{p+1}{p-1}+\frac{2a}{p-1}-n}\\ &\quad\times\left\{\frac{1}{2}\int_{B^+_\lambda} t^{1-2s}|\nabla\overline{u}|^2\,dxdt - \kappa_s\int_{B_\lambda\cap\partial\mathbb{R}^{n+1}_+}\left(\frac{\gamma|x|^{-2s}|\overline{u}|^2}{2} + \frac{|x|^a|\overline{u}|^{p+1}}{p+1}\right)dx\right\}. \end{aligned}$$

Define $\overline{u}^\lambda$ by

$$\overline{u}^\lambda(X) = \lambda^{\frac{2s+a}{p-1}}\overline{u}(\lambda X) \quad \text{for } X = (x,t) \in \mathbb{R}^{n+1}_+.$$

Then $\overline{u}^\lambda$ also solves the equation (2.1), and it holds that

$$\begin{aligned} E_1(\overline{u};\lambda) &= E_1(\overline{u}^\lambda;1) \\ &= \frac{1}{2}\int_{B_1^+} t^{1-2s}|\nabla \overline{u}^\lambda|^2\,dxdt \\ &\quad - \kappa_s \int_{B_1\cap\partial\mathbb{R}^{n+1}_+}\left(\frac{\gamma|x|^{-2s}|\overline{u}^\lambda|^2}{2} + \frac{|x|^a|\overline{u}^\lambda|^{p+1}}{p+1}\right)dx. \end{aligned} \tag{5.2}$$

Differentiating (5.2) with respect to λ and using integration by parts, we get that

$$\begin{aligned} \frac{\partial E_1(\overline{u};\lambda)}{\partial\lambda} &= \int_{B_1^+} t^{1-2s}\nabla\overline{u}^\lambda\cdot\nabla(\partial_\lambda\overline{u}^\lambda)\,dxdt \\ &\quad - \kappa_s\int_{B_1\cap\partial\mathbb{R}^{n+1}_+}\left(\gamma|x|^{-2s}\overline{u}^\lambda + |x|^a|\overline{u}^\lambda|^{p-1}\overline{u}^\lambda\right)\partial_\lambda\overline{u}^\lambda\,dx \\ &= \int_{\partial B_1^+\cap\mathbb{R}^{n+1}_+} t^{1-2s}\partial_r\overline{u}^\lambda\partial_\lambda\overline{u}^\lambda\,dS_{x,t}. \end{aligned}$$

Here we used the fact that $\overline{u}^\lambda$ is a solution to (2.1). Since

$$\lambda\partial_\lambda\overline{u}^\lambda(x,t) = r\partial_r\overline{u}^\lambda(x,t) + \frac{2s+a}{p-1}\overline{u}^\lambda(x,t), \tag{5.3}$$

we deduce that

$$\begin{aligned} \frac{\partial E_1(\overline{u};\lambda)}{\partial\lambda} &= \int_{\partial B_1^+\cap\mathbb{R}^{n+1}_+} t^{1-2s}\left(\lambda\partial_\lambda\overline{u}^\lambda - \frac{2s+a}{p-1}\overline{u}^\lambda\right)\partial_\lambda\overline{u}^\lambda\,dS_{x,t} \\ &= \int_{\partial B_1^+\cap\mathbb{R}^{n+1}_+} t^{1-2s}\lambda^{-1}\left(\lambda\partial_\lambda\overline{u}^\lambda\right)^2 dS_{x,t} \\ &\quad - \frac{\partial}{\partial\lambda}\left[\frac{2s+a}{2(p-1)}\int_{\partial B_1^+\cap\mathbb{R}^{n+1}_+} t^{1-2s}\left(\overline{u}^\lambda\right)^2 dS_{x,t}\right]. \end{aligned}$$

Utilizing (5.3) and scaling back, the monotonicity (5.1) follows. □

6. Homogeneous solution

In order to prove the stability result in the supercritical case, we first derive the following Liouville type theorem for stable homogeneous solutions to the s-harmonic extension problem. Here we impose the conditions $\gamma < \gamma_{n,s,a}(p)$, $p > p_S(n,s,a)$ and (P), and the proof uses a similar argument in Section 5 of [14].

Theorem 6.1. *Assume that* $\gamma < \gamma_{n,s,a}(p)$, $p > p_S(n,s,a)$ *and* (P). *Let* $\overline{u} \in H^1_{\mathrm{loc}}(\overline{\mathbb{R}^{n+1}_+} \setminus \{0\}; t^{1-2s}\,dxdt)$ *with* $u := \overline{u}(\,\cdot\,,0) \in L^{p+1}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$ *be a homogeneous solution of*

$$\begin{cases} -\nabla\cdot(t^{1-2s}\nabla\overline{u}) = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ -\lim_{t\to 0} t^{1-2s}\partial_t\overline{u} = \kappa_s\big(\gamma|x|^{-2s}u + |x|^a|u|^{p-1}u\big) & \text{on } \partial\mathbb{R}^{n+1}_+\setminus\{0\} \end{cases} \tag{6.1}$$

in the distributional sense, that is, for any $\phi \in C_c^\infty(\overline{\mathbb{R}^{n+1}_+} \setminus \{0\})$,

$$\int_{\mathbb{R}^{n+1}_+} t^{1-2s}\nabla\overline{u}\cdot\nabla\phi\,dxdt = \kappa_s\int_{\partial\mathbb{R}^{n+1}_+}\big(\gamma|x|^{-2s}u + |x|^a|u|^{p-1}u\big)\phi(x,0)\,dx. \tag{6.2}$$

If $\overline{u}$ *is stable except the origin in the sense of Lemma 2.4, i.e., for any* $\phi \in C_c^1(\overline{\mathbb{R}^{n+1}_+}\setminus\{0\})$,

$$\kappa_s\int_{\mathbb{R}^n}\big\{p|x|^a|u|^{p-1}\phi^2(x,0) + \gamma|x|^{-2s}\phi^2(x,0)\big\}\,dx \le \int_{\mathbb{R}^{n+1}_+} t^{1-2s}|\nabla\phi|^2\,dxdt, \tag{6.3}$$

then $\overline{u} \equiv 0$.

Proof. We consider standard polar coordinates in $\mathbb{R}^{n+1}$: $X = (x,t) = r\theta$, where $r = |X|$ and $\theta = \frac{X}{|X|}$. Let $\theta_1 = \frac{t}{|X|}$ denote the component of θ in the t direction and $S^n_+ = \{X \in \mathbb{R}^{n+1} : r = 1, \theta_1 > 0\}$ denote the upper half of the unit sphere.

Step 1. Since $\overline{u}$ is a homogeneous solution of (6.1), we may assume that for some $\psi \in H^1(S^n_+;\theta_1^{1-2s})$,

$$\overline{u}(X) = r^{-\frac{2s+a}{p-1}}\psi(\theta). \tag{6.4}$$

Here $H^1(S^n_+;\theta_1^{1-2s})$ is the completion of $C^\infty(\overline{S^n_+})$ with respect to the norm

$$\|\psi\|^2_{H^1(S^n_+;\theta_1^{1-2s})} = \int_{S^n_+}\theta_1^{1-2s}\big\{\psi^2(\theta) + |\nabla_{S^n}\psi|^2\big\}.$$

Since $\overline{u}$ solves (6.1), ψ satisfies

$$\begin{cases} -\operatorname{div}_{S^n}\big(\theta_1^{1-2s}\nabla_{S^n}\psi\big) + \beta\theta_1^{1-2s}\psi = 0 & \text{on } S^n_+, \\ -\lim_{\theta_1\to 0}\theta_1^{1-2s}\partial_{\theta_1}\psi = \kappa_s\big(\gamma\psi + |\psi|^{p-1}\psi\big) & \text{on } \partial S^n_+, \end{cases} \tag{6.5}$$

where $\partial_{\theta_1}\psi$ is the directional derivative of ψ along the inward unit normal vector to ∂S^n_+, the boundary of S^n_+, and a positive constant β is given by

$$\beta = \frac{2s+a}{p-1}\left(n - 2s - \frac{2s+a}{p-1}\right).$$

By multiplying (6.5) by ψ and integrating by parts, we get

$$\int_{S^n_+}\theta_1^{1-2s}|\nabla_{S^n}\psi|^2 + \beta\int_{S^n_+}\theta_1^{1-2s}\psi^2 = \kappa_s\int_{\partial S^n_+}\big(\gamma\psi^2 + |\psi|^{p+1}\big). \tag{6.6}$$

Step 2. We claim that for any $\varphi \in H^1(S^n_+;\theta_1^{1-2s})$,

$$\kappa_s \int_{\partial S^n_+} \big(p|\psi|^{p-1}+\gamma\big)\varphi^2 \le \int_{S^n_+} \theta_1^{1-2s}|\nabla_{S^n}\varphi|^2 + \left(\frac{n-2s}{2}\right)^2 \int_{S^n_+} \theta_1^{1-2s}\varphi^2. \tag{6.7}$$

For a small constant $\varepsilon \in (0,1)$, we choose a standard cut-off function $\eta_\varepsilon \in C_c^\infty(\mathbb{R}_+)$ at the origin and at infinity, i.e., $\chi_{(\varepsilon,1/\varepsilon)}(r) \le \eta_\varepsilon(r) \le \chi_{(\varepsilon/2,2/\varepsilon)}(r)$, and let $\varphi \in H^1(S^n_+;\theta_1^{1-2s}) \cap C^\infty(\overline{S^n_+})$. Then we use the stability (6.3) with

$$\phi(X) = r^{-(n-2s)/2}\eta_\varepsilon(r)\varphi(\theta) \quad \text{for } X \in \mathbb{R}^{n+1}_+$$

to obtain that

$$\begin{aligned}
&\kappa_s \int_{\partial S^n_+} (p|\psi|^{p-1}+\gamma)\varphi^2 \cdot \int_0^\infty \frac{1}{r}\eta_\varepsilon^2(r)\,dr \\
&\le \int_{S^n_+} \theta_1^{1-2s}|\nabla_{S^n}\varphi|^2 \cdot \int_0^\infty \frac{1}{r}\eta_\varepsilon^2(r)\,dr \\
&\quad + \int_{S^n_+} \theta_1^{1-2s}\varphi^2 \cdot \int_0^\infty \frac{1}{r}\left\{r\eta_\varepsilon'(r) - \left(\frac{n-2s}{2}\right)\eta_\varepsilon(r)\right\}^2 dr.
\end{aligned}$$

Since

$$2\log\frac{1}{\varepsilon} \le \int_0^\infty \frac{1}{r}\eta_\varepsilon^2(r)\,dr \le 2\log\frac{2}{\varepsilon}, \quad \forall\, 0<\varepsilon<1$$

and $\int_0^\infty r\big(\eta_\varepsilon'\big)^2(r)\,dr$ is uniformly bounded for any $0<\varepsilon<1$ from the choice of η_ε, the inequality (6.7) holds for any $\varphi \in C^\infty(\overline{S^n_+})$ by letting $\epsilon \to 0$. Since $C^\infty(\overline{S^n_+})$ is dense in $H^1(S^n_+;\theta_1^{1-2s})$, we deduce that (6.7) holds for any $\varphi \in H^1(S^n_+;\theta_1^{1-2s})$. Here we also used the trace inequality [24, Lemma 2.2] and the Fatou lemma.

Step 3. As in [23, Lemma 3.1] by Fall, for $\alpha \in [0,(n-2s)/2)$, let

$$v_\alpha(x) = |x|^{-(n-2s)/2+\alpha}, \quad \forall\, x \in \mathbb{R}^n \setminus \{0\},$$

and $\overline{v}_\alpha$ be its s-harmonic extension given by Theorem 2.1. Then $\overline{v}_\alpha \in C^2(\mathbb{R}^{n+1}_+) \cap C(\overline{\mathbb{R}^{n+1}_+} \setminus \{0\})$ satisfies

$$\begin{cases}
-\nabla\cdot\big(t^{1-2s}\nabla\overline{v}_\alpha\big) = 0 & \text{in } \mathbb{R}^{n+1}_+,\\
\overline{v}_\alpha = v_\alpha & \text{on } \partial\mathbb{R}^{n+1}_+ \setminus \{0\},\\
-\lim_{t\to 0} t^{1-2s}\partial_t\overline{v}_\alpha = \kappa_s\lambda(\alpha)|x|^{-2s}v_\alpha & \text{on } \partial\mathbb{R}^{n+1}_+ \setminus \{0\},
\end{cases}$$

where a constant $\lambda(\alpha)$ is given by (1.12). In light of the proof of [23, Lemma 3.1], we see that a positive function $\overline{v}_\alpha$ is homogeneous, i.e., there exists a function $\phi_\alpha \in H^1(S^n_+;\theta_1^{1-2s}) \cap C(\overline{S^n_+})$ such that

$$\overline{v}_\alpha(X) = r^{-(n-2s)/2+\alpha}\phi_\alpha(\theta), \quad \forall\, X \in \mathbb{R}^{n+1}_+.$$

Thus it can be checked that $\phi_\alpha > 0$ and $\theta_1^{1-2s}\partial_{\theta_1}\phi_\alpha \in C(\overline{S}^n_+)$, and ϕ_α satisfies

$$\begin{cases} -\operatorname{div}_{S^n}\left(\theta_1^{1-2s}\nabla_{S^n}\phi_\alpha\right) + \left\{\left(\frac{n-2s}{2}\right)^2 - \alpha^2\right\}\theta_1^{1-2s}\phi_\alpha = 0 & \text{on } S^n_+, \\ \phi_\alpha = 1 & \text{on } \partial S^n_+, \\ -\lim_{\theta_1\to 0}\theta_1^{1-2s}\partial_{\theta_1}\phi_\alpha = \kappa_s\lambda(\alpha) & \text{on } \partial S^n_+, \end{cases} \tag{6.8}$$

see also [24, Lemma 2.1]. By multiplying the equation (6.8) by φ^2/ϕ_α and integrating by parts, we deduce that for any $\varphi \in H^1(S^n_+;\theta_1^{1-2s})$,

$$\begin{aligned} &\int_{S^n_+}\theta_1^{1-2s}|\nabla_{S^n}\varphi|^2 + \left\{\left(\frac{n-2s}{2}\right)^2 - \alpha^2\right\}\int_{S^n_+}\theta_1^{1-2s}\varphi^2 \\ &= \kappa_s\lambda(\alpha)\int_{\partial S^n_+}\varphi^2 + \int_{S^n_+}\theta_1^{1-2s}\phi_\alpha^2\left|\nabla_{S^n}\left(\frac{\varphi}{\phi_\alpha}\right)\right|^2, \end{aligned} \tag{6.9}$$

where we used the equality

$$\nabla_{S^n}\phi_\alpha\cdot\nabla_{S^n}\left(\frac{\varphi^2}{\phi_\alpha}\right) = \left|\nabla_{S^n}\varphi\right|^2 - \left|\nabla_{S^n}\left(\frac{\varphi}{\phi_\alpha}\right)\right|^2\phi_\alpha^2.$$

Step 4. We first note that $\phi_\alpha \in C^2(S^n_+)\cap C(\overline{S^n_+})$ for $0 \le \alpha < (n-2s)/2$. Since

$$\operatorname{div}\left(\theta_1^{1-2s}\nabla_{S^n}\phi_0\right) = \left(\frac{n-2s}{2}\right)^2\theta_1^{1-2s}\phi_0 \ge \left\{\left(\frac{n-2s}{2}\right)^2 - \alpha^2\right\}\theta_1^{1-2s}\phi_0 \quad \text{on } S^n_+,$$

and $\phi_0 = \phi_\alpha = 1$ on ∂S^n_+, the maximum principle implies that for any $\alpha \in (0,(n-2s)/2)$,

$$\phi_0 \le \phi_\alpha \quad \text{on } S^n_+. \tag{6.10}$$

Step 5. Now let us fix

$$\alpha := \frac{n-2s}{2} - \frac{2s+a}{p-1} \in \left(0, \frac{n-2s}{2}\right). \tag{6.11}$$

With this choice of α, we have that

$$\left(\frac{n-2s}{2}\right)^2 - \alpha^2 = \frac{2s+a}{p-1}\left(n-2s-\frac{2s+a}{p-1}\right) = \beta \quad \text{and} \quad \lambda(\alpha) = \gamma_{n,s,a}(p). \tag{6.12}$$

By applying (6.7) with $\varphi = \psi\phi_0/\phi_\alpha$ with α as in (6.11), it follows that

$$\begin{aligned} \kappa_s\int_{\partial S^n_+}\left(p|\psi|^{p+1} + \gamma\psi^2\right) \le &\int_{S^n_+}\theta_1^{1-2s}\left|\nabla_{S^n}\left(\frac{\psi\phi_0}{\phi_\alpha}\right)\right|^2 \\ &+ \left(\frac{n-2s}{2}\right)^2\int_{S^n_+}\theta_1^{1-2s}\left(\frac{\psi\phi_0}{\phi_\alpha}\right)^2. \end{aligned} \tag{6.13}$$

If $\alpha = 0$, then the equality (6.9) leads to

$$\int_{S^n_+} \theta_1^{1-2s} |\nabla_{S^n} \varphi|^2 + \left(\frac{n-2s}{2}\right)^2 \int_{S^n_+} \theta_1^{1-2s} \varphi^2 = \kappa_s \Lambda_{n,s} \int_{\partial S^n_+} \varphi^2 + \int_{S^n_+} \theta_1^{1-2s} \phi_0^2 \left|\nabla_{S^n}\left(\frac{\varphi}{\phi_0}\right)\right|^2.$$

Using (6.13) and selecting $\varphi = \psi\phi_0/\phi_\alpha$, this equality yields

$$\kappa_s \int_{\partial S^n_+} \left(p|\psi|^{p+1} + \gamma\psi^2\right) \le \kappa_s \Lambda_{n,s} \int_{\partial S^n_+} \psi^2 + \int_{S^n_+} \theta_1^{1-2s} \phi_0^2 \left|\nabla_{S^n}\left(\frac{\psi}{\phi_\alpha}\right)\right|^2.$$

Then by the comparison (6.10), we have that

$$\kappa_s \int_{\partial S^n_+} \left(p|\psi|^{p+1} + \gamma\psi^2\right) \le \kappa_s \Lambda_{n,s} \int_{\partial S^n_+} \psi^2 + \int_{S^n_+} \theta_1^{1-2s} \phi_\alpha^2 \left|\nabla_{S^n}\left(\frac{\psi}{\phi_\alpha}\right)\right|^2,$$

which combines with (6.9) (with $\varphi = \psi$ and α as in (6.11)) and (6.12) to obtain that

$$\kappa_s \int_{\partial S^n_+} \left(p|\psi|^{p+1} + \gamma\psi^2\right) \le \kappa_s \left\{\Lambda_{n,s} - \gamma_{n,s,a}(p)\right\} \int_{\partial S^n_+} \psi^2 + \int_{S^n_+} \theta_1^{1-2s} |\nabla_{S^n}\psi|^2 + \beta \int_{S^n_+} \theta_1^{1-2s}\psi^2.$$

Then in light of (6.6), the above estimate implies that

$$(p-1)\int_{\partial S^n_+} |\psi|^{p+1} \le \left(\Lambda_{n,s} - \gamma_{n,s,a}(p)\right) \int_{\partial S^n_+} \psi^2. \tag{6.14}$$

On the other hand, by utilizing (6.6), (6.9) (with $\varphi = \psi$ and α as in (6.11)) and (6.12), it holds that

$$\int_{\partial S^n_+} |\psi|^{p+1} \ge \left(\gamma_{n,s,a}(p) - \gamma\right) \int_{\partial S^n_+} \psi^2. \tag{6.15}$$

Therefore, by (6.14) and (6.15), we deduce that

$$\left\{p\left(\gamma_{n,s,a}(p) - \gamma\right) - \Lambda_{n,s} + \gamma\right\} \int_{\partial S^n_+} \psi^2 \le 0.$$

From the assumption (P), it follows that $\psi \equiv 0$ on ∂S^n_+. Since ψ solves (6.5) with a positive constant β, the maximum principle implies that $\psi \equiv 0$ in S^n_+ completing the proof of Theorem 6.1. □

7. Blow-down analysis

In this section, we are going to prove Theorem 1.1 for the supercritical case.

Proof of Theorem 1.1. We assume $\gamma < \gamma_{n,s,a}(p)$, $p > p_S(n,s,a)$ and (P). For a solution u of (1.1) which is stable outside B_{R_0}, let $\overline{u}$ be its s-harmonic extension by Theorem 2.1. Then $\overline{u}$ satisfies (2.1) and the inequality (2.4) holds for any $\phi \in C^1_c(\overline{\mathbb{R}^{n+1}_+})$ with $\operatorname{supp}\phi(\,\cdot\,,0) \Subset \mathbb{R}^n \setminus B_{R_0}$. Here we may assume that $R_0 \ge 1$.

Step 1. We first claim that

$$\lim_{\lambda\to+\infty} E(\overline{u};\lambda) < +\infty.$$

Once we have a uniform upper bound of $E(\overline{u};\lambda)$ with respect to $\lambda > 0$, the monotonicity of E in Theorem 5.1 will imply the above claim. In order to prove boundedness of $E(\overline{u};\lambda)$, we decompose $E(\overline{u};\lambda)$ into $E_1(\overline{u};\lambda) + E_2(\overline{u};\lambda)$, where

$$E_1(\overline{u};\lambda) := \lambda^{2s\frac{p+1}{p-1}+\frac{2a}{p-1}-n} \times \left\{ \frac{1}{2}\int_{B_\lambda^+} t^{1-2s}|\nabla\overline{u}|^2\,dxdt - \kappa_s \int_{B_\lambda\cap\partial\mathbb{R}_+^{n+1}} \left(\frac{\gamma|x|^{-2s}|\overline{u}|^2}{2} + \frac{|x|^a|\overline{u}|^{p+1}}{p+1}\right) dx \right\},$$

and

$$E_2(\overline{u};\lambda) := \lambda^{2s\frac{p+1}{p-1}+\frac{2a}{p-1}-n-1} \cdot \frac{2s+a}{2(p-1)} \int_{\partial B_\lambda^+\cap\mathbb{R}_+^{n+1}} t^{1-2s}\overline{u}^2\,dS_{x,t}.$$

With the use of Lemma 2.2, Lemma 3.6 shows that $E_1(\overline{u};\lambda)$ is uniformly bounded for $\lambda > 3R_0$. Since $E(\overline{u};\lambda)$ is nondecreasing by Theorem 5.1, it follows that

$$E(\overline{u};\lambda) \le \frac{1}{\lambda}\int_\lambda^{2\lambda} E(\overline{u};\tau)\,d\tau \le C + \lambda^{2s\frac{p+1}{p-1}+\frac{2a}{p-1}-n-2} \cdot \frac{2s+a}{2(p-1)} \int_{B_{2\lambda}^+\setminus B_\lambda^+} t^{1-2s}\overline{u}^2\,dxdt.$$

The second integral in the above estimate is uniformly bounded for any $\lambda > 3R_0$ by Lemma 3.5. Thus we deduce that $E(\overline{u};\lambda)$ is uniformly bounded from above for any $\lambda > 3R_0$.

Step 2. For $\lambda > 0$, let

$$\overline{u}^\lambda(X) = \lambda^{\frac{2s+a}{p-1}}\overline{u}(\lambda X) \quad \text{for } X = (x,t) \in \mathbb{R}_+^{n+1}.$$

Then direct computation shows that $\overline{u}^\lambda$ satisfies (2.1), and is stable outside $B_{R_0/\lambda}$ in the sense of Lemma 2.4.

In light of the energy estimates in Lemmas 3.5 and 3.6 with Lemma 2.2, $\{\overline{u}^\lambda\}_{\lambda>1}$ is uniformly bounded in $H^1_{\mathrm{loc}}(\overline{\mathbb{R}_+^{n+1}}; t^{1-2s}\,dxdt)$ since for a given $R > 3R_0$ and any $\lambda > 1$,

$$\begin{aligned} &\int_{B_R^+} t^{1-2s}|\overline{u}^\lambda|^2\,dxdt = \lambda^{-n-2+2s\frac{p+1}{p-1}+\frac{2a}{p-1}} \int_{B_{\lambda R}^+} t^{1-2s}\overline{u}^2\,dxdt \le CR^{n+2-2s\frac{p+1}{p-1}-\frac{2a}{p-1}}, \\ &\int_{B_R^+} t^{1-2s}|\nabla\overline{u}^\lambda|^2\,dxdt = \lambda^{-n+2s\frac{p+1}{p-1}+\frac{2a}{p-1}} \int_{B_{\lambda R}^+} t^{1-2s}|\nabla\overline{u}|^2\,dxdt \le CR^{n-2s\frac{p+1}{p-1}-\frac{2a}{p-1}}. \end{aligned} \tag{7.1}$$

Here we notice that $p > p_S(n,s,a)$ and that

$$\int_{B_{2R_0}^+} t^{1-2s}|\nabla\overline{u}|^2\,dxdt < C_0$$

with some constant $C_0 > 0$ by Lemma 2.2. Then by a diagonal argument, there exist a sequence $\{\lambda_i\}$ and a limit function $\overline{u}^\infty$ such that $\lambda_i \to +\infty$ and $\overline{u}^{\lambda_i}$ converges weakly to $\overline{u}^\infty$ in $H^1_{\rm loc}(\overline{\mathbb{R}^{n+1}_+}; t^{1-2s}\, dxdt)$ as $i \to \infty$. By Lemma 2.3 together with (7.1), $\overline{u}^{\lambda_i}$ converges strongly to $\overline{u}^\infty$ in $L^2_{\rm loc}(\overline{\mathbb{R}^{n+1}_+}; t^{1-2s}\, dxdt)$ as $i \to \infty$, up to a subsequence. Moreover, by arguing similarly as for (7.1), Lemma 3.6 implies that

$$\int_{B_R} |x|^a |\overline{u}^\lambda(x,0)|^{p+1} + |x|^{-2s}|\overline{u}^\lambda(x,0)|^2\, dx \le C R^{n-2s\frac{p+1}{p-1}-\frac{2a}{p-1}}$$

for any $R > 3R_0$ and any $\lambda > 1$. This estimate combined with Fatou's lemma yields that a limit $\overline{u}^\infty$ of $\{\overline{u}^{\lambda_i}\}$ as $i \to \infty$ (up to a subsequence) satisfies (6.1) in the distributional sense, and $\overline{u}^\infty$ is stable except the origin in the sense of Lemma 2.4. That is, the equality (6.2) and the inequality (6.3) (with $\overline{u} = \overline{u}^\infty$) hold true for any $\phi \in C^\infty_c(\overline{\mathbb{R}^{n+1}_+} \setminus \{0\})$.

Step 3. Now we will prove that $\overline{u}^\infty$ is homogeneous. Firstly, we recall the scaling property enjoyed by E: for any $\lambda > 0$ and $R > 0$, $E(\overline{u}; \lambda R) = E(\overline{u}^\lambda; R)$. Utilizing the convergence of $E(\overline{u}; \lambda)$ as $\lambda \to \infty$ by Step 1, the scaling property and the monotonicity of E from Theorem 5.1 imply that for any $R_2 > R_1 > 0$,

$$\begin{aligned} 0 &= \lim_{i\to+\infty} \left\{E(\overline{u}; \lambda_i R_2) - E(\overline{u}; \lambda_i R_1)\right\} = \lim_{i\to+\infty} \left\{E(\overline{u}^{\lambda_i}; R_2) - E(\overline{u}^{\lambda_i}; R_1)\right\} \\ &\ge \liminf_{i\to+\infty} \int_{B^+_{R_2}\setminus B^+_{R_1}} t^{1-2s} r^{2s\frac{p+1}{p-1}+\frac{2a}{p-1}-n-2} \left(r\partial_r \overline{u}^{\lambda_i} + \frac{2s+a}{p-1}\overline{u}^{\lambda_i}\right)^2 dxdt. \end{aligned}$$

Thus the convergence of $\{\overline{u}^{\lambda_i}\}$ as $i \to \infty$ by Step 2 yields that for any $R_2 > R_1 > 0$,

$$\int_{B^+_{R_2}\setminus B^+_{R_1}} t^{1-2s} r^{2s\frac{p+1}{p-1}+\frac{2a}{p-1}-n-2} \left(r\partial_r \overline{u}^{\infty} + \frac{2s+a}{p-1}\overline{u}^{\infty}\right)^2 dxdt \le 0,$$

where we used the lower semicontinuity from the weak convergence of $\{\overline{u}^{\lambda_i}\}$ to $\overline{u}^\infty$ in $H^1_{\rm loc}(\overline{\mathbb{R}^{n+1}_+}; t^{1-2s}\, dxdt)$. So, it follows that

$$\partial_r \overline{u}^\infty + \frac{2s+a}{p-1}\frac{\overline{u}^\infty}{r} = 0 \quad \text{a.e. in } \mathbb{R}^{n+1}_+,$$

and hence we deduce that $\overline{u}^\infty(X) = r^{-\frac{2s+a}{p-1}}\psi(\theta)$ for some function $\psi \in H^1(S^n_+; \theta_1^{1-2s})$.

Step 4. Then we conclude that $\overline{u}^\infty \equiv 0$ by Theorem 6.1 since $\overline{u}^\infty$ satisfies the assumptions of Theorem 6.1 in light of Steps 2 and 3.

Step 5. Now we claim that $\overline{u}^\lambda$ converges strongly to 0 in $H^1_{\rm loc}(\overline{\mathbb{R}^{n+1}_+} \setminus \{0\}; t^{1-2s}\, dxdt)$ and $\overline{u}^\lambda(\,\cdot\,, 0)$ converges strongly to 0 in $L^{p+1}_{\rm loc}(\mathbb{R}^n \setminus \{0\})$ as $\lambda \to \infty$. Note that $\overline{u}^\lambda$ satisfies (2.1), and $\overline{u}^\lambda$ is stable outside $B_{R_0/\lambda}$. Let $R > 1$ and $0 < \epsilon < 1$ be any given constants. Arguing similarly for the estimate (3.17), we have that for sufficiently large $\lambda > 1$ such that $B^+_{R_0/\lambda} \subset B^+_{\epsilon/2}$,

$$\int_{B^+_R\setminus B^+_\epsilon} t^{1-2s}|\nabla \overline{u}^\lambda|^2\, dxdt \le C\epsilon^{-2}\int_{B^+_\epsilon} t^{1-2s}|\overline{u}^\lambda|^2\, dxdt + CR^{-2}\int_{B^+_{2R}} t^{1-2s}|\overline{u}^\lambda|^2\, dxdt. \tag{7.2}$$

Then this estimate and the strong convergence of $\{\overline{u}^{\lambda_i}\}$ to $\overline{u}^{\infty} \equiv 0$ in $L^2_{\rm loc}(\overline{\mathbb{R}^{n+1}_+}; t^{1-2s}\,dxdt)$ from Steps 2–4 imply that $\overline{u}^{\lambda_i}$ converges strongly to 0 in $H^1_{\rm loc}(\overline{\mathbb{R}^{n+1}_+} \setminus \{0\}; t^{1-2s}\,dxdt)$ as $\lambda_i \to \infty$ since $R > 1$ and $0 < \epsilon < 1$ are arbitrary. By a similar argument as for (3.16)–(3.19) and (7.2), we deduce the strong convergence of $\{\overline{u}^{\lambda_i}(\,\cdot\,, 0)\}$ to 0 in $L^{p+1}_{\rm loc}(\mathbb{R}^n \setminus \{0\})$ as $\lambda_i \to \infty$. Furthermore, since a sequence $\{\lambda_i\}$ can be arbitrary, the claim follows.

Step 6. Lastly, we will prove that $\overline{u} \equiv 0$. Indeed, direct computation shows that for any $\epsilon \in (0,1)$,

$$\begin{aligned}
E_1(\overline{u};\lambda) &= E_1(\overline{u}^{\lambda};1) \\
&= \frac{1}{2}\int_{B_1^+} t^{1-2s}|\nabla \overline{u}^{\lambda}|^2\,dxdt - \kappa_s \int_{B_1\cap\partial\mathbb{R}^{n+1}_+}\left(\frac{\gamma|x|^{-2s}|\overline{u}^{\lambda}|^2}{2} + \frac{|x|^a|\overline{u}^{\lambda}|^{p+1}}{p+1}\right)dx \\
&= \epsilon^{n-2s\frac{p+1}{p-1}-\frac{2a}{p-1}} E_1(\overline{u};\lambda\epsilon) + \frac{1}{2}\int_{B_1^+\setminus B_\epsilon^+} t^{1-2s}|\nabla \overline{u}^{\lambda}|^2\,dxdt \\
&\quad - \kappa_s \int_{(B_1\setminus B_\epsilon)\cap\partial\mathbb{R}^{n+1}_+}\left(\frac{\gamma|x|^{-2s}|\overline{u}^{\lambda}|^2}{2} + \frac{|x|^a|\overline{u}^{\lambda}|^{p+1}}{p+1}\right)dx.
\end{aligned}$$

Let $\epsilon \in (0,1)$ be given. Since $E_1(\overline{u};\lambda\epsilon)$ is uniformly bounded for any $\lambda\epsilon > 3R_0$ as seen in Step 1, we have that

$$\begin{aligned}
E_1(\overline{u};\lambda) \le C\epsilon^{n-2s\frac{p+1}{p-1}-\frac{2a}{p-1}} &+ \frac{1}{2}\int_{B_1^+\setminus B_\epsilon^+} t^{1-2s}|\nabla \overline{u}^{\lambda}|^2\,dxdt \\
&- \kappa_s \int_{(B_1\setminus B_\epsilon)\cap\partial\mathbb{R}^{n+1}_+}\left(\frac{\gamma|x|^{-2s}|\overline{u}^{\lambda}|^2}{2} + \frac{|x|^a|\overline{u}^{\lambda}|^{p+1}}{p+1}\right)dx.
\end{aligned}$$

Hence by letting $\lambda \to +\infty$ in the estimate above, the strong convergence of $\{\overline{u}^{\lambda}\}$ to 0 from Step 5 (and then letting $\epsilon \to 0$) yields that

$$\lim_{\lambda\to+\infty} E_1(\overline{u};\lambda) \le 0. \tag{7.3}$$

Using the monotonicity of E with the use of (7.1) implies

$$\begin{aligned}
E(\overline{u};\lambda) &\le \frac{1}{\lambda}\int_{\lambda}^{2\lambda} E(\overline{u};\tau)\,d\tau \\
&\le \sup_{\tau\in[\lambda,2\lambda]} E_1(\overline{u};\tau) + C\lambda^{-n-2+2s\frac{p+1}{p-1}+\frac{2a}{p-1}} \int_{B_{2\lambda}^+\setminus B_\lambda^+} t^{1-2s}|\overline{u}|^2\,dxdt \\
&= \sup_{\tau\in[\lambda,2\lambda]} E_1(\overline{u};\tau) + \int_{B_2^+\setminus B_1^+} t^{1-2s}|\overline{u}^{\lambda}|^2\,dxdt.
\end{aligned}$$

Thus we deduce that $\lim_{\lambda\to+\infty} E(\overline{u};\lambda) \le 0$ by (7.3) and the strong convergence of $\{\overline{u}^{\lambda}\}$ to 0 in Step 5. On the other hand, by the continuity of $\overline{u}$ near the origin, it holds that $\liminf_{\lambda\to 0} E(\overline{u};\lambda) \ge 0$. Then, it follows from the monotonicity of $E(\overline{u};\lambda)$ that $E(\overline{u},\lambda) \equiv 0$ for any $\lambda > 0$, and hence $\frac{dE}{d\lambda} \equiv 0$. This combined with the monotonicity formula (5.1) yields that $\overline{u}$ is homogeneous of the form (6.4). Therefore we conclude that $\overline{u} \equiv 0$ by the continuity of u at the origin, which implies $u \equiv 0$. This finishes the proof. □

8. Remark on the condition (P) in the supercritical case

In Theorem 1.1, we impose an implicit condition (P) on p in the supercritical case $p > p_S(n,s,a)$. This section is devoted to the study of the asymptotic behavior of the condition (P) when the order $s \in (0,1)$ of the fractional Laplacian tends to 1. We shall show that as $s \in (0,1)$ tends to 1, the condition (P) provides with a Joseph–Lundgren type exponent given in the results of [1,2,19,25,36]. Here we suppose that $n > 2$, $a > -2$ and $0 \le \gamma < \gamma_{n,1,a}(p) < \Lambda_{n,1}$ in the limit in order to compare with the results of [1,2,19,25,36].

Since the functions $\Gamma|_{(0,\infty)}$ and $\lambda|_{[0,(n-2s)/2)}$ are continuous, it can be easily checked that

$$\Lambda_{n,1} = \frac{(n-2)^2}{4} \quad \text{and} \quad \gamma_{n,1,a}(p) = \frac{2+a}{p-1} \cdot \left(n-2-\frac{2+a}{p-1}\right),$$

where we used the fact that $\Gamma(t+1) = t\Gamma(t)$ for $t > 0$. Hence the limit of the condition (P) (as $s \to 1$):

$$(\mathrm{P}_0) \qquad p > \frac{\Lambda_{n,1}-\gamma}{\gamma_{n,1,a}(p)-\gamma}$$

is equivalent to

$$\begin{aligned} 0 &> \gamma - \left(\frac{p}{p-1}\right)\cdot\left(\frac{2+a}{p-1}\right)\cdot\left(n-2-\frac{2+a}{p-1}\right) + \frac{(n-2)^2}{4(p-1)} \\ &= \gamma - \frac{2+a}{p-1}\cdot\left\{n-2-\frac{(n-2)^2}{4(2+a)}\right\} - \frac{2+a}{(p-1)^2}\cdot(n-4-a) + \frac{(2+a)^2}{(p-1)^3}. \end{aligned} \tag{8.1}$$

Let $m := (2+a)/(p-1) \in (0,(n-2)/2)$. In terms of m, this can be written as

$$h_{n,a,\gamma}(m) := m^3 - (n-4-a)m^2 + \frac{1}{4}(n-2)(n-10-4a)m + (2+a)\gamma < 0.$$

Here we notice that

$$m < \frac{n-2}{2} \quad \text{is equivalent to} \quad p > p_S(n,1,a).$$

The function $h_{n,a,\gamma}$ appears in [1, 2, 19, 36] when calculating the explicit value of the Joseph–Lundgren type exponent. Direct computation shows that

$$\begin{aligned} &h_{n,a,\gamma}(0) = (2+a)\gamma, \quad h'_{n,a,\gamma}(0) = \frac{1}{4}(n-2)(n-10-4a), \\ &h_{n,a,\gamma}\left(\frac{n-2}{2}\right) = (2+a)(-\Lambda_{n,1}+\gamma), \quad h'_{n,a,\gamma}\left(\frac{n-2}{2}\right) = 0, \end{aligned} \tag{8.2}$$

see also the proof of Lemma 5.2 in [36]. If $p > p_S(n,1,a)$ and $0 < \gamma < \Lambda_{n,1}$, there exists a unique zero $m_c(n,a,\gamma)$ of $h_{n,a,\gamma}$ in $(0,(n-2)/2)$. Furthermore, it holds that

$$h_{n,a,\gamma}(m) < 0 \quad \text{is equivalent to} \quad m_c(n,a,\gamma) < m < \frac{n-2}{2},$$

provided that $p > p_S(n,1,a)$ and $0 < \gamma < \Lambda_{n,1}$. Let $p_c(n,a,\gamma)$ be a constant given by $1 + \frac{2+a}{m_c(n,a,\gamma)}$. Then the condition (P_0) corresponds to

$$p_S(n,1,a) < p < p_c(n,a,\gamma),$$

where $p_c(n,a,\gamma)$ is the so-called Joseph–Lundgren type critical exponent in presence of the Hardy term $\gamma|x|^{-2}u$ in the local case [1,2,19,36]. Similarly, if $\gamma = 0$ and $n > 10+4a$, in light of (8.2), there exists a unique zero $m_c(n,a,\gamma)$ of $h_{n,a,\gamma}$ in $(0,(n-2)/2)$, and hence we see that the condition (P_0) leads to $p_S(n,1,a) < p < p_c(n,a,\gamma) = 1+(2+a)/m_c(n,a,\gamma)$. When $\gamma = 0$ and $n \le 10+4a$, the condition (P_0) is equivalent to $p_S(n,1,a) < p < p_c(n,a,\gamma) = \infty$. So our condition (P) on p recovers the local result in $[1,2,19,36]$ as $s \in (0,1)$ tends to 1.

Furthermore, when $a = 0$, the inequality (8.1) is equivalent to

$$0 < (-\gamma)(p-1)^3 + \frac{n-2}{4}(10-n)p^2 + \frac{1}{2}\big\{(n-2)^2 - 4n\big\}p - \frac{(n-2)^2}{4}, \tag{8.3}$$

refer to $[1,2,19,25,36]$. In particular, assuming $n \ge 11$, $p > \frac{n+2}{n-2}$ and $\gamma = 0$, the inequality (8.3) leads to

$$(n-2)(n-10)p^2 - 2\big\{(n-2)^2 - 4n\big\}p + (n-2)^2 < 0$$

which yields

$$\frac{n+2}{n-2} < p < \frac{(n-2)^2 - 4n + 8\sqrt{n-1}}{(n-2)(n-10)} = p_c(n).$$

Here $p_c(n)$ is the Joseph–Lundgren exponent in (1.4) introduced by Farina [25].

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