

Bounds for Hilbert Coefficients

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Abstract. Let (A, \mathfrak{m}) be a noetherian local ring and J an \mathfrak{m} -primary ideal. Elias [3] proved that $\text{depth}(G(J^k))$ is constant for $k \gg 0$ and denoted this number by $\sigma(J)$. In this paper, we prove the non-positivity for the Hilbert coefficients $e_i(J)$ under some conditions for $\sigma(J)$. In case of $J = Q$ is a parameter ideal, we establish bounds for the Hilbert coefficients of Q in terms of the dimension and the first Hilbert coefficient $e_1(Q)$.

1. Introduction

Let (A, \mathfrak{m}) be a noetherian local ring of dimension d and J an \mathfrak{m} -primary ideal of A . Let $\ell(\cdot)$ denote the length of an A -module. The Hilbert–Samuel function of A with respect to J is a function $H_J: \mathbb{Z} \rightarrow \mathbb{N}_0$ given by

$$H_J(n) = \begin{cases} \ell(A/J^n) & \text{if } n \geq 0, \\ 0 & \text{if } n < 0. \end{cases}$$

There exists a unique polynomial $P_J(x) \in \mathbb{Q}[x]$ (called the *Hilbert–Samuel polynomial*) of degree d such that $H_J(n) = P_J(n)$ for $n \gg 0$ and it is written by

$$P_J(n) = \sum_{i=0}^d (-1)^i \binom{n+d-i-1}{d-i} e_i(J).$$

Then, the integers $e_i(J)$ are called *Hilbert coefficients* of J . Let $G(J) = \bigoplus_{n \geq 0} J^n/J^{n+1}$ be the associated graded ring of A with respect to J . In [3], Elias denoted $\sigma_J(k) = \text{depth}(G(J^k))$ and proved that $\sigma_J(k)$ is constant for $k \gg 0$. We call this number $\sigma(J)$.

The aim of this paper is to investigate the non-positivity of $e_i(J)$ under some conditions for $\sigma(J)$. In the case J is a parameter ideal, we establish bounds for the Hilbert coefficients $e_i(J)$, for $i = 2, \dots, d$, in terms of the dimension and the first Hilbert coefficient $e_1(J)$.

First, we study the non-positivity of the Hilbert coefficients. If A is an arbitrary ring, Mandal–Singh–Verma [14] showed that $e_1(Q) \leq 0$ for every parameter ideal Q of

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A. If $\text{depth}(A) \geq d - 1$, Mccune [15] showed that $e_2(Q) \leq 0$ and Saikia–Saloni [20] proved that $e_3(Q) \leq 0$ for every parameter ideal Q . In [15], Mccune also proved that if Q is a parameter ideal such that $\text{depth}(G(Q)) \geq d - 1$, then $e_i(Q) \leq 0$ for $i = 1, \dots, d$. Later, Saikia–Saloni [20] and Linh–Trung [12] proved that if $\text{depth}(A) \geq d - 1$ and Q is a parameter ideal such that $\text{depth}(G(Q)) \geq d - 2$, then $e_i(Q) \leq 0$ for $i = 1, \dots, d$. In [17], Puthenpurakal obtained a remarkable result that if J is an \mathfrak{m} -primary ideal of a Cohen–Macaulay ring with dimension 3 such that $r(J) = 2$, then $e_3(J) \leq 0$.

It is well known that the behavior of Hilbert coefficients $e_i(J)$ depend on $\text{depth}(G(J))$. Elias [3] also proved that $\sigma(J) \geq \text{depth}(G(J))$. The first main result of this paper is the non-positivity of the last Hilbert coefficient $e_d(J)$ under the condition $\sigma(J) \geq d - 2$.

Theorem 1.1. (= Theorem 3.2) *Let (A, \mathfrak{m}) be a noetherian local ring of dimension $d \geq 2$ and $\text{depth}(A) \geq d - 1$. Let J be an \mathfrak{m} -primary ideal such that $r(J) \leq d - 1$. If $\sigma(J) \geq d - 2$, then $e_d(J) \leq 0$.*

Theorem 1.1 implies an early result of Mafi and Nadery [13] that if A is a Cohen–Macaulay ring of dimension 4 and J an \mathfrak{m} -primary asymptotically normal ideal such that $r(J) \leq 3$, then $e_4(J) \leq 0$. From Theorem 1.1, we also get some interesting properties about the non-positivity of $e_3(J)$ and $e_4(J)$.

Theorem 1.1 gives the non-positivity for the last Hilbert coefficient $e_d(J)$, but other Hilbert coefficients may be positive. The next result shows the non-positivity for Hilbert coefficients of an \mathfrak{m} -primary ideal.

Theorem 1.2. (= Theorem 3.8) *Let (A, \mathfrak{m}) be a noetherian local ring with $\dim(A) = d \geq 3$ and $\text{depth}(A) \geq d - 1$. Let J be an \mathfrak{m} -primary ideal of A such that $r(J) \leq 2$. If $\text{depth}(G(J)) \geq d - 2$, then*

$$e_i(J) \leq 0 \quad \text{for } i = 3, \dots, d.$$

Theorem 1.2 is a generalization of an early results of Puthenpurakal [17, Theorem 9.1], Saikia–Saloni [20, Corollary 3.2] and Linh–Trung [12, Theorem 2.9].

Hilbert coefficients reflect the structural information of rings and modules. So, the problem finding bounds for the Hilbert coefficients in terms of several common invariants has attracted the attention of many mathematicians in pass years. If A is Cohen–Macaulay and generalized Cohen–Macaulay, Srinivas and Trivedi [21–23] gave bounds for the Hilbert coefficients of \mathfrak{m} -primary ideals in terms of the dimension and multiplicity. If A is an arbitrary ring, Rossi, Trung and Valla [18] established bounds for the Hilbert coefficients of the maximal ideal in terms of the dimension and an extended degree. Later, Linh [10] extended the result of Rossi, Trung and Valla [18] for \mathfrak{m} -primary ideals. Goto and Ozeki [7] established uniform bounds for the Hilbert coefficients of parameter ideals in a generalized Cohen–Macaulay ring. Recently, Dung and Hoa [2] gave

bounds for $e_{d-t+1}(I), e_{d-t+2}(I), \dots, e_d(I)$ in terms of $e_0(I), e_1(I), \dots, e_{d-t}(I)$ in the case $\text{depth}(A) = t \geq 1$. These bounds obtained in [2] depend on $e_0(I)$.

Question 1.3. Let (A, \mathfrak{m}) be a noetherian local ring of dimension d and $\text{depth}(A) = t$. Does there exist bounds for Hilbert coefficients $e_{d-t+1}(I), \dots, e_d(I)$ in terms of $e_1(I), \dots, e_{d-t}(I)$ which do not depend on $e_0(I)$?

In the case $I = Q$ is a parameter ideal and $t = d - 1$, the problem of the question is to find bounds for $e_2(Q), \dots, e_d(Q)$ in terms of $e_1(Q)$ and these bounds do not depend on $e_0(Q)$. The first Hilbert coefficient $e_1(Q)$ is called Chern number. Recent results on the coefficient $e_1(Q)$ such as [5, 6] show that this coefficient is very important and it reflects clearly structural information. By using the bound for the regularity of the associated graded ring in [11], we establish bounds for $e_2(Q), \dots, e_d(Q)$ in terms of the Hilbert coefficient $e_1(Q)$.

Theorem 1.4. (= Theorem 4.4) *Let A be a noetherian local ring of dimension $d \geq 2$ and $\text{depth}(A) \geq d - 1$. Let Q be a parameter ideal of A . Then*

$$|e_i(Q)| \leq 3 \cdot 2^{i-2} r^{i-1} |e_1(Q)| \quad \text{for } i = 2, \dots, d,$$

where $r = \max\{[-4e_1(Q)]^{(d-1)!} + e_1(Q) - 1, 0\} + 1$.

The paper is divided into three sections. In Section 2, we prepare some facts related to the Hilbert coefficients and regularity. In Section 3, we prove the non-positivity for the Hilbert coefficients of \mathfrak{m} -primary ideals. In Section 4, we establish bounds for the Hilbert coefficients of parameter ideals in terms of the dimension and the first Hilbert coefficient.

2. Preliminaries

Let (A, \mathfrak{m}) be a noetherian local ring of dimension d and J an \mathfrak{m} -primary ideal of A . A numerical function

$$H_J: \mathbb{Z} \longrightarrow \mathbb{N}_0$$

$$n \longmapsto H_J(n) = \begin{cases} \ell(A/J^n) & \text{if } n \geq 0, \\ 0 & \text{if } n < 0 \end{cases}$$

is said to be a *Hilbert–Samuel function* of A with respect to the ideal J . It is well known that there exists a polynomial $P_J \in \mathbb{Q}[x]$ of degree d such that $H_J(n) = P_J(n)$ for $n \gg 0$. The polynomial P_J is called the *Hilbert–Samuel polynomial* of A with respect to the ideal J and it is written in the form

$$P_J(n) = \sum_{i=0}^d (-1)^i \binom{n+d-i-1}{d-i} e_i(J).$$

The integers $e_i(J)$ are called *Hilbert coefficients* of J . In particular, $e(J) = e_0(J)$ and $e_1(J)$ are called the *multiplicity* and *Chern coefficient* of J , respectively. The postulation number of J is defined as the integer

$$n(J) = \max\{n \mid H_J(n) \neq P_J(n)\}.$$

An element $x \in J \setminus \mathfrak{m}J$ is said to be *superficial* for J if there exists a number $c \in \mathbb{N}$ such that $(J^n : x) \cap J^c = J^{n-1}$ for $n > c$. If A/\mathfrak{m} is infinite, then a superficial element for J always exists. A sequence of elements $x_1, \dots, x_r \in J \setminus \mathfrak{m}J$ is said to be a *superficial sequence* for J if x_i is superficial for $J/(x_1, \dots, x_{i-1})$ for $i = 1, \dots, r$.

Suppose that $\dim(A) = d \geq 1$ and $x \in J \setminus \mathfrak{m}J$ is a superficial element for J , then $\ell(0 : Ax) < \infty$ and $\dim(A/(x)) = \dim(A) - 1 = d - 1$. The following lemma give us a relationship between $e_i(J)$ and $e_i(\bar{J})$, where $\bar{J} = J(A/(x))$.

Lemma 2.1. [19, Proposition 1.3.2] *Let A be a noetherian local ring of dimension $d \geq 2$ and J an \mathfrak{m} -primary ideal of A . Let $x \in J \setminus \mathfrak{m}J$ be a superficial element for J and set $\bar{J} = J(A/(x))$. Then*

- (i) $e_i(J) = e_i(\bar{J})$ for $i = 0, \dots, d - 2$,
- (ii) $e_{d-1}(J) = e_{d-1}(\bar{J}) + (-1)^d \ell(0 : x)$.

If we denote by $G(J) = \bigoplus_{n \geq 0} J^n / J^{n+1}$ the associated graded ring of A with respect to J and

$$a_i(G(J)) = \sup\{n \mid H_{G(J)_+}^i(G(J))_n \neq 0\},$$

then the Castelnuovo–Mumford regularity of $G(J)$ is defined by

$$\text{reg}(G(J)) = \max\{a_i(G(J)) + i \mid i \geq 0\}.$$

Lemma 2.2. *Let (A, \mathfrak{m}) be a noetherian local ring of dimension d and J be an \mathfrak{m} -primary ideal of A . Let $x \in J \setminus \mathfrak{m}J$ be a superficial element for J . Set $\bar{A} = A/(x)$ and $\bar{J} = J\bar{A}$. Then*

- (i) $n(J) \leq \text{reg}(G(J))$,
- (ii) $\text{reg}(G(\bar{J})) \leq \text{reg}(G(J))$,
- (iii) $J^{n+1} : x / J^n \cong (0 : x)$ for $n > \text{reg}(G(J))$.

Proof. (i) It is implied from [11, Lemmas 2.1 and 2.2].

(ii) Let x^* be an initial form of x in $G(J)$. Then

$$\text{reg}(G(J)/(x^*)) \leq \text{reg}(G(J)).$$

On the other hand, there is a natural graded epimorphism from $G(J)/(x^*)$ to $G(\bar{J})$ whose kernel is

$$K = \bigoplus_{n \geq 0} (J^{n+1} + x \cap J^n) / (J^{n+1} + xJ^{n-1}).$$

Since x is superficial for J , $x \cap J^{n+1} = xJ^n$ for $n \gg 0$. Hence $K_n = 0$ for $n \gg 0$. Thus K is a module with finite length. Hence

$$\text{reg}(G(\bar{J})) \leq \text{reg}(G(J)/(x^*)).$$

This implies

$$\text{reg}(G(\bar{J})) \leq \text{reg}(G(J)).$$

(iii) From the exact sequence

$$0 \longrightarrow J^{n+1} : x/J^n \longrightarrow A/J^n \xrightarrow{x} A/J^{n+1} \longrightarrow A/(J^{n+1}, x) \longrightarrow 0,$$

we get

$$\begin{aligned} \ell(J^{n+1} : x/J^n) &= \ell(A/J^n) - \ell(A/J^{n+1}) + \ell(\bar{A}/\bar{J}^{n+1}) \\ &= \ell(\bar{A}/\bar{J}^{n+1}) - \ell(J^n/J^{n+1}). \end{aligned}$$

It is well known that $J^{n+1} : x/J^n \cong (0 : x)$ for $n \gg 0$. From (i) and (ii), we have

$$n(J) \leq \text{reg}(G(J)) \quad \text{and} \quad n(\bar{J}) \leq \text{reg}(G(J)).$$

It follows that

$$J^{n+1} : x/J^n \cong (0 : x) \quad \text{for } n > \text{reg}(G(J)). \quad \square$$

Recall that an ideal $K \subseteq J$ is called a reduction of J if $J^{n+1} = KJ^n$ for $n \gg 0$. If K is a reduction of J and no other reduction of J is contained in K , then K is said to be a minimal reduction of J . If K is a minimal reduction of J , then the reduction number of J with respect to K , $r_K(J)$, is given by

$$r_K(J) := \min\{n \mid J^{n+1} = KJ^n\}.$$

The reduction number of J , denoted by $r(J)$, is given by

$$r(J) := \min\{r_K(J) \mid K \text{ is a minimal reduction of } J\}.$$

The following lemma gives a relationship between the reduction number of J and the regularity of $G(J)$.

Lemma 2.3. [24, Proposition 3.2]

$$a_d(G(J)) + d \leq r(J) \leq \text{reg}(G(J)).$$

3. The non-positivity of Hilbert coefficients

Through this section, let (A, \mathfrak{m}) be a noetherian local ring of dimension d and J an \mathfrak{m} -primary ideal of A . In this section, we investigate the non-positivity of Hilbert coefficients $e_i(J)$.

In [3, Proposition 2.2], Elias proved that $\sigma_J(k) = \text{depth}(G(J^k))$ is constant for $k \gg 0$ and call this number $\sigma(J)$. By [8, Lemma 2.4],

$$a_i(G(J^k)) \leq [a_i(G(J))/k] \quad \text{for } i \leq d \text{ and } k \geq 1,$$

where $[a] = \max\{m \in \mathbb{Z} \mid m \leq a\}$. Therefore

$$(3.1) \quad a_i(G(J^k)) \leq 0 \quad \text{for all } i \leq d \text{ and } k \gg 0.$$

By [3, Proposition 2.2],

$$(3.2) \quad \sigma(J) \geq \text{depth}(G(J)).$$

The following lemma gives whenever the number $\sigma(J)$ is positive.

Lemma 3.1. *Let (A, \mathfrak{m}) be a noetherian local ring of dimension $d \geq 1$ and J an \mathfrak{m} -primary ideal of A . If $\text{depth}(A) \geq 1$, then $\sigma(J) \geq 1$.*

Proof. From (3.1), we have $a_i(G(J^k)) \leq 0$ for $k \gg 0$. By [9, Theorem 5.2], $a_0(G(J^k)) < a_1(G(J^k)) \leq 0$. Hence $H_{G(J^k)_+}^0(G(J^k)) = 0$ for $k \gg 0$. This implies that $\sigma(J) = \text{depth}(G(J^k)) \geq 1$ for all $k \gg 0$. □

In the case J is a parameter ideal, Linh [11, Proposition 3.5] proved that if $\sigma(J) \geq d-2$, then $e_d(J) \leq 0$. In the case J is an \mathfrak{m} -primary ideal, we get a generalization for [11, Proposition 3.5].

Theorem 3.2. *Let (A, \mathfrak{m}) be a noetherian local ring of dimension $d \geq 2$ and $\text{depth}(A) \geq d-1$. Let J be an \mathfrak{m} -primary ideal such that $r(J) \leq d-1$. If $\sigma(J) \geq d-2$, then $e_d(J) \leq 0$.*

Proof. For $k \gg 0$, let $I = J^k$. Denote by $R = A[It] = \bigoplus_{n \geq 0} I^n$ the Rees algebra of A with respect to I and $R_+ = \bigoplus_{n > 0} R_n$. From [4, Proposition 2.7], we have $e_d(J) = e_d(I)$. By [1, Theorems 3.8 and 4.1],

$$\begin{aligned} (-1)^d e_d(J) &= (-1)^d e_d(I) = P_I(0) - H_I(0) \\ &= \sum_{i=0}^d (-1)^i \ell(H_{R_+}^i(R)_0) = \sum_{i=0}^d (-1)^i \ell(H_{G(I)_+}^i G(I)_0). \end{aligned}$$

Since $\sigma(J) = \text{depth}(G(I)) \geq d - 2$, it follows that $H_{G(I)_+}^i(G(I)) = 0$ for $i = 0, \dots, d - 3$. On the other hand, we have $a_d(G(I)) + d \leq r(I)$ by Lemma 2.3. From [8, Lemma 2.7],

$$r(I) \leq \frac{\lceil r(J) + 1 - s(J) \rceil}{k} + s(I) - 1 = \frac{\lceil r(J) + 1 - d \rceil}{k} + d - 1 \leq d - 1.$$

Hence $a_d(G(I)) < 0$. Moreover, $a_i(G(I)) \leq 0$ for all $i \geq 0$ from (3.1). By applying [9, Theorem 5.2], we get $a_{d-2}(G(I)) < a_{d-1}(G(I)) \leq 0$. It follows that

$$(-1)^d e_d(J) = (-1)^{d-1} \ell(H_{G(I)_+}^{d-1}(G(I))_0).$$

This implies that $e_d(J) = -\ell(H_{G(I)_+}^{d-1}(G(I))_0) \leq 0$. □

From the proof of Theorem 3.2, $e_d(J) = -\ell(H_{G(I)_+}^{d-1}(G(I))_0)$. If A is Cohen–Macaulay and $\sigma(J) \geq d - 1$, then $a_{d-1}(G(I)) < 0$. This gives us the following corollary.

Corollary 3.3. *Let (A, \mathfrak{m}) be a Cohen–Macaulay ring of dimension $d \geq 2$. Let J be an \mathfrak{m} -primary ideal such that $r(J) \leq d - 1$. If $\sigma(J) \geq d - 1$, then $e_d(J) = 0$.*

An ideal J is said to be asymptotically normal if there exists an integer $k \geq 1$ such that J^n is integrally closed for all $n \geq k$. If J is an asymptotically normal ideal of A , $\sigma(J) \geq 2$ by [16, Theorem 7.3]. Mafi and Naderi [13, Theorem 1.5] proved that if A is a Cohen–Macaulay ring of dimension 4 and J is an \mathfrak{m} -primary asymptotically normal ideal such that $r(J) \leq 3$, then $e_4(J) \leq 0$. By applying Theorem 3.2, we get the following corollary

Corollary 3.4. *Let (A, \mathfrak{m}) be a noetherian local ring of dimension 4 and $\text{depth}(A) \geq 3$. Let J be an \mathfrak{m} -primary asymptotically normal ideal of A such that $r(J) \leq 3$. Then $e_4(J) \leq 0$.*

Notice that the hypothesis of the ring A in Corollary 3.4 is not necessarily Cohen–Macaulay.

Corollary 3.5. *Let (A, \mathfrak{m}) be a noetherian ring of dimension 4 and $\text{depth}(A) \geq 3$. Let J be an \mathfrak{m} -primary ideal of A such that $r(J) \leq 2$. If $\sigma(J) \geq 2$, then*

$$e_i(J) \leq 0 \quad \text{for } i = 3, 4.$$

Proof. From Theorem 3.2, $e_4(J) \leq 0$.

Without loss of generality, assume that A/\mathfrak{m} is infinite and x_1 is a superficial element for J . Let $A_1 = A/(x_1)$ and $J_1 = JA_1$. Then $\dim(A_1) = 3$, J_1 is a \mathfrak{m} -primary ideal of A_1 and $e_3(J) = e_3(J_1)$. Since $\text{depth}(A) \geq 3$, $\text{depth}(A_1) \geq 2$. By Lemma 3.1, $\sigma(J_1) \geq 1$. Moreover, $r(J_1) \leq r(J) \leq 2$. By applying Theorem 3.2, we obtain $e_3(J) = e_3(J_1) \leq 0$. □

In case of A is a Cohen–Macaulay ring of dimension $d = 3$ and $r(I) = 2$, Puthenpurakal [17, Theorem 9.1] proved that $e_3(J) \leq 0$. The following corollary is an extension the result of Puthenpurakal.

Corollary 3.6. *Let (A, \mathfrak{m}) be a noetherian ring with $\dim(A) = d \geq 3$ and $\text{depth}(A) \geq d - 1$. If J is an \mathfrak{m} -primary ideal of A and $r(J) \leq 2$, then $e_3(J) \leq 0$.*

Proof. By Lemma 3.1, one has $\sigma(J) \geq 1$. If $d = 3$, by applying Theorem 3.2 we get $e_3(J) \leq 0$.

If $d > 3$, without loss of generality, assume that A/\mathfrak{m} is infinite and x_1, \dots, x_{d-3} is a superficial sequence for J . Let $\bar{A} = A/(x_1, \dots, x_{d-3})$ and $\bar{J} = J\bar{A}$. Then $\dim(\bar{A}) = 3$, $\text{depth}(\bar{A}) \geq 2$ and $r(\bar{J}) \leq r(J) \leq 2$. Since $\text{depth}(\bar{A}) \geq 2$, $\sigma(\bar{J}) \geq 1$ by Lemma 3.1. Applying Theorem 3.2, we obtain $e_3(\bar{J}) \leq 0$. From Lemma 2.1, we conclude that $e_3(J) = e_3(\bar{J}) \leq 0$. □

Theorem 3.2 gives the non-positivity of the last Hilbert coefficient $e_d(J)$ under assumption $\sigma(J) \geq d - 2$. For this assumption, other Hilbert coefficients may be positive. The following example shows that $e_d \leq 0$, but other Hilbert coefficients are positive.

Example 3.7. Let $A = \mathbb{Q}[x, y, z]_{(x,y,z)}$ and $J = (x^3, y^3, z^3, x^2y + z^3, xz^2, y^2z + x^2z, xyz)$. Then $K = (x^3, y^3, z^3)$ is a minimal reduction of J and $r_K(J) = 2$. Using Macaulay 2, we compute $\text{depth}(G(J)) = 0$ and $\sigma(J) = 1$. On the other hand, the Hilbert series $P_{G(J)}(t)$ of $G(J)$ is

$$P_{G(J)}(t) = \sum_{n \geq 0} \ell(J^n/J^{n+1})t^n = \frac{h(t)}{(1-t)^3},$$

where $h(t) = a_0 + a_1t + \dots + a_s \in \mathbb{Z}[t]$. It follows that

$$h(t) = a_0 + a_1t + \dots + a_s = (1 - 3t + 3t^2 - t^3)P_{G(J)}(t).$$

Hence

$$\begin{aligned} a_0 &= \ell(A/J), \\ a_1 &= \ell(J/J^2) - 3\ell(A/J), \\ a_2 &= \ell(J^2/J^3) - 3\ell(J/J^2) + 3\ell(A/J), \\ a_i &= \ell(J^i/J^{i+1}) - 3\ell(J^{i-1}/J^i) + 3\ell(J^{i-2}/J^{i-1}) - \ell(J^{i-3}/J^{i-2}) \quad \text{for } i \geq 3. \end{aligned}$$

By computing with Macaulay 2, we get

$$a_0 = 13, \quad a_1 = 6, \quad a_2 = 13, \quad a_3 = -6, \quad a_4 = 1, \quad a_5 = a_6 = \dots = 0.$$

This means

$$h(t) = 13 + 6t + 13t^2 - 6t^3 + t^4.$$

So

$$e_0(J) = h(1) = 27, \quad e_1(J) = h'(1) = 18, \quad e_2(J) = \frac{h''(1)}{2!} = 1, \quad e_3(J) = \frac{h^{(3)}(1)}{3!} = -2.$$

The following theorem provides us the non-positivity of other Hilbert coefficients.

Theorem 3.8. *Let (A, \mathfrak{m}) be a noetherian local ring with $\dim(A) = d \geq 3$ and $\text{depth}(A) \geq d - 1$. Let J be an \mathfrak{m} -primary ideal of A such that $r(J) \leq 2$. If $\text{depth}(G(J)) \geq d - 2$, then*

$$e_i(J) \leq 0 \quad \text{for } i = 3, \dots, d.$$

Proof. It is well known that $e_d(J) \leq 0$. If $d \leq 4$, the corollary is proved by Corollary 3.5.

If $d > 4$, we need to prove that $e_{d-i}(J) \leq 0$ for $i = 1, \dots, d - 3$. Indeed, without loss of generality, assume that A/\mathfrak{m} is infinite and x_1, \dots, x_d is a superficial sequence for J . For each $i = 1, \dots, d - 3$, let $A_i = A/(x_1, \dots, x_i)$ and $J_i = JA_i$. By hypothesis, we have $\dim(A_i) = d - i$, $\text{depth}(A_i) \geq d - i - 1$ and $r(J_i) \leq r(J) \leq 2$. Since $\text{depth}(G(J)) \geq d - 2$, $\text{depth}(G(J_i)) \geq d - i - 2$. From (3.2), we have

$$\sigma(J_i) \geq \text{depth}(G(J_i)) \geq d - i - 2.$$

By applying Theorem 3.2, we get

$$e_{d-i}(J) = e_{d-i}(J_i) \leq 0 \quad \text{for } i = 1, \dots, d - 3.$$

It follows $e_i(J) \leq 0$ for $i = 3, \dots, d - 1$. So, we conclude that $e_i(J) \leq 0$ for $i = 3, \dots, d$. \square

Remark 3.9. Theorem 3.8 is a generalization of early results of Puthenpurakal [17, Theorem 9.1], Saikia–Saloni [20, Corollary 3.2] and Linh–Trung [12, Theorem 2.9].

4. Bound for Hilbert coefficients of parameter ideals

Let (A, \mathfrak{m}) be a noetherian local ring of dimension d and $\text{depth}(A) \geq d - 1$. In this section, we will establish bounds for the Hilbert coefficients of parameter ideals.

Lemma 4.1. *Let A be a noetherian local ring of dimension $d \geq 2$ and $\text{depth}(A) \geq d - 1$. Let Q be a parameter ideal of A and x a superficial element for Q . For all $n \geq 1$, we have*

$$\ell(Q^{n+1} : x/Q^n) \leq -\binom{n + d - 3}{d - 2} e_1(Q).$$

Proof. Suppose that $Q = (x_1, \dots, x_d)$ and $x = x_d$ is superficial for Q . Setting $J = (x_1, \dots, x_{d-1})$, we have

$$\begin{aligned} Q^{n+1} : x/Q^n &= ((xQ^n + J^nQ) : x)/Q^n \\ &= (Q^n + (J^nQ : x))/Q^n \\ &\cong (J^nQ : x)/(Q^n \cap (J^nQ : x)). \end{aligned}$$

Since $J^n \subseteq Q^n \cap (J^n Q : x)$,

$$\ell(Q^{n+1} : x/Q^n) \leq \ell(J^n : x/J^n).$$

By [11, Corollary 4.4],

$$\ell(J^n : x/J^n) \leq -\binom{n+d-3}{d-2} e_1(Q).$$

This implies that

$$\ell(Q^{n+1} : x/Q^n) \leq -\binom{n+d-3}{d-2} e_1(Q). \quad \square$$

Lemma 4.2. *Let A be a noetherian local ring of dimension $d \geq 2$ and $\text{depth}(A) \geq 1$. Let I be an \mathfrak{m} -primary ideal of A and x a superficial element for I . Then*

$$(-1)^d e_d(I) = \sum_{k=0}^r (H_{\bar{I}}(k) - P_{\bar{I}}(k)) - \sum_{k=0}^r \ell(I^{k+1} : x/I^k),$$

where some $r \geq \text{reg}(G(I)) + 1$, $\bar{A} = A/(x)$ and $\bar{I} = I\bar{A}$.

Proof. From [15, Lemma 3.2], we have

$$(-1)^d e_d(I) = \sum_{k=0}^{\infty} (H_{\bar{I}}(k) - P_{\bar{I}}(k)) - \sum_{k=0}^{\infty} \ell(I^{k+1} : x/I^k).$$

By Lemma 2.2,

$$n(\bar{I}) \leq \text{reg}(G(\bar{I})) \leq \text{reg}(G(I)) < r \quad \text{and} \quad \ell(I^{k+1} : x/I^k) = \ell(0 :_A x) = 0$$

for $k \geq r$. Thus

$$(-1)^d e_d(I) = \sum_{k=0}^r (H_{\bar{I}}(k) - P_{\bar{I}}(k)) - \sum_{k=0}^r \ell(I^{k+1} : x/I^k). \quad \square$$

In [11], the author gave a bound for the regularity of associated graded ring with respect to parameter ideals in terms of the first coefficient $e_1(Q)$.

Theorem 4.3. [11, Theorem 4.5] *Let A be a noetherian local ring of dimension $d \geq 1$ and $\text{depth}(A) \geq d - 1$. Let Q be a parameter ideal of A . Then*

$$\text{reg}(G(Q)) \leq \begin{cases} \max\{-e_1(Q) - 1, 0\} & \text{if } d = 1, \\ \max\{[-4e_1(Q)]^{(d-1)!} + e_1(Q) - 1, 0\} & \text{if } d \geq 2. \end{cases}$$

Using the bound for the regularity of $G(Q)$ in Theorem 4.3, we will establish bounds for Hilbert coefficients $e_i(Q)$.

Theorem 4.4. *Let A be a noetherian local ring of dimension $d \geq 2$ and $\text{depth}(A) \geq d - 1$. Let Q be a parameter ideal of A . Then*

$$|e_i(Q)| \leq 3 \cdot 2^{i-2} r^{i-1} |e_1(Q)| \quad \text{for } i = 2, \dots, d,$$

where $r = \max\{[-4e_1(Q)]^{(d-1)!} + e_1(Q) - 1, 0\} + 1$.

Proof. Suppose that $Q = (x_1, \dots, x_d)$. Without loss of generality, we may assume that the residue field A/\mathfrak{m} is infinite. Let $x = x_d$ be a superficial element for Q . Set $\bar{A} = A/(x)$ and $\bar{Q} = Q\bar{A}$. By Lemma 4.2, we have

$$\begin{aligned} (-1)^d e_d(Q) &= \sum_{k=0}^r [H_{\bar{A}}(k) - P_{\bar{A}}(k)] - \sum_{k=0}^r \ell(Q^{k+1} : x/Q^k) \\ &= \sum_{k=0}^r \left[\ell(\bar{A}/\bar{Q}^k) - \sum_{i=0}^{d-1} (-1)^i \binom{k+d-i-2}{d-i-1} e_i(\bar{Q}) \right] - \sum_{k=0}^r \ell(Q^{k+1} : x/Q^k) \\ &= \sum_{k=0}^r \left[\ell(\bar{A}/\bar{Q}^k) - \binom{k+d-2}{d-1} e_0(\bar{Q}) - \sum_{i=1}^{d-1} (-1)^i \binom{k+d-i-2}{d-i-1} e_i(\bar{Q}) \right] \\ &\quad - \sum_{k=0}^r \ell(Q^{k+1} : x/Q^k). \end{aligned}$$

By [11, Lemma 4.1],

$$0 \leq \ell(\bar{A}/\bar{Q}^k) - \binom{k+d-2}{d-1} e_0(\bar{Q}) \leq -\binom{k+d-3}{d-2} e_1(\bar{Q}).$$

On the other hand, from [11, Corollary 4.3],

$$\ell(Q^{k+1} : x/Q^k) \leq -\binom{k+d-3}{d-2} e_1(Q) = \binom{k+d-3}{d-2} |e_1(Q)|.$$

Thus

$$\begin{aligned} |e_d(Q)| &\leq 3 \sum_{k=0}^r \binom{k+d-3}{d-2} |e_1(Q)| + \sum_{k=0}^r \sum_{i=2}^{d-1} \binom{k+d-i-2}{d-i-1} |e_i(\bar{Q})| \\ &\leq 3 \binom{r+d-2}{d-1} |e_1(Q)| + \sum_{i=2}^{d-1} \sum_{k=0}^r \binom{k+d-i-2}{d-i-1} |e_i(\bar{Q})| \\ &= 3 \binom{r+d-2}{d-1} |e_1(Q)| + \sum_{i=2}^{d-1} \binom{r+d-i-1}{d-i} |e_i(\bar{Q})|. \end{aligned}$$

Notice that

$$\binom{r+d-2}{d-1} \leq r^{d-1} \quad \text{and} \quad \binom{r+d-i-1}{d-i} \leq r^{d-i}.$$

Hence

$$|e_d(Q)| \leq 3 \cdot r^{d-1} |e_1(Q)| + \sum_{i=2}^{d-1} r^{d-i} e_i(\overline{Q}).$$

By induction on d , we may assume that

$$|e_i(\overline{Q})| \leq 3 \cdot 2^{i-2} \cdot r^{i-1} |e_1(\overline{Q})| \quad \text{for } i = 2, \dots, d-1.$$

But $e_i(Q) = e_i(\overline{Q})$ for $i = 1, \dots, d-1$, from Lemma 2.1. This implies that

$$|e_i(Q)| \leq 3 \cdot 2^{i-2} \cdot r^{i-1} |e_1(\overline{Q})| = 3 \cdot 2^{i-2} \cdot r^{i-1} |e_1(Q)| \quad \text{for } i = 2, \dots, d-1.$$

It remains to prove the bound for $e_d(Q)$. Indeed, from inductive hypothesis we have

$$\begin{aligned} |e_d(Q)| &\leq 3 \cdot r^{d-1} |e_1(Q)| + \sum_{i=2}^{d-1} r^{d-i} \cdot 3 \cdot 2^{i-2} \cdot r^{i-1} |e_1(Q)| \\ &= 3 \cdot r^{d-1} |e_1(Q)| + \sum_{i=2}^{d-1} 3 \cdot r^{d-1} \cdot 2^{i-2} |e_1(Q)| \\ &= 3 \cdot r^{d-1} |e_1(Q)| + 3 \cdot r^{d-1} |e_1(Q)| \left(\sum_{i=2}^{d-1} 2^{i-2} \right) \\ &= 3 \cdot r^{d-1} |e_1(Q)| + 3 \cdot r^{d-1} |e_1(Q)| \cdot (2^{d-2} - 1) \\ &= 3 \cdot 2^{d-2} \cdot r^{d-1} |e_1(Q)|. \end{aligned}$$

This finishes the proof. □

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