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Concrete L^2-spectral Analysis of a Bi-weighted Γ-automorphic Twisted Laplacian

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Abstract. We consider a twisted Laplacian $\Delta_{\nu,\mu}$ on the n-complex space associated with the sub-Laplacian of the Heisenberg group $\mathbb{C}\times_\omega\mathbb{C}^n$ realized as a central extension of the real Heisenberg group H_{2n+1}. The main results to which is aimed this paper concern the spectral theory of $\Delta^\Gamma_{\nu,\mu}$ when acting on some L^2 space of Γ-automorphic functions of biweight (ν,μ) associated to given cocompat discrete subgroup Γ of the additive group $\mathbb{C}^n$. We describe its spectrum proving a stability theorem. Using the Selberg's approach, we give the explicit dimension formula for the corresponding L^2-eigenspaces. We also construct a concrete basis of such L^2-eigenspaces.

1. Introduction

The elliptic second order differential operator

$$(1.1)\quad \Delta_{\nu,\mu} = \sum_{j=1}^{n}\left\{4\frac{\partial^2}{\partial z_j\partial\overline{z}_j} + 2(\mu+i\nu)z_j\frac{\partial}{\partial z_j} - 2(\mu-i\nu)\overline{z}_j\frac{\partial}{\partial\overline{z}_j} - (\nu^2+\mu^2)|z_j|^2 + 2i\nu\right\},$$

where ν and μ are real numbers such that $\mu>0$, has been considered recently in [4]. It is shown there that $\Delta_{\nu,\mu}$ is closely connected to the sub-Laplacian of a group of Heisenberg type $\mathbb{C}\times_\omega\mathbb{C}^n$ using partial Fourier transform in (s,t) with (ν,μ) as dual arguments. Such a group is realized as a central extension of the standard Heisenberg group $H_{2n+1} = (\mathbb{R}\times\mathbb{C}^n, \cdot_{\Im m\omega})$. It can also be regarded as a Schrödinger operator with a uniform magnetic field on the complete oriented Riemannian manifold $\mathbb{C}^n=\mathbb{R}^{2n}$ and associated to a special Berry phase. The concrete stationary spectral analysis of $\Delta_{\nu,\mu}$ is described there, when acting on both the Frechet space $\mathcal{C}^\infty(\mathbb{C}^n)$ of $\mathcal{C}^\infty$-complex-valued functions on $\mathbb{C}^n$ endowed with the compact-open topology and the free Hilbert space $L^2(\mathbb{C}^n;dm)$ of square integrable complex-valued functions on $\mathbb{C}^n$ with respect to the usual Lebesgue measure dm (see the next section for a brief review).

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The considered Laplacian $\Delta_{\nu,\mu}$ can also be viewed as an operator on the space $\mathcal{F}^{\nu,\mu}_{\Gamma}(\mathbb{C}^n)$ of Γ-automorphic functions of biweight (ν,μ) defined as the set of smooth functions $F \in \mathcal{C}^\infty(\mathbb{C}^n)$ satisfying the functional equation

$$F(z+\gamma) = e^{i\mu\Im m\langle z,\gamma\rangle - i\nu\Re e\langle z,\gamma\rangle} F(z) \tag{1.2}$$

for all $z \in \mathbb{C}^n$ and all γ in a given co-compact discrete subgroup Γ of $(\mathbb{C}^n,+)$. This was possible since $\Delta_{\nu,\mu}$ is invariant with respect to the unitary transformations

$$\left[T^{\nu,\mu}_{[A,b]} f\right](z) = \overline{e^{i\mu\Im m\langle z,A^*b\rangle - i\nu\Re\langle z,A^*b\rangle}} f(Az+b) \tag{1.3}$$

with $[A,b]$ an element of the the group of rigid motions of the n-complex Hermitian space which is known to be the semi-direct product $G = \mathbb{C}^n \rtimes U(n)$, $U(n)$ being the unitary group of $\mathbb{C}^n$. Namely, $A \in U(n)$ and $b \in \mathbb{C}^n$. Thus, we denote by $\Delta^{\Gamma}_{\nu,\mu}$ the action of $\Delta_{\nu,\mu}$ on $\mathcal{F}^{\nu,\mu}_{\Gamma}(\mathbb{C}^n)$. Notice here that we always confound the differential operators with their Friedrichs extensions.

In the present paper, we focus our study on the spectral properties of the Laplacian $\Delta^{\Gamma}_{\nu,\mu}$ acting on $\mathcal{F}^{\nu,\mu}_{\Gamma}(\mathbb{C}^n)$ and on the associated L^2-Hilbert space $L^2_{\nu,\mu}(\mathbb{C}^n/\Gamma)$ of all bi-weighted Γ-automorphic functions obeying the functional equation in (1.2) and subject to the norm boundedness

$$\|F\|^2 := \langle F,F\rangle = \int_{\Lambda(\Gamma)} |F(z)|^2\, dm(z) < +\infty,$$

where $\Lambda(\Gamma)$ is a fundamental domain of Γ. To this end, we begin by showing the non-triviality of $\mathcal{F}^{\nu,\mu}_{\Gamma}(\mathbb{C}^n)$ (and then of $L^2_{\nu,\mu}(\mathbb{C}^n/\Gamma)$) which requires a specific quantization condition on the triplet $(\Gamma;\nu,\mu)$. We next provide two kinds of concrete geometrical realizations of $\mathcal{F}^{\nu,\mu}_{\Gamma}(\mathbb{C}^n)$. The first one as sections of a line bundle over $\mathbb{C}^n/\Gamma$ and the second one as equivariant functions of a principal bundle (see Section 3). Our main results are proved in Sections 4 and 5. Mainely, we show a stability theorem for the spectrum of the free Laplacian $\Delta_{\nu,\mu}$ on $L^2(\mathbb{C}^n;dm)$ and its perturbations $\Delta^{\Gamma}_{\nu,\mu}$ by lattices Γ (Theorem 4.1). In fact, we show that the spectrum consists of the eigenvalues $-2\mu(2\ell+n)$ with $\ell = 0,1,2,\ldots$. Moreover, under the quantization condition, the operator $\Delta^{\Gamma}_{\nu,\mu}$ admits a compact resolvent in $L^2_{\nu,\mu}(\mathbb{C}^n/\Gamma)$. Therefore, the eigenspaces

$$\mathcal{F}^2_{\ell}(\Delta^{\Gamma}_{\nu,\mu},\mathbb{C}^n) = \{F \in L^2_{\nu,\mu}(\mathbb{C}^n/\Gamma);\Delta^{\Gamma}_{\nu,\mu}F = -2\mu(2\ell+n)F\}$$

are of finite dimension, for Γ being a co-compact lattice of $\mathbb{C}^n$ (i.e., $\mathbb{C}^n/\Gamma$ is a compact torus). Therefore, the index Atiyah–Singer theorem might be used to give the dimension of the space $\mathcal{F}^2_0(\Delta^{\Gamma}_{\nu,\mu},\mathbb{C}^n)$ associated with the first eigenvalue of the operator $\Delta^{\Gamma}_{\nu,\mu}$ viewed as an elliptic differential operator on the compact manifold $\mathbb{C}^n/\Gamma$, acting on sections of a line bundle over $\mathbb{C}^n/\Gamma$ (see the next section). The general case of $\mathcal{F}^2_{\ell}(\Delta^{\Gamma}_{\nu,\mu},\mathbb{C}^n)$ with

$\ell = 1, 2, \ldots$, is not easy to handel in an abstract way. However, using the Selberg approach we will be able to give the explicit dimension formula for the L^2-eigenspaces $\mathcal{F}^2_\ell(\Delta^\Gamma_{\nu,\mu}, \mathbb{C}^n)$ (Theorem 4.2).

The obtained results generalize in some sort the ones previously obtained in [6]. Moreover, we provide an explicit basis of the holomorphic L^2-eigenspace $\mathcal{F}^2_0(\Delta^\Gamma_{\nu,\mu}, \mathbb{C}^n)$, i.e., the one associated to the lowest Landau level $\ell = 0$. Their respective eigenfunctions are useful for different physical pictures. We will use them to generate the basis of the non-holomorphic L^2-eigenspace $\mathcal{F}^2_\ell(\Delta^\Gamma_{\nu,\mu}, \mathbb{C}^n)$ with $\ell \neq 0$ leading to a Hilbertian decomposition of $L^2_{\nu,\mu}(\mathbb{C}^n/\Gamma)$. The higher dimensions are also discussed.

2. Preliminaries

The operator $\Delta_{\nu,\mu}$ in (1.1) is essentially the known special Hermite operator

$$H_\mu := 4\sum_{j=1}^{n} \frac{\partial^2}{\partial z_j \partial \overline{z}_j} + 2\mu(E - \overline{E}) - \mu^2|z|^2, \quad \mu > 0, \tag{2.1}$$

where E is the Euler operator on the n-complex space $\mathbb{C}^n$ and $\overline{E}$ its complex conjugate, perturbed by the first order differential operator $D_\nu := 2i\nu(E + \overline{E} + n) - \nu^2|z|^2$. The operator H_μ describes in physics the quantum behavior of a charged spinless particle on the configuration space $\mathbb{C}^n$ under the influence of a constant magnetic field [1,5,7,10,11]. The realization of $\Delta_{\nu,\mu}$ as a Schrödinger operator corresponds to the isotropic magnetic field $\overrightarrow{B}_\mu = id\theta_{\nu,\mu}$ of constant strength μ, and associated to the specific differential one-form

$$\theta_{\nu,\mu}(z) := -\frac{\mu - i\nu}{2}\sum_{j=1}^{n} \overline{z}_j dz_j + \frac{\mu + i\nu}{2}\sum_{j=1}^{n} z_j d\overline{z}_j.$$

The additional gauge function $\varphi(z) = \frac{i\nu}{2}|z|^2$ in the gauge transformation $\theta_{\nu,\mu} = \theta^s_\mu + \frac{i\nu}{2} d|z|^2$, is equivalent to multiplying the eigenstates of the Landau Hamiltonian H_μ; $\nu = 0$, by the phase factor $e^{-\frac{i\nu}{2}|z|^2}$.

From the general theory of Schrödinger operators on non-compact manifolds, the operator $\Delta_{\nu,\mu}$, viewed as an unbounded operator in $L^2(\mathbb{C}^n; dm)$, is essentially self-adjoint for any smooth measure dm (see for example [9]). It is proved in [4] that the L^2-spectrum of $\Delta_{\nu,\mu}$ acting on the free Hilbert space $L^2(\mathbb{C}^n; dm)$ is purely discrete, independent of the parameter ν and coincides with the Landau energy levels

$$E_\ell := -2\mu(2\ell + n), \quad \ell = 0, 1, 2, \ldots$$

of the twisted Laplacian in (2.1). Moreover, each eigenvalue occurs with infinite degeneracy, in the sense that the L^2-eigenspace

$$\mathcal{F}^2_\ell(\Delta_{\nu,\mu}, \mathbb{C}^n) := \{F \in L^2(\mathbb{C}^n; dm); \Delta_{\nu,\mu} F = -2\mu(2\ell + n)F\}$$

is of infinite dimension. Moreover, the explicit expression of the reproducing kernel of the L^2-eigenspaces $\mathcal{F}^2_\ell(\Delta_{\nu,\mu},\mathbb{C}^n)$ in terms of the confluent hypergeometric function is given by [4, Proposition 5.11]

$$K^{\nu,\mu}_\ell(z,w) = \left(\frac{\mu}{\pi}\right)^n \frac{(n-1+\ell)!}{(n-1)!\ell!} e^{-\frac{i\nu}{2}(|z|^2-|w|^2)+i\mu\Im m\langle z,w\rangle} e^{-\frac{\mu}{2}|z-w|^2} {}_1F_1(-\ell;n;\mu|z-w|^2).$$

The operator $\Delta_{\nu,\mu}$ satisfies the interesting invariance property

$$\Delta_{\nu,\mu} T^{\nu,\mu}_g = T^{\nu,\mu}_g \Delta_{\nu,\mu} \tag{2.2}$$

with respect the unitary transformations $T^{\nu,\mu}_g F(z) = \overline{j^{\nu,\mu}(g,z)} F(g\cdot z)$ in (1.3) for g in the semi-direct product

$$G := \mathbb{C}^n \rtimes U(n) = \left\{ g = \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} =: [A,b]; A \in U(n), b \in \mathbb{C}^n \right\}$$

that acts transitively on $\mathbb{C}^n$ via the mappings $g\cdot z = Az + b$ for $g = [A,b] \in G$. The mapping

$$j^{\nu,\mu}(g,z) = \exp\left\{ i\nu \Re e(\langle z, g^{-1}\cdot 0\rangle) - i\mu \Im m(\langle z, g^{-1}\cdot 0\rangle)\right\} \tag{2.3}$$

on $G \times \mathbb{C}^n$ satisfies the automorphic equation

$$j^{\nu,\mu}(\gamma\gamma', z) = j^{\nu,\mu}(\gamma, \gamma' z) j^{\nu,\mu}(\gamma', z) \tag{2.4}$$

for every γ, γ', if and only if

$$\nu \Re e\langle \gamma,\gamma'\rangle - \mu \Im m\langle\gamma,\gamma'\rangle \in 2\pi\mathbb{Z}. \tag{2.5}$$

Definition 2.1. The triplet $(\Gamma;\nu,\mu)$, of real numbers ν, μ and discrete subgroup Γ in $(\mathbb{R}^{2n},+) = (\mathbb{C}^n,+)$, is said to be quantized if it obeys the Riemann–Dirac quantization (RDQ) condition (2.5) which equivalently reads

$$\nu \Re e\langle\gamma,\gamma'\rangle \in 2\pi\mathbb{Z} \quad \text{and} \quad \mu \Im m\langle\gamma,\gamma'\rangle \in 2\pi\mathbb{Z} \tag{2.6}$$

for all $\gamma,\gamma' \in \Gamma$.

From now on we assume that the triplet $(\Gamma;\nu,\mu)$ is quantized in the sense of Definition 2.1. It turns out that the quantization condition we require seems to be very restrictive. However, we provide below examples of such triplets where the discrete subgroup Γ is of full rank.

Example 2.2. Let $n = 1$ and b, c be integer numbers such that $c > 0$ and $b^2 - 4c < 0$. Let τ be the solution of the algebraic equation $\tau^2 - b\tau + c = 0$ so that $\operatorname{Im}\tau > 0$. Then the lattices $\Gamma_\tau = \mathbb{Z} + \tau\mathbb{Z}$ in $\mathbb{C} = \mathbb{R}^2$ are of special kind and include the Gauss numbers $\Gamma_i = \mathbb{Z} + i\mathbb{Z}$ and $\Gamma_j = \mathbb{Z} + j\mathbb{Z}$. Making varying b and c, we can get other lattices Γ_τ in $\mathbb{C}$ such as those associated to $\tau = \frac{-1+i\sqrt{7}}{2}, \frac{-1+i\sqrt{11}}{2}, \frac{-1+i\sqrt{15}}{2}, \frac{-1+i\sqrt{19}}{2}, \dots$ and corresponding to $b = -1$ and $c = 2, 3, 4, 5, \dots$, respectively. The choice of the parameters (ν, μ) such as $\nu = 2\pi k_0$ and $\mu = \pi l_0/\operatorname{Im}\tau$ for some $k_0, l_0 \in \mathbb{Z}$, leads to a quantized triplet $(\Gamma_\tau; \nu, \mu)$.

Example 2.3. Starting from Γ_τ as in Example 2.2, we can build a lattice in $\mathbb{C}^n = \mathbb{R}^{2n}$ of maximal rank $2n$, say $\widetilde{\Gamma^n_\tau} = \Gamma_\tau \times \Gamma_\tau \times \cdots \times \Gamma_\tau$ (n-times) such that $\mathbb{C}^n/\widetilde{\Gamma^n_\tau}$ is a torus of $\mathbb{R}^{2n} = \mathbb{C}^n$. Moreover, the triplet $(\widetilde{\Gamma^n_\tau}; \nu, \mu)$ is subject to the (RDQ) assumption for suitable conditions on (ν, μ) like the ones provided in the previous example, i.e., $\nu = 2\pi k_0$ and $\mu = \pi l_0/\operatorname{Im}\tau$ for some $k_0, l_0 \in \mathbb{Z}$.

The $T^{\nu,\mu}_g$-invariance property shows in particular that $\Delta_{\nu,\mu}$ leaves invariant the functional space of $\mathcal{C}^\infty$-functions f satisfying the functional equation $T^{\nu,\mu}_\gamma f = f$, for every γ belonging to given discrete subgroup Γ of G. Accordingly, the corresponding eigenvalue problem is well-defined. In the sequel, we will treat the co-compact case, i.e., Γ is a discrete subgroup of $(\mathbb{C}^n, +)$ of maximal rank.

3. Γ-automorphic functions of bi-weight (ν, μ) on $\mathbb{C}^n$

In this section, we introduce the class of the so-called Γ-automorphic functions of bi-weight (ν, μ) on $\mathbb{C}^n$. We focuss our discussion on some of their elementary properties, including their concrete geometrical realizations as a space of sections of some rank-one bundles over $\mathbb{C}^n/\Gamma$ under the assumption that the (RDQ) "quantization rule" in (2.6) holds. In fact, we will provide two kinds of such realizations, the first one as sections of a line bundle and the second one as equivariant functions of a principal bundle. To this end, let Γ be a uniform lattice (co-compact discrete subgroup) of the additive group $(\mathbb{R}^{2n}, +) = (\mathbb{C}^n, +)$, so that the quotient space is the compact torus $\mathbb{C}^n/\Gamma$. Thus, there exists some $\mathbb{R}$-linearly independent vectors $u_1, u_2, \dots, u_{2n}$ in $\mathbb{C}^n$ such that

$$\Gamma = \mathbb{Z}u_1 + \mathbb{Z}u_2 + \cdots + \mathbb{Z}u_{2n}.$$

Associated to such Γ and given $(\nu, \mu) \in \mathbb{R}^2$, we consider the functional space $\mathcal{F}^{\nu,\mu}_\Gamma(\mathbb{C}^n)$ as defined in the introduction and consisting of all $\mathcal{C}^\infty$-complex-valued functions on $\mathbb{C}^n$ satisfying the pseudo-periodicity (1.2).

Definition 3.1. The space $\mathcal{F}^{\nu,\mu}_\Gamma(\mathbb{C}^n)$ is called here the space of Γ-periodic functions of bi-weight (ν, μ) on $\mathbb{C}^n$.

The particular space $\mathcal{F}^{0,0}_{\Gamma}(\mathbb{C}^n)$, corresponding to $(\nu,\mu) = (0,0)$, is nothing than the space of Γ-periodic functions which is known to be a non-zero vector space. However, for arbitrary given $(\nu,\mu) \neq (0,0)$ and Γ, there is no guarantee that the corresponding space $\mathcal{F}^{\nu,\mu}_{\Gamma}(\mathbb{C}^n)$ is not trivial. Thus, additional assumption on the triplet $(\Gamma;\nu,\mu)$ must be imposed in order to ensure the non-triviality of the space $\mathcal{F}^{\nu,\mu}_{\Gamma}(\mathbb{C}^n)$.

Lemma 3.2. *Assume that* $(\Gamma;\nu,\mu)$ *is quantized. Then, the space* $\mathcal{F}^{\nu,\mu}_{\Gamma}(\mathbb{C}^n)$ *is an infinite dimensional vector space over* $\mathbb{C}$.

Proof. Let $\varphi(z)$ be a compactly supported $\mathcal{C}^\infty$-complex-valued function on $\mathbb{C}^n$ with support contained in the interior of a fundamental domain $\Lambda(\Gamma)$ of Γ. Making a sort of $(\Gamma;\nu,\mu)$-periodization *à la* Poincaré

$$[\mathcal{P}^{\nu,\mu}_{\Gamma}\varphi](z) = \sum_{\gamma\in\Gamma} e^{i[-\mu\Im m(\langle z,\gamma\rangle)+\nu\Re e(\langle z,\gamma\rangle)]}\varphi(z+\gamma),$$

it is clear that $\mathcal{P}^{\nu,\mu}_{\Gamma}\varphi|_{\Lambda(\Gamma)}(z) = \varphi(z)$. Moreover, we can check easily that $\mathcal{P}^{\nu,\mu}_{\Gamma}\varphi \in \mathcal{F}^{\nu,\mu}_{\Gamma}(\mathbb{C}^n)$ thanks to the (RDQ)-assumption. Hence, the vector space $\mathcal{F}^{\nu,\mu}_{\Gamma}(\mathbb{C}^n)$ contains all the elements $\mathcal{P}^{\nu,\mu}_{\Gamma}\varphi$ for $\varphi \in \mathcal{C}^\infty_c(\Lambda(\Gamma))$. It follows that $\mathcal{F}^{\nu,\mu}_{\Gamma}(\mathbb{C}^n)$ is not-trivial and it is of infinite dimension. □

The above lemma can be reproved by realizing $\mathcal{F}^{\nu,\mu}_{\Gamma}(\mathbb{C}^n)$ as a space of sections of a "homogeneous" $\mathcal{C}^\infty$-line bundle over $\mathbb{C}^n/\Gamma$. Indeed, under the (RDQ)-assumption (2.6), the automorphic factor $j^{\nu,\mu}(\gamma,z)$ in (2.3) satisfies the chain rule (2.4). Therefore, the Γ-action on $\mathbb{C}^n$ can be extended to $\mathbb{C}^n\times\mathbb{C}$ by considering the mappings

$$\widetilde{\gamma}(z,v) := (\gamma+z; j^{\nu,\mu}(\gamma,z)v)$$

for varying $\gamma\in\Gamma$. The mapping $\widetilde{\gamma}$ is linear on the fiber $\mathbb{C}_v$ and hence the space $\mathbb{C}^n\times\mathbb{C}$ becomes a homogeneous line bundle over $\mathbb{C}^n$ whose fiber is $p^{-1}(z) = \{z\}\times\mathbb{C} \simeq \mathbb{C}$, where p is the projection of $\mathbb{C}^n\times\mathbb{C}$ onto $\mathbb{C}^n$. If π denotes the canonical quotient projection, then the above claims can be pictured by the following commutative diagrams

$$\begin{tikzcd}
 & & \mathbb{C}^n\times\mathbb{C} \\
\mathbb{C}^n\times\mathbb{C} \arrow[urr, "\widetilde{\gamma+\gamma'}"] \arrow[rr, "\widetilde{\gamma}"] \arrow[d, "p"'] \arrow[drr, phantom, "\circlearrowleft"] & \arrow[ur, phantom, "\circlearrowleft"] & \mathbb{C}^n\times\mathbb{C} \arrow[u, "\widetilde{\gamma}'"'] \arrow[d, "p"] \\
\mathbb{C}^n \arrow[rr, "\gamma"] \arrow[dr, "\pi"'] \arrow[drr, phantom, "\circlearrowleft"] & & \mathbb{C}^n \arrow[dl, "\pi"] \\
 & \mathbb{C}^n/\Gamma &
\end{tikzcd}$$

Accordingly, there is a line bundle over $\mathbb{C}^n/\Gamma$ (where the fiber is $\mathbb{C}$) so that its cross sections are indeed our space $\mathcal{F}^{\nu,\mu}_{\Gamma}(\mathbb{C}^n)$ of Γ-periodic functions of quantified bi-weight (ν,μ). Therefore, the space $\mathcal{F}^{\nu,\mu}_{\Gamma}(\mathbb{C}^n)$ is of infinite dimension.

We conclude this section by providing another concrete geometrical realization of $\mathcal{F}^{\nu,\mu}_{\Gamma}(\mathbb{C}^n)$ as equivariant functions of a rank-one principal bundle over $\mathbb{C}^n/\Gamma$. To this end, we invoke the group $N_\omega := \mathbb{C}\times_\omega \mathbb{C}^n$ constructed in [4] and realized as a central extension of the Heisenberg group $H_{2n+1} := \mathbb{R}\times_{\Im m\omega}\mathbb{C}^n$, where the mapping ω denotes the standard Hermitian form on $\mathbb{C}^n$ given by $\omega(z,w)=\langle z,w\rangle$. Assume that we are given a lattice Γ_0 in $(\mathbb{C},+)=(\mathbb{R}^2,+)$ and a lattice Γ in $(\mathbb{C}^n,+)=(\mathbb{R}^{2n},+)$ such that ω sends $\Gamma\times\Gamma$ to Γ_0, i.e., for every γ, γ', we have

$$\omega(\gamma,\gamma')=\langle\gamma,\gamma'\rangle\in\Gamma_0. \tag{3.1}$$

Then under the condition (3.1), it is easy to check that the following facts hold.

Lemma 3.3. *$\Gamma_0\times_\omega\Gamma$ is a discrete subgroup of $N_\omega=\mathbb{C}\times_\omega\mathbb{C}^n$ and it acts freely by left translations on $N_\omega=\mathbb{C}\times_\omega\mathbb{C}^n$. The resulting quotient space $\mathbb{P}^{n+1}_\omega := \Gamma_0\times_\omega\Gamma\setminus\mathbb{C}\times_\omega\mathbb{C}^n$ is a smooth manifold.*

Obviously, the functions $f\in\mathcal{C}^\infty(\mathbb{P}^{n+1}_\omega)$ are those on $\mathbb{C}\times\mathbb{C}^n$ that are $\Gamma_0\times_\omega\Gamma$-left invariants, i.e.,

$$f((\gamma_0;\gamma)\cdot_\omega(z_0;z))=f(\gamma_0+z_0+\omega(\gamma,z);\gamma+z)=f(z_0;z)$$

for all $(\gamma_0;\gamma)\in\Gamma_0\times\Gamma$ and $(z_0;z)\in\mathbb{C}\times\mathbb{C}^n$.

Lemma 3.4. *The group $(\mathbb{C},+)$ is isomorphic to the center $Z(N_\omega)=\mathbb{C}\times_\omega\{0\}$ of N_ω.*

Subsequently, the group $(\mathbb{C},+)$ acts on $N_\omega=\mathbb{C}\times_\omega\mathbb{C}^n$ via the rule (left translations) as follows

$$\lambda\cdot(z_0;z)=(\lambda+z_0;z). \tag{3.2}$$

The action in (3.2) can be restricted to $\lambda\in\Gamma_0$ and $(z_0,z)=(\gamma_0,\gamma)\in\Gamma_0\times\Gamma$, so that we can see that the torus $\mathbb{C}^n/\Gamma$ acts freely and effectively on the quotient space $(\mathbb{C}/\Gamma_0)\setminus\mathbb{P}^{n+1}_\omega$.

Definition 3.5. A function $f(z_0;z)\in\mathcal{C}^\infty(\mathbb{C}\times\mathbb{C}^n)$ is said to be equivriant with respect to given pair of real numbers (ν,μ) if it satisfies

$$f(\lambda+z_0;z)=e^{i[\nu\Re e(\lambda)+\mu\Im m(\lambda)]}f(z_0;z)$$

for every $\lambda\in\mathbb{C}$ and $(z_0;z)\in\mathbb{C}\times\mathbb{C}^n$.

Accordingly, equivariant functions $f(z_0, z)$ are necessary of the form

$$f(z_0; z) = e^{i[\nu\Re e(z_0)+\mu\Im m(z_0)]}F(z), \tag{3.3}$$

where $F(z)$ is a $\mathcal{C}^\infty$ function on $\mathbb{C}^n$.

Proposition 3.6. *Let Γ_0 and Γ be lattices in $(\mathbb{C}, +)$ and $(\mathbb{C}^n, +)$ respectively, and assume that the condition* (3.1) *holds. Then, we have*

(i) *Let $f(z_0; z)$ be in $\mathcal{C}^\infty(\mathbb{P}^{n+1}_\omega)$ such that it is equivariant. Then the function $F(z)$ occurring in* (3.3) *must satisfy the following "pseudo-periodic condition" with respect to the Γ-lattice of $(\mathbb{C}, +)$. Namely, for every $\gamma \in \Gamma$ and $z \in \mathbb{C}^n$, we have*

$$F(z + \gamma) = e^{i\mu\Im m\langle z,\gamma\rangle - i\nu\Re e\langle z,\gamma\rangle}F(z). \tag{3.4}$$

(ii) *Let ν, μ be in Γ_0^* the dual lattice of Γ_0 in $\mathbb{R}^2 = \mathbb{C}$ defined by*

$$\Gamma_0^* = \{\gamma_0^* \in \mathbb{R}^2; \Re e(\gamma_0^* \cdot \overline{\gamma_0}) \in 2\pi\mathbb{Z} \text{ for all } \gamma_0 \in \Gamma_0\}.$$

Then, a function $F(z)$ on $\mathbb{C}^n$ satisfying (3.4) *with respect to Γ gives rise, through the formula* (3.2), *to a function $f(z_0; z)$ on $\mathbb{P}^{n+1}_\omega$ which is equivariant.*

To conclude, we picture the involved algebraic and geometrical objects by the following commutative diagrams keeping in mind the previous notations.

$$\begin{tikzcd}
{} \arrow[r] & \Gamma_0 \times_\omega \{0\} \arrow[r, "\mathrm{inj}"] \arrow[d, "i"'] \arrow[dr, phantom, "\circlearrowleft"] & \Gamma_0 \times_\omega \Gamma \arrow[r, "p"] \arrow[d, "i"'] \arrow[dr, phantom, "\circlearrowleft"] & \Gamma \arrow[r] \arrow[d, "i"'] & {} \\
{} \arrow[r] & \mathbb{C} \times_\omega \{0\} \arrow[r, "\mathrm{inj}"] \arrow[d, "q"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{C} \times_\omega \mathbb{C}^n \arrow[r, "p"] \arrow[d, "q"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{C}^n \arrow[r] \arrow[d, "q"'] & {} \\
{} \arrow[r] & \mathbb{C}/\Gamma_0 = T^2 \arrow[r, "\mathrm{inj}"] \arrow[d] & \mathbb{P}^{n+1}_\omega \arrow[r, "\pi"] \arrow[d] & \mathbb{C}^n/\Gamma \arrow[r] \arrow[d] & {} \\
& {} & {} & {} &
\end{tikzcd}$$

Above inj is the natural injection morphism from the subgroup $\Gamma_0 \times_\omega \{0\}$ into $\Gamma_0 \times_\omega \Gamma$ and similarly for $\mathbb{C} \times_\omega \{0\}$ into the group $N_\omega = \mathbb{C} \times_\omega \mathbb{C}^n$. The mapping p is the projection on the second factor and it is a morphism of groups. The mapping q is the natural quotient projection from $\mathbb{C}\times_\omega\{0\}$ onto the quotient group $\Gamma_0\times_\omega\{0\}\backslash\mathbb{C}\times_\omega\{0\}$ which is isomorphic as a group to the torus $T^2 = \mathbb{C}/\Gamma_0$. To precise the projection mapping π in above, we should notice here that T^2 acts on $\mathbb{P}^{n+1}_\omega$ so that the set of its orbits $\mathcal{O}(T^2, \mathbb{P}^{n+1}_\omega)$ is diffeomorphic to the torus $\mathbb{C}^n/\Gamma$. More precisely, we have $\mathcal{O}(T^2, \mathbb{P}^{n+1}_\omega) = \mathbb{P}^{n+1}_\omega/T^2$ and π is then the canonical projection of the action T^2 on $\mathbb{P}^{n+1}_\omega$ for which $\Gamma_0 \times_\omega \{0\} \setminus \mathbb{C} \times_\omega \{0\} = T^2$ are the fibers of the principal bundle $\mathbb{P}^{n+1}_\omega$ over $\mathbb{C}^n/\Gamma$.

In the next section, we investigate some spectral properties of the Laplacian $\Delta_{\nu,\mu}$ when acting on the space $\mathcal{F}^{\nu,\mu}_\Gamma(\mathbb{C}^n)$ for given co-compact discrete subgroup Γ.

4. Spectrum and dimension formula

From the definition of the space $\mathcal{F}^{\nu,\mu}_{\Gamma}(\mathbb{C}^n)$ and the $T^{\nu,\mu}_g$-invariance property of the Laplacian $\Delta_{\nu,\mu}$ in (2.2), we can consider $\Delta_{\nu,\mu}$ as an operator acting on the space $\mathcal{F}^{\nu,\mu}_{\Gamma}(\mathbb{C}^n)$, carrying $\mathcal{F}^{\nu,\mu}_{\Gamma}(\mathbb{C}^n)$ to $\mathcal{F}^{\nu,\mu}_{\Gamma}(\mathbb{C}^n)$. Thus, for given $F_1, F_2 \in \mathcal{F}^{\nu,\mu}_{\Gamma}(\mathbb{C}^n)$, the function $\psi(z) = F_1(z)\overline{F_2(z)}$ is Γ-periodic, in the sense that $\psi(z+\gamma) = \psi(z)$ for every $\gamma \in \Gamma$. Therefore, the quantity

$$\langle F_1, F_2\rangle := \int_{\Lambda(\Gamma)} F_1(z)\overline{F_2(z)}\, dm(z), \tag{4.1}$$

where $\Lambda(\Gamma)$ is a fundamental domain of Γ in $\mathbb{C}^n$, is well-defined and is independent of the choice of the fundamental domain. Moreover, it defines a Hermitian scalar product on $\mathcal{F}^{\nu,\mu}_{\Gamma}(\mathbb{C}^n)$. Hence, all the elements of $\mathcal{F}^{\nu,\mu}_{\Gamma}(\mathbb{C}^n)$ are necessarily bounded functions, for Γ being a lattice in $\mathbb{C}^n = \mathbb{R}^{2n}$. We define $L^2_{\nu,\mu}(\mathbb{C}^n/\Gamma)$ to be the Hilbert space obtained as completion of the pre-hilbertian space $\mathcal{F}^{\nu,\mu}_{\Gamma}(\mathbb{C}^n)$ with respect to the scalar product (4.1). The operator $\Delta_{\nu,\mu}$, acting with densely domain in $L^2_{\nu,\mu}(\mathbb{C}^n/\Gamma)$, is essentially self-adjoint. Thus, we denote by $\Delta^{\Gamma}_{\nu,\mu}$ its unique self-adjoint realization in the Hilbert space $L^2_{\nu,\mu}(\mathbb{C}^n/\Gamma)$.

Within the above notation, we can state and prove the main results on the spectral properties of $\Delta^{\Gamma}_{\nu,\mu}$ for lattices in $\mathbb{R}^{2n} = \mathbb{C}^n$ and reals ν, μ such that $(\Gamma;\nu,\mu)$ is quantized. The first one determinates the spectrum of $\Delta^{\Gamma}_{\nu,\mu}$ on $L^2_{\nu,\mu}(\mathbb{C}^n/\Gamma)$.

Theorem 4.1 (Spectrum stability)**.** *Let Γ be a lattice of $\mathbb{C}^n = \mathbb{R}^{2n}$ such that the triplet $(\Gamma;\nu,\mu)$ satisfies the (RDQ) condition. Then, the spectrum of $\Delta^{\Gamma}_{\nu,\mu}$ acting on $L^2_{\nu,\mu}(\mathbb{C}^n/\Gamma)$ coincides with the spectrum of $\Delta_{\nu,\mu}$ when acting on the free Hilbert space $L^2(\mathbb{C}^n; dm)$. More precisely, we have*

$$\mathrm{Sp}(\Delta^{\Gamma}_{\nu,\mu}) = \{E_\ell := -2\mu(2\ell+n); \ell = 0,1,2,\ldots\} = \mathrm{Sp}(\Delta_{\nu,\mu}).$$

However, each eigenvalue $-2\mu(2\ell+n)$ occurs here with finite degeneracy in contrary to those of $\Delta_{\nu,\mu}$ on $L^2(\mathbb{C}^n; dm)$.

Proof. For any given $g \in G = \mathbb{C}^n \rtimes U(n)$ and $F(z) \in \mathcal{C}^\infty(\mathbb{C}^n)$, we consider the function defined on $\mathbb{C}^n$ by the "averaging" formula over $U(n)$,

$$\mathcal{A}^{\nu,\mu}_g F(z) = \int_{U(n)} T^{\nu,\mu}_{gh} F(z)\, dh = \int_{U(n)} \overline{j^{\nu,\mu}(gh,z)} F(gh\cdot z)\, dh,$$

where $j^{\nu,\mu}(\,\cdot\,,z)$ is the automorphic factor given by (2.3) and dh stands for the normalized (left) Haar measure on the unitary group $U(n)$. By the compactness of the Lie group $U(n)$, it is easy to see that the output function $\mathcal{A}^{\nu,\mu}_g F$ shares the same smoothness as that of the input function F. For instance, $\mathcal{A}^{\nu,\mu}_g F$ is a $\mathcal{C}^\infty$-complex-valued function on $\mathbb{C}^n$ if F is $\mathcal{C}^\infty$ and $\mathcal{A}^{\nu,\mu}_g F$ is a bounded function on $\mathbb{C}^n$ if F is bounded. We also have $\mathcal{A}^{\nu,\mu}_g F(0) = F(g\cdot 0)$

and $\mathcal{A}_g^{\nu,\mu}F(z)$ is $U(n)$-invariant in the sense that $\mathcal{A}_g^{\nu,\mu}F(k\cdot z)=\mathcal{A}_g^{\nu,\mu}F(z)$ for all $k\in U(n)$ and $z\in\mathbb{C}^n$. This is to say that $\mathcal{A}_g^{\nu,\mu}F(z)$ is a radial function on $\mathbb{C}^n$. Moreover, the averaging operator $\mathcal{A}_g^{\nu,\mu}$ leaves invariant the eigenspace

$$E_\lambda(\Delta_{\nu,\mu})=\{f\in\mathcal{C}^\infty(\mathbb{C}^n);\Delta_{\nu,\mu}f=-2\mu(2\lambda+n)f\}.$$

This relies on the co-cycle condition satisfied by the automorphic factor $j^{\nu,\mu}(g,z)$ as well as on the invariance property satisfied by $\Delta_{\nu,\mu}$ with respect to the $T_g^{\nu,\mu}$-action defined in (1.3). Accordingly, the explicit expression of $\mathcal{A}_g^{\nu,\mu}F(z)$ is given in terms of the confluent hypergeometric function [4, Proposition 5.2]

$$\mathcal{A}_g^{\nu,\mu}F(z)=F(g\cdot 0)e^{\frac{i\nu-\mu}{2}|z|^2}{}_1F_1(-\lambda;n;\mu|z|^2)=F(g\cdot 0)\varphi_\lambda^{\nu,\mu}(z).$$

The spherical eigenfuction $\varphi_\lambda^{\nu,\mu}(z)$ is then bounded on $\mathbb{C}^n$ if and only if λ is a nonnegative integer. This immediately follows from the well-known asymptotic behavior of the confluent hypergeometric function ${}_1F_1(\alpha;\gamma;x)$, near $x=+\infty$, given by [8, page 332]

$${}_1F_1(a;c;x)=\Gamma(c)\left\{\frac{(-x)^{-a}}{\Gamma(c-a)}+\frac{e^xx^{a-c}}{\Gamma(a)}\right\}\left(1+O\left(\frac{1}{x}\right)\right),$$

where $\Gamma(z)$ is the Gamma Euler function on $\mathbb{C}$. Subsequently, the eigenspace

$$\mathcal{E}_\lambda^b(\Delta_{\nu,\mu})=\{F\in E_\lambda(\Delta_{\nu,\mu});F\text{ is bounded on }\mathbb{C}^n\}$$

formed by bounded eigenfunctions with $-2\mu(2\lambda+n)$ as eigenvalue, is a nonzero space if and only if λ is a nonnegative integer, $\lambda=\ell=0,1,2,\dots$. To conclude for Theorem 4.1, we need only to observe that for given lattice Γ in $\mathbb{C}^n$ the L^2-eigenspace

$$\mathcal{F}_\lambda^2(\Delta_{\nu,\mu}^\Gamma,\mathbb{C}^n)=\{F\in L^2_{\nu,\mu}(\mathbb{C}^n/\Gamma);\Delta_{\nu,\mu}^\Gamma F=-2\mu(2\lambda+n)F\}$$

is indeed contained in $\mathcal{E}_\lambda^b(\Delta_{\nu,\mu})$. □

Then next result concerns the dimension formula of the L^2-eigenspaces $\mathcal{F}_\ell^2(\Delta_{\nu,\mu}^\Gamma,\mathbb{C}^n)$.

Theorem 4.2 (Dimension formula). *Let $(\Gamma;\nu,\mu)$ be a quantized triplet and $\ell=0,1,2,\dots$. The dimension formula of $\mathcal{F}_\ell^2(\Delta_{\nu,\mu}^\Gamma,\mathbb{C}^n)$ is given by*

$$\dim\mathcal{F}_\ell^2(\Delta_{\nu,\mu}^\Gamma,\mathbb{C}^n)=\left(\frac{\mu}{\pi}\right)^n\frac{(n-1+\ell)!}{(n-1)!\ell!}\operatorname{vol}(\Lambda(\Gamma)),$$

where $\operatorname{vol}(\Lambda(\Gamma))$ is the Lebesgue volume measure in $\mathbb{C}^n=\mathbb{R}^{2n}$ of a fundamental domain $\Lambda(\Gamma)$ of Γ.

Proof. For $\ell = 0, 1, 2, \ldots$, let $P_\ell(\Delta^\Gamma_{\nu,\mu})$ be the orthogonal projection of $L^2_{\nu,\mu}(\mathbb{C}^n/\Gamma)$ onto the L^2-eigenspace $\mathcal{F}^2_\ell(\Delta^\Gamma_{\nu,\mu}, \mathbb{C}^n)$. Then, under the (RDQ)-condition, the Schwartz kernel $P^{\nu,\mu;\Gamma}_\ell(z,w)$ of the orthogonal projector operator $P_\ell(\Delta^\Gamma_{\nu,\mu})$ from $L^2_{\nu,\mu}(\mathbb{C}^n/\Gamma)$ to $\mathcal{F}^2_\ell(\Delta^\Gamma_{\nu,\mu}, \mathbb{C}^n)$ can be proved to be given explicitly by the formula

$$\begin{aligned} P^{\nu,\mu;\Gamma}_\ell(z,w) &= \left(\frac{\mu}{\pi}\right)^n \frac{(n-1+\ell)!}{(n-1)!\ell!} e^{-\frac{i\nu}{2}(|z|^2-|w|^2)+\frac{\mu}{2}(\langle z,w\rangle - \overline{\langle z,w\rangle})} \\ &\quad \times \sum_{\gamma\in\Gamma} e^{-\frac{i\nu}{2}|\gamma|^2+\frac{\mu}{2}(\langle z+w,\gamma\rangle-\overline{\langle z+w,\gamma\rangle})} e^{-\frac{\mu}{2}|z-w-\gamma|^2} {}_1F_1(-\ell; n; \mu|z-w-\gamma|^2). \end{aligned}$$

The kernel function $P^{\nu,\mu;\Gamma}_\ell(z,w)$ is in fact the $T^{\nu,\mu}$-periodization à la Poincaré of the Schwartz kernel $P^{\nu,\mu}_\ell$ of the orthogonal projection on the free Hilbert space $L^2(\mathbb{C}^n; dm)$ on the L^2-eigenspace $\mathcal{F}^2_\ell(\Delta_{\nu,\mu}, \mathbb{C}^n) := \mathrm{Ker}\,|_{L^2(\mathbb{C}^n;dm)}(\Delta_{\nu,\mu} - E_\ell)$ given explicitly in [4]. Therefore, the dimension of $\mathcal{F}^2_\ell(\Delta^\Gamma_{\nu,\mu}, \mathbb{C}^n)$ is given by

$$\begin{aligned} \dim \mathcal{F}^2_\ell(\Delta^\Gamma_{\nu,\mu}, \mathbb{C}^n) &= \mathrm{Trace}(P_\ell(\Delta^\Gamma_{\nu,\mu})) \\ &= \int_{\Lambda(\Gamma)} P^{\nu,\mu;\Gamma}_\ell(z,z)\, dm(z) \\ &= \left(\frac{\mu}{\pi}\right)^n \frac{(n-1+\ell)!}{(n-1)!\ell!} \mathrm{vol}(\Lambda(\Gamma)). \end{aligned}$$

The last equality follows making use of the identity

$$\int_{\Lambda(\Gamma)} e^{\mu(\langle z,\gamma\rangle - \overline{\langle z,\gamma\rangle})}\, dm(z) = \delta_{0,\gamma}\, \mathrm{vol}(\Lambda(\Gamma)). \qquad \square$$

Remark 4.3. The dimension of $\mathcal{F}^2_\ell(\Delta^\Gamma_{\nu,\mu}, \mathbb{C}^n)$ is clearly independent of the parameter ν. Moreover, for $n = 1$, the eigenspaces $\mathcal{F}^2_\ell(\Delta^\Gamma_{\nu,\mu}, \mathbb{C})$ are all of the same dimension $\left(\frac{\mu}{\pi}\right) \mathrm{vol}(\Lambda(\Gamma))$. An explicit basis is constructed in the next section.

5. Explicit basis of $\mathcal{F}^2_\ell(\Delta^\Gamma_{\nu,\mu}, \mathbb{C}^n)$ for high levels and higher dimensions

The differential operator $\Delta^\Gamma_{\nu,\mu}$ that we dealt with above and which is given by (1.1) is a particular case of the operators $\square^{\sigma_0}_\xi = 4\{\Delta + \xi E - \overline{\xi E} - |\xi z|^2\} + \sigma_0$, for given complex numbers ξ, σ_0 (not necessarily reals) with $\xi = \frac{\mu+i\nu}{2}$ and $\sigma_0 = 2i\nu n$. If M_ξ denotes the multiplication operator $M_\xi F = e^{\xi|z|^2} F$, then direct computation shows that $M_{\frac{\mu+i\nu}{2}} \Delta^\Gamma_{\nu,\mu} M_{-\frac{\mu+i\nu}{2}} = 4\Delta_\mu - 2\mu n$, so that the spectral theory of $\Delta^\Gamma_{\nu,\mu}$ on $L^2_{\nu,\mu}(\mathbb{C}^n/\Gamma)$ reduces further to the one of

$$\Delta^{\Gamma,\chi}_\mu := \sum_{j=1}^n \frac{\partial^2}{\partial z_j \partial \overline{z}_j} - \mu \overline{E}, \tag{5.1}$$

acting on the Hilbert space $L^{2,\mu}_{\Gamma,\chi_\nu}(\mathbb{C}^n) := L^2(\mathbb{C}^n/\Gamma; e^{-\mu|z|^2}dm)$ in [6] defined as the space of (Γ,χ_ν)-automorphic functions on $\mathbb{C}^n$ satisfying the pseudo-periodicity

$$g(z+\gamma) = \chi_\nu(\gamma) e^{\frac{\mu}{2}|\gamma|^2+\mu\langle z,\gamma\rangle} g(z); \quad z\in\mathbb{C}^n,\ \gamma\in\Gamma, \tag{5.2}$$

and subject to the norm boundedness

$$\int_{\Lambda(\Gamma)} |f(z)|^2 e^{-\mu|z|^2}\, dm(z) < +\infty.$$

Here the given mapping χ_ν is defined on the lattice Γ by $\chi_\nu(\gamma) := e^{\frac{i\nu}{2}|\gamma|^2}$. It takes values in the unit circle $\{\lambda\in\mathbb{C}; |\lambda|=1\} = U(1)$ and turns out to be a character on Γ.

More precisely, to any L^2-automorphic eigenfunction $\Delta^{\Gamma,\chi_\nu}_\mu f = E_\lambda f$ in $L^{2,\mu}_{\Gamma,\chi_\nu}(\mathbb{C}^n)$ corresponds a unique L^2-automorphic eigenfunction $F = e^{-\frac{\mu+i\nu}{2}|z|^2} f$ of the operator $\Delta^\Gamma_{\nu,\mu}$ belonging to $L^2_{\nu,\mu}(\mathbb{C}^n/\Gamma)$ with $\lambda = 4E_\lambda - 2\mu n$ as associated eigenvalue. This reproves the stability of the spectrum, since $\mathrm{Sp}(\Delta^{\Gamma,\chi_\nu}_\mu) = \{-\mu\ell; \ell=0,1,2,\dots\}$, see [6]. Hence

$$\mathrm{Sp}(\Delta^\Gamma_{\nu,\mu}) = 4\,\mathrm{Sp}(\Delta^{\Gamma,\chi_\nu}_\mu) - 2\mu n = \{-2\mu(2\ell+n); \ell=0,1,2,\dots\}.$$

Accordingly, to give the explicit basis of the L^2-eigenspaces $\mathcal{F}^2_\ell(\Delta^\Gamma_{\nu,\mu},\mathbb{C}^n)$ of the bi-weighted automorphic functions one might start by constructing an explicit basis of the L^2-eigenspaces of the invariant operator $\Delta^{\Gamma,\chi}_\mu$ in (5.1) acting on $L^{2,\mu}_{\Gamma,\chi_\nu}(\mathbb{C}^n)$.

5.1. Explicit basis of $\mathcal{E}^2_\ell(\Delta^{\Gamma,\chi}_\mu,\mathbb{C}^n) = \mathrm{Ker}(\Delta^{\Gamma,\chi}_\mu - \mu\ell)$ with given pseudo-character

Associated to these data (Γ,χ,μ), we define the functional space $\mathcal{O}^{2,\mu}_{\Gamma,\chi}(\mathbb{C}^n)$ to be the space of all L^2-holomorphic functions obeying the functional equation (5.2) and belonging to $L^{2,\mu}_{\Gamma,\chi}(\mathbb{C}^n)$. The non-triviality of $\mathcal{O}^{2,\mu}_{\Gamma,\chi}(\mathbb{C}^n)$ requires that

$$\chi(\gamma_1+\gamma_2) = \chi(\gamma_1)\chi(\gamma_2) e^{i\nu\Im\langle\gamma_1,\gamma_2\rangle} \tag{RDQ}$$

holds for every $\gamma_1,\gamma_2\in\Gamma$. Moreover, $\mathcal{O}^{2,\mu}_{\Gamma,\chi}(\mathbb{C}^n)$ is of finite dimension with [6]

$$\dim \mathcal{O}^{2,\mu}_{\Gamma,\chi}(\mathbb{C}^n) = \left(\frac{\mu}{\pi}\right)^n \mathrm{vol}(\Lambda(\Gamma)) = N_n.$$

In fact $\mathcal{O}^{2,\mu}_{\Gamma,\chi}(\mathbb{C}^n)$ appears here as the first L^2-eigenspace of the invariant operator $\Delta^{\Gamma,\chi}_\mu$ in (5.1) acting on $L^{2,\mu}_{\Gamma,\chi}(\mathbb{C}^n)$. The construction of the basis for $\mathcal{O}^{2,\mu}_{\Gamma,\chi}(\mathbb{C}^n)$ and $\mathcal{E}^2_\ell(\Delta^{\Gamma,\chi}_\mu,\mathbb{C}^n) = \mathrm{Ker}(\Delta^{\Gamma,\chi}_\mu - \mu\ell)$ is based on the fact that $\Delta^{\Gamma,\chi}_\mu$ can be factorized in terms of the first order operators $A_{-,j} = \partial_{\bar z_j}$ and their formal adjoint $A_{+,j} = (-\partial_{z_j} + \mu\bar z_j)$ with respect to the Gaussian measure. Namely, we have

$$\Delta^{\Gamma,\chi}_\mu = \sum_{j=1}^n A_{+,j}A_{-,j}.$$

We provide below an explicit basis of $\mathcal{E}^2_\ell(\Delta^{\Gamma,\chi}_\mu, \mathbb{C}^n)$ in terms of the modified Riemann theta function with characteristics

$$\vartheta_{a,b}(z|\tau) = \sum_{n\in\mathbb{Z}} \mathrm{e}^{i\pi\tau(n+a)^2+2i\pi(n+a)(z+b)} \tag{5.3}$$

and its derivatives. To this end, we begin by giving a concrete basis of $\mathcal{O}^{2,\mu}_{\Gamma,\chi}(\mathbb{C}^n)$.

Theorem 5.1. *For given reals α, β, we set*

$$\varphi^{\alpha,\beta}_k(z) := e^{\frac{\mu}{2}z^2+2i\pi(\alpha+m)z}\vartheta_{0,(\alpha+m)\tau-\beta}(Nz|N\tau).$$

Then, there exist some reals $\alpha_1,\beta_1,\dots,\alpha_n,\beta_n$ such that the functions φ_K; $K=(k_1,\dots,k_n) \in \{0,1,\dots,N-1\}^n$, defined by

$$\varphi_K(z) := \varphi^{\alpha_1,\beta_1}_{k_1}(z_1)\cdots\varphi^{\alpha_n,\beta_n}_{k_n}(z_n) \tag{5.4}$$

form an orthogonal basis of the functional space $\mathcal{O}^{2,\mu}_{\Gamma,\chi}(\mathbb{C}^n)$.

Proof. We first consider the special case of $n=1$ and $\Gamma_\tau = \mathbb{Z}+\tau\mathbb{Z}$, $\Im m\tau > 0$. Let α and β be two real numbers such that $\chi(1) = e^{2i\pi\alpha}$ and $\chi(\tau) = e^{2i\pi\beta}$, and set $P(z) = -\frac{\mu}{2}z^2 - 2i\pi\alpha z$. Thus, for every $f \in \mathcal{O}^{2,\mu}_{\Gamma_\tau,\chi}(\mathbb{C})$, the function $h_f(z) := e^{P(z)}f(z)$ satisfies $h_f(z+1) = h_f(z)$. Moreover, we have

$$h_f(z+\gamma) = \chi(\gamma)e^{-2i\pi\alpha\gamma}e^{-\mu\left(z+\frac{\gamma}{2}\right)(\gamma-\overline{\gamma})}h_f(z).$$

So that for the particular case $\gamma=\tau$, we get

$$h_f(z+\tau) = \chi(\tau)e^{-2i\pi\alpha\tau}e^{-\mu\left(z+\frac{\tau}{2}\right)(\tau-\overline{\tau})}h_f(z) = e^{2i\pi(\beta-\alpha\tau)}e^{-iN\pi(2z+\tau)}h_f(z), \tag{5.5}$$

where $N = \dim\mathcal{O}^{2,\mu}_{\Gamma_\tau,\chi}(\mathbb{C}) = \frac{\nu}{\pi}\operatorname{Im}(\tau)$. Hence, we can expand h_f as

$$h_f(z) = \sum_{k\in\mathbb{Z}} a_k e^{ik^*z} = \sum_{k\in\mathbb{Z}} a_k e^{2i\pi kz}$$

by means of [2, page 79]. Therefore,

$$f(z) = e^{\frac{\mu}{2}z^2+2i\pi\alpha z}\sum_{k\in\mathbb{Z}} a_k e^{2i\pi kz} = e^{\frac{\nu}{2}z^2}q_z^\alpha\sum_{k\in\mathbb{Z}} a_k q_z^k,$$

where q_z and q_τ stand for $q_z = e^{2i\pi z}$ and $q_\tau = e^{2i\pi\tau}$. Then, (5.5) reduces further to the following

$$h_f(z+\tau) = e^{2i\pi(\beta-\alpha\tau)}q_z^{-N}q_\tau^{-\frac{N}{2}}h_f(z)$$

which is clearly equivalent to

$$\sum_{k\in\mathbb{Z}} a_k q_z^k q_\tau^k = \sum_{k\in\mathbb{Z}}\left(a_{k+N}e^{2i\pi(\beta-\alpha\tau)}q_\tau^{-\frac{N}{2}}\right)q_z^k.$$

This yields $a_{k+N} = q_\tau^{k+\frac{N}{2}} e^{-2i\pi(\beta-\alpha\tau)} a_k$ by identification of power series in q_τ^k. Accordingly, f is completely determined by knowing $a_0, a_1, \dots, a_{N-1}$. The other coefficients are obtained by the recurrence formula

$$a_{m+kN} = q_\tau^{kj+k^2\frac{N}{2}} e^{-2i\pi k(\beta-\alpha\tau)} a_m.$$

Subsequently

$$f(z) = e^{\frac{\mu}{2}z^2} q_z^{\alpha_1} \sum_{k\in\mathbb{Z}} a_k q_z^k = \sum_{m=0}^{N-1} \left(e^{\frac{\nu}{2}z^2} q_z^{\alpha} \sum_{k\in\mathbb{Z}} a_{m+kN} q_z^{m+kN} \right) = \sum_{m=0}^{N-1} a_m \varphi_m^{\alpha,\beta}(z),$$

where we can reexpress $\varphi_m^{\alpha,\beta}(z)$; $m = 0, 1, \dots \leq N-1$, in terms of $\vartheta_{a,b}(\,\cdot\,|\,\cdot\,)$ in (5.3) as

$$\begin{aligned}
\varphi_m^{\alpha,\beta}(z) &= e^{\frac{\mu}{2}z^2} q_z^{\alpha} \sum_{k\in\mathbb{Z}} e^{-2i\pi k(\beta-\alpha\tau)} q_\tau^{mk+k^2\frac{N}{2}} q_z^{m+kN} \\
&= e^{\frac{\mu}{2}z^2+2i\pi(\alpha+m)z} \sum_{k\in\mathbb{Z}} e^{2i\pi[Nz+(\alpha+m)\tau-\beta]k} e^{i\pi k^2 N\tau} \\
&= e^{\frac{\mu}{2}z^2+2i\pi(\alpha+m)z} \vartheta_{0,(\alpha+m)\tau-\beta}(Nz|N\tau).
\end{aligned}$$

These functions are orthogonal in $L^{2,\mu}_{\Gamma_\tau,\chi}(\mathbb{C})$ and therefore form a basis of the functional space $\mathcal{O}^{2,\mu}_{\Gamma_\tau,\chi}(\mathbb{C})$.

The result for higher dimensions readily follows by taking the product of n copies of $\varphi_m^{\alpha,\beta}$, namely the functions $\varphi_J(z)$ in (5.4) form an orthogonal basis of $\mathcal{O}^{2,\mu}_{\Gamma,\chi}(\mathbb{C}^n)$. $\square$

The next result concerns the eigenspace $\mathcal{E}^2_\ell(\Delta_\mu^{\Gamma,\chi}, \mathbb{C}^n) = \mathrm{Ker}(\Delta_\mu^{\Gamma,\chi} - \mu\ell)$ for $\ell > 0$. To this end, we denote by H_k^σ; $\sigma > 0$, the rescaled complex Hermite polynomial defined by

$$H_k^\sigma(\xi) = (-1)^k e^{\sigma\xi^2} \frac{d^k}{d\xi^k} e^{-\sigma\xi^2}, \quad \xi \in \mathbb{C}.$$

Theorem 5.2. *Let φ_K be as in (5.4). Then, the functions $A_{+,J}\varphi_K := A_{+,j_1}A_{+,j_2}\cdots A_{+,j_l}\varphi_K$ for varying $1 \leq j_1 \leq \cdots \leq j_l \leq n$ and $K = (k_1, \dots, k_n) \in \{0, 1, \dots, N-1\}^n$ is an orthogonal basis of $\mathcal{E}^2_\ell(\Delta_\mu^{\Gamma,\chi}, \mathbb{C}^n)$ for any $\ell = 0, 1, 2, \dots,$ and $n = 1, 2, \dots.$ For the particular case of $n = 1$, they reduces further to*

$$\begin{aligned}
\varphi_{\ell,m}^{\alpha,\beta}(z) &:= (-1)^\ell e^{\frac{\mu}{2}z^2+2i\pi(\alpha+m)z} \\
&\quad \times \sum_{k=0}^{\ell} \binom{\ell}{k} i^{\ell-k} H_{\ell-k}^{\mu/2}\left(2\Im mz + \frac{2\pi}{\mu}(\alpha+m)\right) \partial_z^k \vartheta_{0,(\alpha\tau-\beta)+m\tau}(Nz|N\tau)
\end{aligned} \tag{5.6}$$

with varying $m = 0, 1, \dots, N-1$, and constitute an orthogonal basis of $\mathcal{E}^2_\ell(\Delta_\mu^{\Gamma,\chi}, \mathbb{C})$ for every ℓ.

The proof of Theorem 5.2 is contained in Lemmas 5.3, 5.4 and 5.6 below.

Lemma 5.3. *Fix $n = 1$ and let ℓ be a nonnegative integer and set $A_+^\ell = (-\partial_z + \mu\overline{z})^\ell$ and $A_-^\ell = \partial_{\overline{z}}^\ell$.*

(i) *The operators A_+^ℓ and A_-^ℓ leave invariant the space $\mathcal{C}^\infty_{\Gamma,\chi}(\mathbb{C})$ of $\mathcal{C}^\infty$ functions satisfying the functional equation in* (5.2)*.*

(ii) *We have $\Delta_\mu A_+^\ell = A_+^\ell(\Delta_\mu + \mu\ell)$. Moreover, the A_+ is injective from $\mathcal{E}^2_\ell(\Delta_\mu^{\Gamma,\chi}, \mathbb{C})$ into $\mathcal{E}^2_{\ell+1}(\Delta_\mu^{\Gamma,\chi}, \mathbb{C})$.*

Proof. The proof is straightforward. □

Subsequently, if f is a given eigenfunction of $\Delta_\mu^{\Gamma,\chi}$ in $L^{2,\mu}_{\Gamma,\chi}(\mathbb{C}^n)$ with λ as the corresponding eigenvalue, then $A_+^\ell f$ is again an eigenfunction of $\Delta_\mu^{\Gamma,\chi}$ but with $\lambda + \mu\ell$ as the associated eigenvalue. This is to say that for $n = 1$ the basis of the eigenspace $\mathcal{E}^2_\ell(\Delta_\mu^{\Gamma,\chi}, \mathbb{C})$ can be constructed from the ground states (the elements of $\mathcal{O}^{2,\mu}_{\Gamma,\chi}(\mathbb{C}) = \mathcal{E}^2_0(\Delta_\mu^{\Gamma,\chi})$) by applying A_+^ℓ, since in this case all the eigenspaces $\mathcal{E}^2_\ell(\Delta_\mu^{\Gamma,\chi}, \mathbb{C})$ have the same dimension. More precisely, we assert

Lemma 5.4. *For $n = 1$, the functions $A_+^\ell \varphi_m^{\alpha,\beta}$; $m = 0, 1, \dots \leq N-1$, form a basis of $\mathcal{E}^2_\ell(\Delta_\mu^{\Gamma,\chi}, \mathbb{C})$ for every ℓ. Moreover, they are given by* (5.6)*, i.e., $A_+^\ell \varphi_m^{\alpha,\beta}(z) =: \varphi_{\ell,m}^{\alpha,\beta}(z)$.*

Proof. We need only to prove the $\mathbb{R}$-independency of $A_+^\ell \varphi_m^{\alpha,\beta}$; $m = 0, 1, \dots \leq N-1$, which immediately follows since this family is orthogonal, indeed we have

$$\left\langle A_+^\ell \varphi_m^{\alpha,\beta}, A_+^\ell \varphi_{m'}^{\alpha,\beta} \right\rangle_{L^{2,\mu}_{\Gamma_\tau,\chi}(\mathbb{C})} = \mu^\ell \ell! \left\langle \varphi_m^{\alpha,\beta}, \varphi_{m'}^{\alpha,\beta} \right\rangle_{L^{2,\mu}_{\Gamma_\tau,\chi}(\mathbb{C})} = \mu^\ell \ell! \left\| \varphi_m^{\alpha,\beta} \right\|^2_{L^{2,\mu}_{\Gamma_\tau,\chi}(\mathbb{C})} \delta_{m,m'}.$$

The explicit expression of $A_+^\ell \varphi_m^{\alpha,\beta}$ can be handled by rewriting the operator A_+^ℓ as $A_+^\ell f = (-1)^\ell e^{\mu|z|^2} \partial_z^\ell (e^{-\mu|z|^2} f)$ and therefore

$$\begin{aligned} &A_+^\ell \varphi_m^{\alpha,\beta}(z) \\ &= e^{\frac{\nu}{2} z^2 + 2i\pi(\alpha+m)z} \sum_{k=0}^{\ell} (-1)^k \binom{\ell}{k} G^{\mu,\frac{\mu}{2}}_{\ell-k}(z, \overline{z} | 2i\pi(\alpha+m)) \partial_z^k \vartheta_{(0,(\alpha\tau-\beta)+m\tau)}(Nz|N\tau), \end{aligned}$$

where the polynomials

$$G_k^{\mu,\sigma}(z, \overline{z} | \xi) = (-1)^k e^{\mu|z|^2 - \sigma z^2 - \xi z} \partial_z^k e^{-\mu|z|^2 + \sigma z^2 + \xi z}; \quad \mu > 0, \ \sigma \neq 0, \ \xi \in \mathbb{C}$$

are the Intissar–Hermite polynomials introduced and studied in [3] and showed to can reexpressed as

$$G_k^{\mu,\sigma}(z, \overline{z} | \xi) = (-i)^k H_k^\sigma\left(\frac{2\sigma z - \mu\overline{z} + \xi}{2i\sigma} \right).$$

□

Remark 5.5. The functions involved in the right hand-side of (5.6) can be seen as a special kind of generalization of the Jacobi theta functions to the non-holomorphic setting for satisfying the functional equation (5.2) and that for $\ell = 0$ they reduce further to Jacobi theta function $\vartheta(Nz + (\alpha\tau - \beta) + m\tau|N\tau)$.

Lemma 5.6. *For $n \geq 2$, the set of functions $A_{+,j_1}A_{+,j_2}\cdots A_{+,j_l}\varphi_K$ for varying $1 \leq j_1 \leq \cdots \leq j_l \leq n$ and $K = (k_1, \dots, k_n) \in \{0, 1, \dots, N-1\}^n$ is an orthogonal basis of $\mathcal{E}^2_\ell(\Delta^{\Gamma,\chi}_\mu, \mathbb{C}^n)$.*

Proof. Notice first that Lemma 5.3 remains valid for high dimension and the assertions holds true if we replace there A_+ by $A_{+,j}$ and A_- by $A_{-,j}$. Thus, for every fixed $j = 1, \dots, n$, the map $A_{+,j} \colon \mathcal{E}^2_\ell(\Delta^{\Gamma,\chi}_\mu, \mathbb{C}^n) \longrightarrow \mathcal{E}^2_{\ell+1}(\Delta^{\Gamma,\chi}_\mu, \mathbb{C}^n)$ is injective and therefore the function

$$A_{+,J_\ell}\varphi := A_{+,j_1}A_{+,j_2}\cdots A_{+,j_l}\varphi$$

belongs to $\mathcal{E}^2_\ell(\Delta^{\Gamma,\chi}_\mu, \mathbb{C}^n)$ for every given $\varphi \in \mathcal{E}^2_0(\Delta^{\Gamma,\chi}_\mu, \mathbb{C}^n)$. Moreover, by means of Lemma 5.4 and the definition of φ_K (see (5.4)) and applying Fubini's theorem, it is clear that the family of functions $A_{+,J_\ell}\varphi_K$ for varying $J_\ell = (j_1, \dots, j_l) \in \{1, \dots, n\}^\ell$ and $K = (k_1, \dots, k_n) \in \{0, 1, \dots, N-1\}^n$ is orthogonal in $\mathcal{E}^2_\ell(\Delta^{\Gamma,\chi}_\mu, \mathbb{C}^n)$. Then the result of Lemma 5.6 follows since

$$\sharp\{A_{+,J_\ell}\varphi_K; J_\ell, K\} = \left[\left(\frac{\mu}{\pi}\right)^n \mathrm{vol}(\Lambda(\Gamma))\right] \times \frac{(n-1+\ell)!}{(n-1)!\ell!} = \dim \mathcal{E}^2_\ell(\Delta^{\Gamma,\chi}_\mu, \mathbb{C}^n),$$

where $\sharp$ denotes the cardinality of a set. $\square$

5.2. On the basis of $\mathcal{F}^2_\ell(\Delta^\Gamma_{\nu,\mu}, \mathbb{C}^n)$

The key observation in obtaining our basis for $\mathcal{F}^2_\ell(\Delta^\Gamma_{\nu,\mu}, \mathbb{C}^n)$ is that the multiplication operator

$$M^{\nu,\mu} \colon f \longmapsto e^{\frac{i\nu+\mu}{2}|z|^2} f$$

defines an isometric bijection from $L^2_{\nu,\mu}(\mathbb{C}^n/\Gamma)$, the L^2-space of bi-weighted automorphic functions, onto the space $L^{2,\mu}_{\Gamma,\chi_\nu}(\mathbb{C}^n)$ of simple (Γ,χ)-automorphic functions satisfying (5.2) and associated to the character $\chi_\nu(\gamma) := e^{\frac{i\nu}{2}|\gamma|^2}$; $\gamma \in \Gamma$.

According to the discussion in the beginning of this section, the basis for the eigenspaces of $\Delta^\Gamma_{\nu,\mu}$ are inherited from those of the eigenspaces of the Laplacian $\Delta^{\Gamma,\chi_\nu}_\mu$ acting on $L^2_{\nu,\mu}(\mathbb{C}^n/\Gamma)$. We formulate the result of the above discussion as follows.

Corollary 5.7. *Keep notations as above. The functions*

$$e^{-\frac{i\nu+\mu}{2}|z|^2} A_{+,J}\varphi_K = e^{-\frac{i\nu+\mu}{2}|z|^2} A_{+,j_1}A_{+,j_2}\cdots A_{+,j_l}\varphi_K$$

for varying $1 \leq j_1 \leq \cdots \leq j_l \leq n$ *and* $K = (k_1, \dots, k_n) \in \{0, 1, \dots, N-1\}^n$*, form an orthogonal basis of* $\mathcal{F}^2_\ell(\Delta^\Gamma_{\nu,\mu}, \mathbb{C}^n)$*. Here* φ_K *is the function in* (5.4)*. In particular, for* $n = 1$*, the functions* $e^{-\frac{i\nu+\mu}{2}|z|^2}\varphi^{\alpha,\beta}_{\ell,m}(z)$*,* $m = 0, 1, \dots, N-1$*, constitute an orthogonal basis of* $\mathcal{F}^2_\ell(\Delta^\Gamma_{\nu,\mu}, \mathbb{C})$*, where* $\varphi^{\alpha,\beta}_{\ell,m}$ *are the functions given by* (5.6)*.*

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