

Degree Bipartite Ramsey Numbers

Ye Wang, Yusheng Li and Yan Li*

Abstract. Let $H \xrightarrow{s} G$ denote that any edge-coloring of H by s colors contains a monochromatic G . The degree Ramsey number $r_{\Delta}(G; s)$ is defined to be $\min\{\Delta(H) : H \xrightarrow{s} G\}$, and the degree bipartite Ramsey number $br_{\Delta}(G; s)$ is defined to be $\min\{\Delta(H) : H \xrightarrow{s} G \text{ and } \chi(H) = 2\}$. In this note, we show that $r_{\Delta}(K_{m,n}; s)$ is linear on n with fixed m . We also evaluate $br_{\Delta}(G; s)$ for paths and other trees.

1. Introduction

Ramsey theory is a fascinating branch of combinatorics. There are many difficult open problems in this area. Ramsey theory can be viewed as a generalization of the Pigeonhole Principle. A typical result in Ramsey theory states that if some mathematical object is partitioned into finite many parts, then one of the parts must contain a sub-object of particular property. The smallest size of the large object to guarantee the property is called Ramsey number. For more, see Graham, Rothschild and Spencer [11].

For graphs G and H , let $H \xrightarrow{s} G$ denote that any edge-coloring of H by s colors contains a monochromatic G . The Ramsey number $r(G; s)$ is the smallest N such that $K_N \xrightarrow{s} G$. More generally, for any monotone graph parameter ρ , the ρ -Ramsey number is defined as

$$r_{\rho}(G; s) = \min\{\rho(H) : H \xrightarrow{s} G\}.$$

This generalizes the Ramsey number since $r_{\rho}(G; s) = r(G; s)$ if $\rho(H)$ denotes the order of H . When $\rho(H)$ denotes the size of H , it becomes the size Ramsey number $\widehat{r}(G; s)$, see [2, 3, 6, 7, 10, 18]. For cases $\rho(H)$ of being the clique number and the chromatic number of H , we refer the reader to [9, 16, 17] and [5, 21, 22], respectively.

The degree Ramsey number is defined as

$$r_{\Delta}(G; s) = \min\{\Delta(H) : H \xrightarrow{s} G\},$$

Received March 16, 2020; Accepted November 25, 2020.

Communicated by Sen-Peng Eu.

2020 *Mathematics Subject Classification*. 05C55.

Key words and phrases. Ramsey number, degree bipartite Ramsey number, bipartite graph.

This paper was supported in part by NSFC.

*Corresponding author.

where $\Delta(H)$ is the maximum degree of H . Kinnersley, Milans and West [14], and Jiang, Milans and West [12] obtained bounds for degree Ramsey numbers of trees and cycles. Kang and Perarnau [13] proved that $r_\Delta(C_4; s) = \Theta(s^2)$, and Tait [19] proved that $r_\Delta(C_6; s) = \Theta(s^{3/2})$ and $r_\Delta(C_{10}; s) = \Theta(s^{5/4})$.

In this note, we define the degree bipartite Ramsey number $br_\Delta(G; s)$ as

$$br_\Delta(G; s) = \min\{\Delta(H) : H \text{ is bipartite and } H \xrightarrow{s} G\}.$$

Obviously, for any bipartite graph G , we have

$$(1.1) \quad r_\Delta(G; s) \leq br_\Delta(G; s).$$

Note that (1.1) holds with equality for trees in Theorem 1.1. Now we consider the degree bipartite Ramsey numbers of trees, including stars, paths, and complete bipartite graphs. We show that $r_\Delta(K_{m,n}; s)$ is linear on n with m fixed.

Theorem 1.1. *If T is a tree in which one vertex has degree k and all others have degree at most $\lceil k/2 \rceil$, then*

$$br_\Delta(T; s) = s(k - 1) + 1.$$

Kinnersley, Milans and West [14] showed $r_\Delta(T; s) = s(k-1)+1$ for any tree T satisfying conditions in Theorem 1.1 with odd k , and thus the inequality in (1.1) is sharp. They also proved

$$(1.2) \quad r_\Delta(T; s) \leq 2s(\Delta(T) - 1)$$

for any tree T . We shall generalize (1.2) to the bipartite version. Hence if (1.2) holds with equality, then the inequality in Theorem 1.2 becomes an equality from (1.1).

Theorem 1.2. *If T is a tree, then*

$$br_\Delta(T; s) \leq 2s(\Delta(T) - 1).$$

The above bound is sharp since Alon, Ding, Oporowski and Vertigan [1] showed that $r_\Delta(P_n; s) = 2s$ for fixed s and large n , where P_n is a path on n vertices.

Let us have more notation. The Turán number of a graph G , denoted by $ex(N; G)$, is the maximum number of edges in a graph of order N that contains no G . For a bipartite graph G , the Zarankiewicz number $z(N; G)$ is defined [20] to be the maximum number of edges in a subgraph of $K_{N,N}$ that contains no G . For two positive functions $f(t)$ and $g(t)$, we write that $f(t) \leq O(g(t))$ or $g(t) \geq \Omega(f(t))$ if there exists a positive constant c so that $f(t) \leq cg(t)$ for large t , and $f(t) = \Theta(g(t))$ if $\Omega(g(t)) \leq f(t) \leq O(g(t))$.

We now turn to even cycles and complete bipartite graphs. The results $r_\Delta(C_{2m}; s) = \Theta(s^{1+1/(m-1)})$ in [13, 19] and the upper bound in (1.1) imply $br_\Delta(C_{2m}; s) \geq \Omega(s^{1+1/(m-1)})$

for cycles C_{2m} with $m = 2, 3, 5$ and $s \rightarrow \infty$. It is well known that $ex(2n; C_{2m}) \leq O(n^{1+1/m})$ shown by Bondy and Simonovits [4] for fixed m and $n \rightarrow \infty$. On the other hand, if the edges of $K_{n,n}$ are colored by s colors, then at least n^2/s edges are monochromatic. Therefore, if $n^2/s \geq \Omega(n^{1+1/m})$, equivalently $n \geq \Omega(s^{1+1/(m-1)})$, then $n^2/s \geq ex(2n; C_{2m})$ and $K_{n,n} \xrightarrow{s} C_{2m}$, hence $br_{\Delta}(C_{2m}; s) \leq O(s^{1+1/(m-1)})$ for $m \geq 2$ and $s \rightarrow \infty$. Combining with the lower bound as mentioned, we have

$$br_{\Delta}(C_{2m}; s) = \Theta(s^{1+1/(m-1)})$$

for $m = 2, 3, 5$ and $s \rightarrow \infty$.

The following result differs from the result in [19] which pointed out $r_{\Delta}(K_{m,n}; s) = \Theta(s^m)$ for fixed m and n with $n > (m-1)!$ and $s \rightarrow \infty$.

Theorem 1.3. *If m and s are fixed integers and $\epsilon > 0$, then*

$$e^{-2s^{(mn-1)/(m+n)}}n \leq r_{\Delta}(K_{m,n}; s) \leq br_{\Delta}(K_{m,n}; s) \leq (1 + \epsilon)s^m n$$

for all large n .

2. Proofs of main results

Lemma 2.1. *For any integers $n, s \geq 2$, $br_{\Delta}(K_{1,n}; s) = s(n-1) + 1$.*

Proof. Since $K_{1,s(n-1)+1} \xrightarrow{s} K_{1,n}$, we have $br_{\Delta}(K_{1,n}; s) \leq s(n-1)+1$. For the lower bound, for any bipartite graph H with maximum degree $s(n-1)$, let H' be an $s(n-1)$ -regular bipartite supergraph of H . By Hall's Theorem, H' decomposes into 1-factors. Taking each of s color classes to be the union of $n-1$ of these 1-factors yields an edge-coloring of H' by s colors with degree $n-1$ in each color at each vertex. \square

A classic result of Erdős and Sachs [8] for the existence of regular graphs with large degree and girth can be modified to yield a bipartite form easily.

Lemma 2.2. (see Erdős and Sachs [8]) *For any positive integers g and k , there is a k -regular bipartite graph with girth at least g .*

Proof of Theorem 1.1. For the lower bound, for any tree T in which one vertex has degree k , it is obtained that $K_{1,k} \subseteq T$ and $br_{\Delta}(K_{1,k}; s) \leq br_{\Delta}(T; s)$, so $br_{\Delta}(T; s) \geq s(k-1) + 1$ by Lemma 2.1.

For the upper bound, by Lemma 2.2, let H be a regular bipartite graph having degree $s(k-1) + 1$ and girth more than $|V(T)|$. In any edge-coloring of H by s colors, by the pigeonhole principle, some color class has average degree more than $k-1$, which yields a monochromatic bipartite subgraph H_1 with average degree more than $k-1$. If H_1 has a

subgraph with a vertex u of degree at most $r - 1$ with $r = \lceil k/2 \rceil$, as $k - 1 \geq 2(r - 1)$, graph $H_1 \setminus \{u\}$ has average degree more than $k - 1$. Thus there must be a subgraph H_2 in H with minimum degree at least r and average degree more than $k - 1$. Then H_2 also has a vertex of degree at least k , denoted by v . In such a graph H_2 , we can “grow” T from v by adding children. When we want to grow from a current leaf, it has $r - 1$ neighbors in H_2 that (by the girth condition) are not in the tree yet, and then we get the desired tree T , finishing the proof. \square

The following lemma is a well known fact, and we shall use it to prove Theorem 1.2. Here we sketch the proof. For graph H with average degree $d > 0$, when we delete the vertices of degrees less than $d/2$ repeatedly if any, then the resulting graphs have non-decreasing average degrees and minimum degrees.

Lemma 2.3. *For positive integers δ and d with $d \geq 2(\delta - 1)$, if graph H has average degree at least d , then H contains a subgraph with minimum degree at least δ and average degree at least d .*

Proof of Theorem 1.2. Let $r = \Delta(T)$. And we can construct a $2s(r - 1)$ -regular bipartite graph H with girth more than $|V(T)|$ which is known to be possible in various ways. See for instance [8] for constructing a $2s(r - 1)$ -regular graph G with girth more than $|V(T)|$, then the direct product $H = G \times K_2$ is a $2s(r - 1)$ -regular bipartite graph with girth $g(H) \geq g(G) \geq |V(T)|$.

Consider an edge-coloring of bipartite graph H by s colors, then by the pigeonhole principle, some color class has average degree at least $2(r - 1)$, which yields a monochromatic bipartite subgraph H_1 with average degree at least $2(r - 1)$. By Lemma 2.3, H_1 contains a subgraph H_2 with minimum degree at least r . First we choose a vertex from $V(H_2)$ as the root of tree and then “grow” T from this vertex by adding children. When we want to grow from the current leaf, it has $r - 1$ neighbors in H_2 that (by the girth condition) are not in the tree yet. Thus, we have the desired monochromatic tree T . \square

The following lemma appeared in [13] firstly, and then it was restated by Tait [19] in a more general way. Before stating it, we need some notations. For $v \in V(G)$, denote by $N_G(v)$ the set of all neighbors of v in G . For graphs G and H , a homomorphism ϕ from G to H is an edge preserving mapping from $V(G)$ to $V(H)$. A homomorphism from G to H is locally injective if $N_G(v)$ is mapped to $N_H(\phi(v))$ injectively for every $v \in V(G)$. A graph is L_G -free if it does not contain any graph in L_G as a subgraph, where L_G is the set of all graphs H such that there is a locally injective homomorphism from G to H .

To avoid confusion, let us clarify a decomposition of a graph G means a partition of the edge set of G in this note.

Lemma 2.4. [13,19] *Let G be a graph with at least one cycle and H a graph of maximum degree Δ . If K_N can be decomposed into $O(N^{1-\xi})$ L_G -free graphs for fixed $\xi > 0$, then H can be decomposed into $O(\Delta^{1-\xi})$ graphs which are G -free.*

Proof of Theorem 1.3. For the lower bound, we shall show that $r_\Delta(K_{m,n}; s) \geq e^{-2} s^{(mn-1)/(m+n)} n$, which is equivalent to showing that any graph of maximum degree Δ can be decomposed into $(e^2 \Delta/n)^{(m+n)/(mn-1)}$ graphs which are $K_{m,n}$ -free. By taking $\xi = 1 - (m+n)/(mn-1)$ and $N = e^{-2} s^{(mn-1)/(m+n)} n$, we have

$$(e^2 \Delta/n)^{(m+n)/(mn-1)} = O(\Delta^{1-\xi}).$$

By Lemma 2.4, it suffices to show that K_N can be decomposed into $(e^2 N/n)^{(m+n)/(mn-1)} = O(N^{1-\xi})$ graphs which are $K_{m,n}$ -free. Let us consider a random edge-coloring of K_N by s colors such that each edge is colored independently with probability $1/s$. Let p be the probability that there is a monochromatic $K_{m,n}$. Then $p \leq s \binom{N}{m+n} \binom{m+n}{m} / s^{mn}$. For $s = (e^2 N/n)^{(m+n)/(mn-1)}$, we have

$$\begin{aligned} p &\leq \left(\frac{eN}{m+n}\right)^{m+n} \left(\frac{e(m+n)}{m}\right)^m \left(\frac{1}{s}\right)^{mn-1} = \frac{e^{2m+n} N^{m+n}}{m^m (m+n)^n s^{mn-1}} \\ &= \frac{n^{m+n}}{m^m e^n (m+n)^n} = \frac{n^m}{m^m e^n} \left(1 - \frac{m}{m+n}\right)^n \leq \frac{n^m}{m^m e^n} e^{-mn/(m+n)} \\ &\leq \frac{n^m}{(\sqrt{em})^m e^n}, \end{aligned}$$

which implies $p < 1$ for finitely many n . Hence $K_N \not\rightarrow K_{m,n}$ and the desired lower bound follows from Lemma 2.4.

For the upper bound, let $M = br_\Delta(K_{m,n}; s) - 1$. Then there is an edge-coloring of $K_{M,M}$ by s colors that contains no monochromatic $K_{m,n}$. A well known argument of Kővári, Sós and Turán [15] shows that the Zarankiewicz number

$$z(M; K_{m,n}) \leq (n-1)^{1/m} M^{2-1/m} + (m-1)M.$$

Let $c = (1 + \epsilon)^{1/m} - 1 > 0$. Then we have

$$M^2 \leq s z(M; K_{m,n}) \leq (1 + c) s n^{1/m} M^{2-1/m},$$

which yields $M \leq (1 + \epsilon) s^m n$. □

Acknowledgments

We are grateful to the editors and reviewers for their invaluable comments and suggestions which improve the manuscript greatly.

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Ye Wang

College of Mathematical Sciences, Harbin Engineering University, Harbin 150001, China
E-mail address: wima.wy@gmail.com

Yusheng Li

School of Mathematical Sciences, Tongji University, Shanghai 200092, China
E-mail address: li_yusheng@tongji.edu.cn

Yan Li

School of Mathematical Sciences, Tongji University, Shanghai 200092, China
and
College of Science, University of Shanghai for Science and Technology, Shanghai 200093, China
E-mail address: 1610521@tongji.edu.cn