TAIWANESE JOURNAL OF MATHEMATICS
Vol. 24, No. 6, pp. 1383–1397, December 2020
DOI: 10.11650/tjm/200403

Maximal Density of Sets with Missing Differences and Various Coloring Parameters of Distance Graphs

Ram Krishna Pandey* and Neha Rai

Abstract. For a given set M of positive integers, a well-known problem of Motzkin asked to determine the maximal asymptotic density of M-sets, denoted by $\mu(M)$, where an M-set is a set of non-negative integers in which no two elements differ by an element in M. In 1973, Cantor and Gordon found $\mu(M)$ for $|M| \leq 2$. Partial results are known in the case $|M| \geq 3$ including results in the case when M is an infinite set. This number theory problem is also related to various types of coloring problems of the distance graphs generated by M. In particular, it is known that the reciprocal of the fractional chromatic number of the distance graph generated by M is equal to the value $\mu(M)$ when M is finite. Motivated by the families $M = \{a, b, a+b\}$ and $M = \{a, b, a+b, b-a\}$ discussed by Liu and Zhu, we study two families of sets M, namely, $M = \{a, b, b-a, n(a+b)\}$ and $M = \{a, b, a+b, n(b-a)\}$. For both of these families, we find some exact values and some bounds on $\mu(M)$. We also find bounds on the fractional and circular chromatic numbers of the distance graphs generated by these families. Furthermore, we determine the exact values of chromatic number of the distance graphs generated by these two families.

1. Introduction

For a given set M of positive integers, a problem of Motzkin asked to find the maximal upper density of sets S of non-negative integers in which no two elements of S are allowed to differ by an element of M. Following the question of Motzkin, if M is a given set of positive integers, a set S of non-negative integers is said to be an *M-set* if $a, b \in S$ implies $a - b \notin M$. For $x \in \mathbb{R}$ and a set S of non-negative integers, let $S(x)$ be the number of elements $n \in S$ such that $n \leq x$. We define the upper and lower densities of S, denoted respectively by $\overline{\delta}(S)$ and $\underline{\delta}(S)$, as follows:

$$\overline{\delta}(S) = \limsup_{x\to\infty} \frac{S(x)}{x}, \quad \underline{\delta}(S) = \liminf_{x\to\infty} \frac{S(x)}{x}.$$

Received December 24, 2019; Accepted April 16, 2020.
Communicated by Daphne Der-Fen Liu.
2010 *Mathematics Subject Classification*. Primary: 11B05; Secondary: 05C15.
Key words and phrases. asymptotic density, distance graph, fractional chromatic number, circular chromatic number.
This research was supported by the grant no. SB/FTP/MS-005/2013 by SERB, DST New Delhi, India.
*Corresponding author.

We say that S has density $\delta(S)$ when $\overline{\delta}(S) = \underline{\delta}(S) = \delta(S)$. The parameter of interest is the maximal density of an M-set, defined by

$$\mu(M) := \sup \overline{\delta}(S),$$

where the supremum is taken over all M-sets S. Motzkin [19] posed the problem of finding the quantity $\mu(M)$. In 1973, Cantor and Gordon [2] proved that there exists a set S such that $\delta(S) = \mu(M)$, when M is finite. The following two lemmas proved in [2] and [10], respectively, are useful for bounding $\mu(M)$.

Lemma 1.1. *Let $M = \{m_1, m_2, m_3, \ldots\}$ and c and m be positive integers such that $\gcd(c,m) = 1$. Then*

$$\mu(M) \geq \kappa(M) := \sup_{(c,m)=1} (1/m) \min_{k \geq 1} |cm_k|_m,$$

where for an integer x and a positive integer m, $|x|_m = |r|$ if $x \equiv r \pmod{m}$ with $0 \leq |r| \leq m/2$.

Lemma 1.2. *Let α be a real number, $\alpha \in [0,1]$. If for any M-set S with $0 \in S$ there exists a positive integer k such that $S(k) \leq (k+1)\alpha$, then $\mu(M) \leq \alpha$.*

For a finite set M, by a remark of Haralambis [10, Remark 1], we can write $\kappa(M)$ as

$$\kappa(M) = \max_{\substack{m=m_i+m_j \\ 1 \leq k \leq m/2}} (1/m) \min_i |km_i|_m, \tag{1.1}$$

where m_i, m_j are distinct elements of M.

Motzkin's maximal density problem is closely related to several coloring parameters of distance graphs generated by M. Moreover, the parameter $\kappa(M)$, which serves as a lower bound for $\mu(M)$, is related to the "lonely runner conjecture". The lonely runner conjecture is a long standing open conjecture on the diophantine approximations, which was first posed by Wills [27] and then independently by Cusick [6].

The study of Motzkin's density problem is equivalent to the study of the fractional chromatic number of distance graphs. A *fractional coloring* of a graph G is a mapping c which assigns to each independent set I of G a non-negative weight $c(I)$ such that for each vertex x, $\sum_{x \in I} c(I) \geq 1$. The *fractional chromatic number* of G, denoted by $\chi_f(G)$, is the least total weight of a fractional coloring of G.

Let M be a set of positive integers. The *distance graph* generated by M, denoted by $G(\mathbb{Z}, M)$, has the set $\mathbb{Z}$ of all integers as the vertex set, and two vertices x and y are adjacent whenever $|x-y| \in M$. It was proved by Chang et al. [3] that for any finite set M, the fractional chromatic number of the distance graph generated by M is the reciprocal of the maximal density of M-sets. Precisely, they proved the next theorem.

Theorem 1.3. *For any finite set M of positive integers, $\mu(M) = 1/\chi_f(G(\mathbb{Z}, M))$.*

The fractional chromatic number of a graph is related to another coloring parameter called the *circular chromatic number* defined as follows: Let $k \geq 2d$ be positive integers. A (k, d)-coloring of a graph G is a mapping, $c\colon V(G) \to \{0, 1, \ldots, k-1\}$, such that $d \leq |c(u) - c(v)| \leq k - d$ for any $uv \in E(G)$. The circular chromatic number of G, denoted by $\chi_c(G)$, is the minimum ratio k/d such that G admits a (k, d)-coloring. Zhu [28] proved that for any graph G,

$$\chi_f(G) \leq \chi_c(G) \leq \chi(G) = \lceil \chi_c(G) \rceil.$$

Moreover, for a distance graph $G(\mathbb{Z}, M)$, the following theorem connects the circular chromatic number of $G(\mathbb{Z}, M)$ with $\kappa(M)$.

Theorem 1.4. [29] *For any finite set M of positive integers, $\chi_c(G(\mathbb{Z}, M)) \leq \frac{1}{\kappa(M)}$.*

Notice that $\kappa(M)$ gives a lower bound for $\mu(M)$ and the reciprocal of $\kappa(M)$ gives an upper bound for $\chi_c(G(\mathbb{Z}, M))$.

The values and bounds of $\mu(M)$ for several special families of sets M (see [2, 3, 5, 7–10, 14–18, 20–22, 24, 25]) have been studied. But, in general, complete solutions are only known when $|M| \leq 2$ [2].

A set M is called *almost difference closed* if it holds that $\omega(G(\mathbb{Z}, M)) \geq |M|$, where $\omega(G)$ is the clique size of a graph G. Kemnitz and Marangio [12] characterized almost difference closed sets into three types as in the following result.

Theorem 1.5. *Let M be a finite set of positive integers with $|M| = m$ and $\gcd(M) = 1$. Then M is almost difference closed if and only if M is one of the following three sets:*

(i) $M = \{a, 2a, 3a, \ldots, (m-1)a, b\}$,

(ii) $M = \{a, b, a+b\}$,

(iii) $M = \{a, b, b-a, a+b\}$ *for some $b > a$.*

The chromatic number of the distance graphs generated by the three types of sets M in Theorem 1.5 were determined by several authors (see [12] for type (i), [4, 26] for type (ii) and [11, 13, 17] for type (iii)). The values of $\mu(M)$, $\kappa(M)$, $\chi_f(G(\mathbb{Z}, M))$, and $\chi_c(G(\mathbb{Z}, M))$ were determined by Liu and Zhu [17] except for a single case in type (iii), namely, when both a and b are odd, for which only the value of $\mu(M)$ was not determined but bounds were presented. These bounds are tight enough to compute the chromatic number of $G(\mathbb{Z}, M)$. We mention a theorem of Liu and Zhu [17, Theorem 3.1] which is applied at several places in this paper. This result (stated below) confirmed a conjecture of Rabinowitz and Proulx [23] in which one direction of the inequality was proved.

Theorem 1.6. *Suppose $M = \{x, y, x+y\}$, where $0 < x < y$ and $\gcd(x, y) = 1$. Then*

$$\mu(M) = \begin{cases} \frac{1}{3} & \text{if } y - x \equiv 0 \pmod 3, \\ \frac{2x+y-1}{3(2x+y)} & \text{if } y - x \equiv 1 \pmod 3, \\ \frac{x+2y-1}{3(x+2y)} & \text{if } y - x \equiv 2 \pmod 3. \end{cases}$$

In this article, we consider the two four-element families $M = \{a, b, b-a, n(a+b)\}$ and $M = \{a, b, a+b, n(b-a)\}$. For these two four-element families, we study the parameters $\mu(M)$, $\kappa(M)$, $\chi_f(G(\mathbb{Z}, M))$, $\chi_c(G(\mathbb{Z}, M))$, and the chromatic number $\chi(G(\mathbb{Z}, M))$. Some bounds and some exact values of these parameters are determined.

We let $\mathbb{N}$ denote the set of positive integers. Using definition (1.1) of $\kappa(M)$, we give lower bounds for $\kappa(M)$ for most of the sets in the families $M = \{a, b, b-a, n(a+b)\}$ and $M = \{a, b, a+b, n(b-a)\}$ in Sections 2 and 3, respectively. In Section 4, we investigate and compare $\chi(G(\mathbb{Z}, M))$, $\chi_f(G(\mathbb{Z}, M))$, and $\chi_c(G(\mathbb{Z}, M))$ for the families $M = \{a, b, b-a, n(a+b)\}$ and $M = \{a, b, a+b, n(b-a)\}$. In this investigation, we completely determine $\chi(G(\mathbb{Z}, M))$ for both families of sets M. Finally, in Section 5, we present some concluding remarks.

Since we have $\mu(M) = \mu(kM)$ for any positive integer k, it is sufficient to consider the case $\gcd(M) = 1$. Thereby, for both the families of sets M we assume $\gcd(a, b) = 1$.

2. The family $M = \{a, b, b-a, n(a+b)\}$

We find lower bounds for $\kappa(M)$, where $M = \{a, b, b-a, n(a+b)\}$. We divide the study according to the nature of $a + b \pmod 3$. Theorems 2.1, 2.3, and 2.5 give lower bounds for $\kappa(M)$ according as $a + b \equiv 0$ or 1 or $2 \pmod 3$, respectively. Theorem 2.1 holds for all $n \geq 1$ whereas, Theorems 2.3 and 2.5 hold for all but finitely many values of n.

Theorem 2.1. *Let $M = \{a, b, b-a, n(a+b)\}$, where $a < b$, $\gcd(a, b) = 1$ and $a + b \equiv 0 \pmod 3$. Then for $n \geq 1$,*

$$\kappa(M) \geq \begin{cases} \frac{n(a+b)}{3m} & \text{if } b > 2a, \text{ where } m = (b-a) + n(a+b), \\ \frac{n(a+b)}{3m} & \text{if } b < 2a, \text{ where } m = a + n(a+b). \end{cases}$$

Proof. To find the lower bound on $\kappa(M)$, we consider two cases, $b > 2a$ and $b < 2a$. In both cases, we consider two subcases, one each for $a \equiv 1 \pmod 3$ and $a \equiv 2 \pmod 3$. Notice that since $a + b \equiv 0 \pmod 3$ and $\gcd(a, b) = 1$, so the case $b = 2a$ and the subcase $a \equiv 0 \pmod 3$ are not possible.

Case (1): $b > 2a$. Let $m = (b-a) + n(a+b)$.

Subcase (i): Let $a \equiv 1 \pmod 3$. Let $x = (m-1)/3$. Then

$$ax \equiv \frac{b - 2a + n(a+b)}{3} \pmod m \quad \text{and} \quad n(a+b)x \equiv -\frac{n(a+b)}{3} \pmod m.$$

Hence,

$$(b-a)x \equiv -n(a+b)x \equiv \frac{n(a+b)}{3} \pmod{m} \quad \text{and} \quad bx \equiv -\frac{2b-a+n(a+b)}{3} \pmod{m}.$$

Now, using the fact that $b > 2a$, we see that

$$\frac{n(a+b)}{3} < \frac{b-2a+n(a+b)}{3} < \frac{2b-a+n(a+b)}{3} \leq \frac{m}{2}.$$

Therefore,

$$\min\{|ax|_m, |bx|_m, |(b-a)x|_m, |n(a+b)x|_m\} = \frac{n(a+b)}{3}. \tag{2.1}$$

Subcase (ii): Let $a \equiv 2 \pmod 3$. Let $x = (m+1)/3$. Then

$$ax \equiv -\frac{b-2a+n(a+b)}{3} \pmod{m} \quad \text{and} \quad n(a+b)x \equiv \frac{n(a+b)}{3} \pmod{m}.$$

Hence,

$$(b-a)x \equiv -n(a+b)x \equiv -\frac{n(a+b)}{3} \pmod{m} \quad \text{and} \quad bx \equiv \frac{2b-a+n(a+b)}{3} \pmod{m}.$$

Therefore,

$$\min\{|ax|_m, |bx|_m, |(b-a)x|_m, |n(a+b)x|_m\} = \frac{n(a+b)}{3}. \tag{2.2}$$

From (2.1) and (2.2) we get that if $b > 2a$, then

$$\kappa(M) \geq \frac{n(a+b)}{3m}.$$

Remark 2.2. Before we go to Case (2), we would like to notice here from Case (1) that to get a lower bound for $\kappa(M)$, which is actually calculated mostly in this paper, the key point of the proof is to find the appropriate x and m. Then find the minimum of $|xM|_m$, and then find the lower bound of $\kappa(M)$. To avoid similar repeated calculations, we construct tables presenting $|xM|_m$ corresponding to a given x and m under different conditions, from now onwards throughout the paper, wherever we calculate a lower bound for $\kappa(M)$.

Case (2): $b < 2a$. Let $m = a + n(a+b)$. Then, we have the following table.

	$a \equiv 1 \pmod 3, x = \frac{m-1}{3}$	$a \equiv 2 \pmod 3, x = \frac{m+1}{3}$
$\lvert ax\rvert_m = \lvert n(a+b)x\rvert_m$	$\frac{n(a+b)}{3}$	$\frac{n(a+b)}{3}$
$\lvert bx\rvert_m$	$\frac{(n+1)(a+b)}{3}$	$\frac{(n+1)(a+b)}{3}$
$\lvert (b-a)x\rvert_m$	$\frac{n(a+b)+2a-b}{3}$	$\frac{n(a+b)+2a-b}{3}$
$\min \lvert xM\rvert_m$	$\frac{n(a+b)}{3}$	$\frac{n(a+b)}{3}$

From the table, we see that $\kappa(M) \geq \frac{n(a+b)}{3m}$. This completes the proof. $\square$

Theorem 2.3. *Let $M = \{a, b, b-a, n(a+b)\}$, where $a < b$, $\gcd(a,b) = 1$, and $(a+b) \equiv 1 \pmod 3$. Then for $n \geq \frac{b-(2a+1)}{3}$,*

$$\kappa(M) \geq \frac{n(a+b-1)}{3(a+n(a+b))}.$$

Proof. Let $m = a + n(a+b)$ and $d = \gcd(a,m)$. Then $\gcd(a/d, m/d) = 1$. Let x be an integer such that

$$\frac{a}{d}x \equiv \frac{m/d - (a/d + n/d)}{3} \pmod{m/d}.$$

Then, we have the following table.

$\lvert ax\rvert_m = \lvert n(a+b)x\rvert_m$	$\frac{n(a+b-1)}{3}$
$\lvert bx\rvert_m$	$\frac{(n+1)(a+b-1)}{3}$
$\lvert(b-a)x\rvert_m$	$\frac{(n-1)(a+b-1)}{3} + a + n$
$\min \lvert xM\rvert_m$	$\frac{n(a+b-1)}{3}$

From the table, we see that $\kappa(M) \geq \frac{n(a+b-1)}{3(a+n(a+b))}$. This completes the proof. $\square$

Observation 2.4. Let a and b be positive integers with $a < b$, $a + b \equiv 2 \pmod 3$, and l be a non-negative integer. Set

$$N_1^{(l)} = \left\{ l(2b-a) + \left\lfloor \frac{a}{3} \right\rfloor + 1 + t_1 : 0 \leq t_1 \leq \frac{2b-a-7}{3} - \left\lfloor \frac{a}{3} \right\rfloor \right\},$$
$$N_2^{(l)} = \left\{ l(2b-a) + \frac{2b-a-1}{3} + t_2 : 0 \leq t_2 \leq \frac{2b-a-1}{3} \right\},$$
$$N_3^{(l)} = \left\{ l(2b-a) + \frac{2(2b-a)+1}{3} + t_3 : 0 \leq t_3 \leq \frac{2b-a-1}{3} + \left\lfloor \frac{a}{3} \right\rfloor \right\}.$$

Then $N_1^{(l)}$, $N_2^{(l)}$, $N_3^{(l)}$ are pairwise disjoint sets and $\bigcup_{l \geq 0} N_1^{(l)} \cup N_2^{(l)} \cup N_3^{(l)} = \mathbb{N} \setminus \{1, 2, \ldots, \lfloor a/3 \rfloor\}$.

Theorem 2.5. *Let $M = \{a, b, b-a, n(a+b)\}$, where $a < b$, $\gcd(a,b) = 1$, and $(a+b) \equiv 2 \pmod 3$. Set $m = b + n(a+b)$. Then for $n \geq \lfloor a/3 \rfloor + 1$,*

$$\kappa(M) \geq \begin{cases} \frac{m-b(3l+1)+n}{3m} & \text{if } n \in N_1^{(l)}, \\ \frac{m+(3l+1)(b-a)-2n-1}{3m} & \text{if } n \in N_2^{(l)} \cup N_3^{(l)}. \end{cases}$$

Proof. Let $\gcd(b,m) = d$. Then $\gcd(b/d, m/d) = 1$. Let x be an integer such that

$$\frac{b}{d}x \equiv \frac{m/d - (b/d + 3lb/d - n/d)}{3} \pmod{m/d}.$$

Then, we have the following table.

$\lvert bx\rvert_m = \lvert n(a+b)x\rvert_m$	$\frac{m-b(3l+1)+n}{3}$
$\lvert ax\rvert_m$	$\frac{m-b(3l+1)+n}{3} + l(a+b) + \frac{a+b+1}{3}$
$\lvert(b-a)x\rvert_m$	$\frac{m+(3l+1)(b-a)-2n-1}{3}$
$\min \lvert xM\rvert_m$ if $n \in N_1^{(l)}$	$\frac{m-b(3l+1)+n}{3}$
$\min \lvert xM\rvert_m$ if $n \in N_2^{(l)} \cup N_3^{(l)}$	$\frac{m+(3l+1)(b-a)-2n-1}{3}$

From the table, we see that if $n \in N_1^{(l)}$, then $\kappa(M) \geq \frac{m-b(3l+1)+n}{3m}$; and if $n \in N_2^{(l)} \cup N_3^{(l)}$, then $\kappa(M) \geq \frac{m+(3l+1)(b-a)-2n-1}{3m}$. This completes the proof. □

Corollary 2.6. *Let $M = \{a, b, b-a, n(a+b)\}$, where $a < b$, $\gcd(a,b) = 1$, $(a+b) \equiv 2 \pmod 3$, and $n \in N_2^{(l)}$. Then*

$$\mu(M) = \kappa(M) = \frac{2b-a-1}{3(2b-a)} \quad \textit{if } b > 2a$$

and

$$\frac{2b-a-1}{3(2b-a)} \leq \kappa(M) \leq \mu(M) \leq \frac{a+b-1}{3(a+b)} \quad \textit{if } b < 2a.$$

Proof. Let $m = 2b-a$. Then $m \equiv 1 \pmod 3$. Since $\gcd(b, 2b-a) = 1$, suppose that x is an integer such that $bx \equiv -(m-1)/3 \pmod m$. Then

$$(b-a)x \equiv -bx \equiv \frac{m-1}{3} \pmod m \quad \text{and} \quad ax = bx - (b-a)x \equiv \frac{m+2}{3} \pmod m.$$

We have $(a+b)x \equiv 1 \pmod m$. So $n(a+b)x \equiv n \equiv (m-1)/3 + t_2 \pmod m$. If $0 \leq t_2 \leq (m+2)/6$, then $(m-1)/3 + t_2 \leq m/2$. Let $(m+2)/6 \leq t_2 \leq (m-1)/3$. Then rewrite the congruence for $n(a+b)x$ as

$$n(a+b)x \equiv -\frac{2m+1}{3} + t_2 \pmod m.$$

Now we also have

$$\frac{m+2}{3} \leq \frac{2m+1}{3} - t_2 \leq \frac{m}{2}.$$

So,

$$\min\{\lvert ax\rvert_m, \lvert bx\rvert_m, \lvert(b-a)x\rvert_m, \lvert n(a+b)x\rvert_m\} = \frac{m-1}{3}.$$

Hence,

$$\mu(M) \geq \kappa(M) \geq \frac{m-1}{3m} = \frac{2b-a-1}{3(2b-a)}.$$

To get the upper bound for $\mu(M)$, we first let $b > 2a$. Setting $x = a$, $y = b - a$, we have $y - x \equiv 2 \pmod 3$. Hence, using Theorem 1.6, we get

$$\mu(M) \leq \mu(\{a, b-a, b\}) = \mu(\{x, y, x+y\}) = \frac{2y + x - 1}{3(2y + x)} = \frac{2b - a - 1}{3(2b - a)}.$$

Secondly, let $b < 2a$. Setting $x = b - a$, $y = a$, we have $y - x \equiv 2 \pmod 3$. Hence, again using Theorem 1.6, we get

$$\mu(M) \leq \mu(\{b-a, a, b\}) = \mu(\{x, y, x+y\}) = \frac{2y + x - 1}{3(2y + x)} = \frac{b + a - 1}{3(b + a)}.$$

This completes the proof of the corollary. □

3. The family $M = \{a, b, a+b, n(b-a)\}$

We find lower bounds for $\kappa(M)$, where $M = \{a, b, a+b, n(b-a)\}$. We divide the study according to the nature of $b - a \pmod 3$. Theorems 3.1, 3.3, and 3.5 give lower bounds for $\kappa(M)$ according as $b - a \equiv 0$ or 1 or $2 \pmod 3$, respectively. Theorems 3.1 and 3.5 hold for all $n \geq 1$ whereas, Theorem 3.3 holds for all but finitely many values of n.

Theorem 3.1. *Let $M = \{a, b, a+b, n(b-a)\}$, where $a < b$, $\gcd(a, b) = 1$, and $a \equiv b \pmod 3$. Then for $n \geq 1$,*

$$\kappa(M) \geq \frac{n(b-a)}{3(b + n(b-a))}.$$

Proof. Let $m = b + n(b-a)$. Then, we have the following table.

	$b \equiv 1 \pmod 3, x = \frac{m-1}{3}$	$b \equiv 2 \pmod 3, x = \frac{m+1}{3}$
$\lvert bx \rvert_m = \lvert n(b-a)x \rvert_m$	$\frac{n(b-a)}{3}$	$\frac{n(b-a)}{3}$
$\lvert (a+b)x \rvert_m$	$\frac{(m+a+b)}{3}$	$\frac{(m+a+b)}{3}$
$\lvert ax \rvert_m$	$\frac{m-a}{3}$	$\frac{m-a}{3}$
$\min \lvert xM \rvert_m$	$\frac{n(b-a)}{3}$	$\frac{n(b-a)}{3}$

From the table, we see that $\kappa(M) \geq \frac{n(b-a)}{3(b+n(b-a))}$. This completes the proof. □

Observation 3.2. Let a and b be positive integers with $a < b$, $b - a \equiv 1 \pmod 3$, and k be a non-negative integer. Set

$$N_1^{(k)} = \left\{ k(2a+b) + \frac{2a + b - 1}{3} + t_1 : 0 \leq t_1 \leq \frac{2a + b - 1}{3} \right\},$$
$$N_2^{(k)} = \left\{ k(2a+b) + \frac{4a + 2b + 1}{3} + t_2 : 0 \leq t_2 \leq \frac{4a + 2b - 5}{3} \right\}.$$

Then $N_1^{(k)}$, $N_2^{(k)}$ are pairwise disjoint sets and $\bigcup_{k\geq 0} N_1^{(k)} \cup N_2^{(k)} = \mathbb{N} \setminus \{1, 2, \ldots, (2a+b-4)/3\}$.

Theorem 3.3. *Let $M = \{a, b, a+b, n(b-a)\}$, where $a < b$, $\gcd(a, b) = 1$, and $(b-a) \equiv 1 \pmod 3$. Then for $n \geq (2a+b-1)/3$,*

$$\mu(M) = \kappa(M) = \frac{m-1}{3m} \quad \text{if } n \in N_1^{(k)}, \text{ where } m = 2a+b$$

and

$$\kappa(M) \geq \frac{m-(n-2a-3ak)}{3m} \quad \text{if } n \in N_2^{(k)}, \text{ where } m = a+n(b-a).$$

Proof. Case (i): $n \in N_1^{(k)}$. Let $m = 2a+b$. Then $m \equiv 1 \pmod 3$. Since $\gcd(a, m) = 1$, suppose that x is an integer such that $ax \equiv (m-1)/3 \pmod m$. Then

$$(a+b)x \equiv -ax \equiv -\frac{m-1}{3} \pmod m \quad \text{and} \quad bx = (a+b)x - ax \equiv \frac{m+2}{3} \pmod m.$$

Since $(b-a)x \equiv 1 \pmod m$, we have $n(b-a)x \equiv n \equiv (m-1)/3 + t_1 \pmod m$. From now on the proof is similar to that as in Corollary 2.6, so we omit the details. Thus, we get

$$\mu(M) \geq \kappa(M) \geq \frac{m-1}{3m}.$$

On the other hand, by Theorem 1.6, we have $\mu(M) \leq \mu(\{a, b, a+b\}) = \frac{2a+b-1}{3(2a+b)}$. Therefore,

$$\mu(M) = \kappa(M) = \frac{2a+b-1}{3(2a+b)} = \frac{m-1}{3m}.$$

Case (ii): $n \in N_2^{(k)}$. Let $m = a+n(b-a)$. Let $d = \gcd(a, m)$. Then $\gcd(a/d, m/d) = 1$. Let x be an integer such that

$$\frac{a}{d}x \equiv \frac{m/d - (n/d - 2a/d - 3ak/d)}{3} \pmod{m/d}.$$

Then, we have the following table.

$\lvert ax\rvert_m = \lvert n(b-a)x\rvert_m$	$\frac{m-(n-2a-3ak)}{3}$
$\lvert bx\rvert_m$	$\frac{m-(n-2a-3ak)}{3} + (k+1)(b-a) - \frac{b-a-1}{3}$
$\lvert (a+b)x\rvert_m$	$\frac{m-(n-2a-3ak)}{3} + n - k(2a+b) - \frac{4a+2b+1}{3}$
$\min \lvert xM\rvert_m$ if $n \in N_2^{(k)}$	$\frac{m-(n-2a-3ak)}{3}$

From the table, we see that if $n \in N_2^{(k)}$, then $\kappa(M) \geq \frac{m-(n-2a-3ak)}{3m}$. This completes the proof. $\square$

Observation 3.4. Let a and b be positive integers with $a < b$, $b - a \equiv 2 \pmod 3$, and k be a non-negative integer. Set

$$P_1^{(k)} = \left\{ k(a+2b) - \frac{2a+b-2}{3} + t_1 : 0 \le t_1 \le a+b-1 \right\},$$
$$P_2^{(k)} = \left\{ k(a+2b) + \frac{a+2b+2}{3} + t_2 : 0 \le t_2 \le \frac{a+2b-4}{3} \right\},$$
$$P_3^{(k)} = \left\{ k(a+2b) + \frac{2(a+2b)+1}{3} + t_3 : 0 \le t_3 \le \frac{b-a-2}{3} \right\}.$$

Then $P_1^{(k)}$, $P_2^{(k)}$, $P_3^{(k)}$ are pairwise disjoint sets and $\bigcup_{k \ge 0} P_1^{(k)} \cup P_2^{(k)} \cup P_3^{(k)} \supset \mathbb{N}$.

Theorem 3.5. *Let $M = \{a, b, a+b, n(b-a)\}$, where $a < b$, $\gcd(a,b) = 1$, and $(b-a) \equiv 2 \pmod 3$. Set $m = b + n(b-a)$. Then*

$$\kappa(M) \ge \begin{cases} \frac{m-(3k+1)b+n}{3m} & \text{if } n \in P_1^{(k)}, \\ \frac{m+(3k+1)(a+b)-(2n+1)}{3m} & \text{if } n \in P_2^{(k)} \cup P_3^{(k)}. \end{cases}$$

Proof. Let $d = \gcd(b, m)$. Then $\gcd(b/d, m/d) = 1$. Let x be an integer such that

$$\frac{b}{d} x \equiv \frac{m/d - (3kb/d + b/d - n/d)}{3} \pmod{m/d}.$$

Then, we have the following table.

$\lvert bx\rvert_m = \lvert n(b-a)x\rvert_m$	$\frac{m-(3bk+b-n)}{3}$
$\lvert ax\rvert_m$	$\frac{m-(3bk+b-n)}{3} + k(b-a) + \frac{b-a+1}{3}$
$\lvert (a+b)x\rvert_m$	$\frac{m-(3bk+b-n)}{3} - n + k(a+2b) + \frac{a+2b-1}{3}$
$\min \lvert xM\rvert_m$ if $n \in P_1^{(k)}$	$\frac{m-(3bk+b-n)}{3}$
$\min \lvert xM\rvert_m$ if $n \in P_2^{(k)} \cup P_3^{(k)}$	$\frac{m-(3bk+b-n)}{3} - n + k(a+2b) + \frac{a+2b-1}{3}$

From the table, we see that if $n \in P_1^{(k)}$, then $\kappa(M) \ge \frac{m-(3kb+b-n)}{3m}$; and if $n \in P_2^{(k)} \cup P_3^{(k)}$, then $\kappa(M) \ge \frac{m-(3kb+b-n)+(3k+1)(a+2b)-3n-1}{3m}$. This completes the proof. □

Corollary 3.6. *Let $M = \{a, b, a+b, n(b-a)\}$, where $a < b$, $\gcd(a,b) = 1$, and $(b-a) \equiv 2 \pmod 3$. Then for $n \in P_2^{(k)}$ and $m = a + 2b$, we have*

$$\mu(M) = \kappa(M) = \frac{m-1}{3m}.$$

Proof. Since $\gcd(b, m) = 1$, suppose that x is an integer such that

$$bx \equiv \frac{m-1}{3} \pmod m.$$

The rest of the congruences can be written exactly as in Corollary 2.6, and hence the proof similarly follows. □

4. Various coloring parameters of $G(\mathbb{Z}, M)$

In this section, we investigate and compare several coloring parameters of $G(\mathbb{Z}, M)$ for $M = \{a, b, a+b, n(b-a)\}$ and $M = \{a, b, b-a, n(a+b)\}$. For $n = 1$, Liu and Zhu [17, Corollary 5.2] have shown that if a and b are of distinct parity, then $\chi_f(G(\mathbb{Z}, M)) = \chi_c(G(\mathbb{Z}, M)) = \chi(G(\mathbb{Z}, M)) = 4$; and if a and b are both odd, then $\chi(G(\mathbb{Z}, M)) = 5$. For $n \geq 2$, Barajas and Serra [1, Theorem 3] proved that $\chi(G(\mathbb{Z}, M)) \leq 4$.

In the next lemma, we find $\chi(G(\mathbb{Z}, M))$ for both of these families for all $n \geq 2$. Consequently, we are able to completely determine the chromatic numbers of these two families (see Theorem 4.2). To make the discussion easier in the lemma, we change the problem into three families $M = \{a, b, a+b, n(b-a)\}$, $M = \{a, b, a+b, n(2a+b)\}$, and $M = \{a, b, a+b, n(a+2b)\}$. As $\{a, b, b-a\}$ can be written as $\{a, b', a+b'\}$ by letting $b' = b-a$. If $b > 2a$, then $n(a+b) = n(2a+b')$; if $a < b < 2a$, then $n(a+b) = n(a'+2b')$ by letting $b' = a$ and $a' = b-a$. The benefit of changing into these families is that we have now the first three elements same in all the three families, which helps in combining the discussion of $\chi(G(\mathbb{Z}, M))$. Now, we state and prove the lemma.

Lemma 4.1. *Let $M = \{a, b, a+b, n(b-a)\}$ or $M = \{a, b, a+b, n(2a+b)\}$ or $M = \{a, b, a+b, n(a+2b)\}$ with $a < b$, $\gcd(a, b) = 1$, and $n \geq 2$. Then $\chi(G(\mathbb{Z}, M)) = 4$.*

Proof. We consider two cases.

Case (i): $a \equiv b \pmod 3$. Using Theorem 1.6, we have $\mu(\{a, b, a+b\}) = 1/3$. Furthermore, since each of $b-a$, $2a+b$, and $a+2b$ is a multiple of 3 and none of a, b or $a+b$ is a multiple of 3, we have $\mu(\{a, b, a+b\})$ $(= 1/3) > \mu(M)$. Hence, we get

$$3 = \frac{1}{\mu(\{a, b, a+b\})} < \frac{1}{\mu(M)} = \chi_f(G(\mathbb{Z}, M)) \leq \chi_c(G(\mathbb{Z}, M)) \leq \chi(G(\mathbb{Z}, M)) \leq 4,$$

which implies $\chi(G(\mathbb{Z}, M)) = 4$.

Case (ii): $b - a \not\equiv 0 \pmod 3$. Using again Theorem 1.6, we have

$$3 < \frac{1}{\mu(\{a, b, a+b\})} \leq \frac{1}{\mu(M)} = \chi_f(G(\mathbb{Z}, M)) \leq \chi_c(G(\mathbb{Z}, M)) \leq \chi(G(\mathbb{Z}, M)) \leq 4,$$

which implies $\chi(G(\mathbb{Z}, M)) = 4$. $\square$

Using Lemma 4.1 and the corollary (for $n = 1$) of Liu and Zhu [17, Corollary 5.2], we obtain the complete solution of $\chi(G(\mathbb{Z}, M))$ for all $n \geq 1$.

Theorem 4.2. *Let $M = \{a, b, a+b, n(b-a)\}$ or $M = \{a, b, b-a, n(a+b)\}$ with $a < b$ and $\gcd(a, b) = 1$. Then*

$$\chi(G(\mathbb{Z}, M)) = \begin{cases} 5 & \text{if } n = 1 \text{ and } a \equiv b \equiv 1 \pmod 2, \\ 4 & \text{otherwise.} \end{cases}$$

In the following corollary, we observe that $\chi_c(G(\mathbb{Z}, M)) < \chi(G(\mathbb{Z}, M))$ for infinitely many sets in both the families.

Corollary 4.3. *If $M = \{a, b, a+b, n(b-a)\}$ or $M = \{a, b, b-a, n(a+b)\}$, then there are infinitely many sets of these families (i.e., for infinitely many n) with*

$$\chi_c(G(\mathbb{Z}, M)) < \chi(G(\mathbb{Z}, M)).$$

Proof. Applying Theorems 2.1, 2.3, and 2.5 for $M = \{a, b, a+b, n(b-a)\}$; and Theorems 3.1, 3.3, and 3.5 for $M = \{a, b, b-a, n(a+b)\}$, we observe that $1/\kappa(M) < \chi(G(\mathbb{Z}, M))$. Further, we always have $\chi_c(G(\mathbb{Z}, M)) \leq \min\{1/\kappa(M), \chi(G(\mathbb{Z}, M))\}$ for any distance graph $G(\mathbb{Z}, M)$. Hence, we get $\chi_c(G(\mathbb{Z}, M)) < \chi(G(\mathbb{Z}, M))$. $\square$

5. Conclusion

In this concluding remark, we mention the cases where we believe that the given lower bounds for $\kappa(M)$ are sharp.

Case	Condition for the given lower bound for $\kappa(M)$ to be sharp
Theorem 2.1	$b > 2a$
Theorem 2.3	$n > n_0$ for some $n_0 \geq \frac{2b-a-2}{3}$
Theorem 2.5	$n \in N_1^{(l)}$; $\kappa(M) = \frac{2b-a-1}{3(2b-a)}$ if $n \in N_2^{(l)}$ (proved for $b > 2a$ in Corollary 2.6)
Theorem 3.1	$n > n_0$ for some positive integer n_0
Theorem 3.3	$n \in N_1^{(k)}$
Theorem 3.5	$n \in P_1^{(k)}$ for $k > k_0$ for some positive integer k_0; $\kappa(M) = \frac{a+2b-1}{3(a+2b)}$ if $n \in P_2^{(k)}$ (proved in Corollary 3.6)

Further, we notice that these results for $\kappa(M)$ align with the known results when $n = 1$ [17] only in Theorem 2.1 for $b > 2a$.

Acknowledgments

The authors are very much thankful to the anonymous referees for their careful readings and extremely useful scientific comments that led us to considerably improve the paper.

The authors are also thankful to the anonymous referee (from the previous submission to a journal) for his/her meticulous review and useful suggestions to improve the paper.

References

[1] J. Barajas and O. Serra, *Distance graphs with maximum chromatic number*, Discrete Math. **308** (2008), no. 8, 1355–1365.

[2] D. G. Cantor and B. Gordon, *Sequences of integers with missing differences*, J. Combinatorial Theory Ser. A **14** (1973), 281–287.

[3] G. J. Chang, D. D.-F. Liu and X. Zhu, *Distance graphs and T-colorings*, J. Combin. Theory Ser. B **75** (1999), no. 2, 259–269.

[4] J.-J. Chen, G. J. Chang and K.-C. Huang, *Integral distance graphs*, J. Graph Theory **25** (1997), no. 4, 287–294.

[5] D. Collister and D. D.-F. Liu, *Study of $\kappa(D)$ for $D = \{2, 3, x, y\}$*, in: *Combinatorial Algorithms*, 250–261, Lecture Notes in Computer Science, Springer, Cham, 2015.

[6] T. W. Cusick, *View-obstruction problems in n-dimensional geometry*, J. Combinatorial Theory Ser. A **16** (1974), 1–11.

[7] J. R. Griggs and D. D.-F. Liu, *The channel assignment problem for mutually adjacent sites*, J. Combin. Theory Ser. A **68** (1994), no. 1, 169–183.

[8] S. Gupta, *Sets of integers with missing differences*, J. Combin. Theory Ser. A **89** (2000), no. 1, 55–69.

[9] S. Gupta and A. Tripathi, *Density of M-sets in arithmetic progression*, Acta Arith. **89** (1999), no. 3, 255–257.

[10] N. M. Haralambis, *Sets of integers with missing differences*, J. Combinatorial Theory Ser. A **23** (1977), no. 1, 22–33.

[11] A. Kemnitz and H. Kolberg, *Coloring of integer distance graphs*, Discrete Math. **191** (1998), no. 1-3, 113–123.

[12] A. Kemnitz and M. Marangio, *Chromatic numbers of integer distance graphs*, Discrete Math. **233** (2001), no. 1-3, 239–246.

[13] ———, *Colorings and list colorings of integer distance graphs*, Congr. Numer. **151** (2001), 75–84.

[14] D. D.-F. Liu, *From rainbow to the lonely runner: A survey on coloring parameters of distance graphs*, Taiwanese J. Math. **12** (2008), no. 4, 851–871.

[15] D. D.-F. Liu and G. Robinson, *Sequences of integers with three missing separations*, European J. Combin. **85** (2020), 103056, 11 pp.

[16] D. D.-F. Liu and A. Sutedja, *Chromatic number of distance graphs generated by the sets* $\{2, 3, x, y\}$, J. Comb. Optim. **25** (2013), no. 4, 680–693.

[17] D. D.-F. Liu and X. Zhu, *Fractional chromatic number and circular chromatic number for distance graphs with large clique size*, J. Graph Theory **47** (2004), no. 2, 129–146.

[18] ———, *Fractional chromatic number of distance graphs generated by two-interval sets*, European J. Combin. **29** (2008), no. 7, 1733–1743.

[19] T. S. Motzkin, *Unpublished problems collection.*

[20] R. K. Pandey and A. Tripathi, *A note on a problem of Motzkin regarding density of integral sets with missing differences*, J. Integer Seq. **14** (2011), no. 6, Article 11.6.3, 8 pp.

[21] ———, *On the density of integral sets with missing differences from sets related to arithmetic progressions*, J. Number Theory **131** (2011), no. 4, 634-647.

[22] ———, *A note on the density of M-sets in geometric sequence*, Ars Combin. **119** (2015), 221–224.

[23] J. H. Rabinowitz and V. K. Proulx, *An asymptotic approach to the channel assignment problem*, SIAM J. Algebraic Discrete Methods **6** (1985), no. 3, 507–518.

[24] A. Srivastava, R. K. Pandey and O. Prakash, *On the maximal density of integral sets whose differences avoiding the weighted Fibonacci numbers*, Integers **17** (2017), A48, 19 pp.

[25] ———, *Motzkin's maximal density and related chromatic numbers*, Unif. Distrib. Theory **13** (2018), no. 1, 27–45.

[26] M. Voigt, *Colouring of distance graphs*, Ars Combin. **52** (1999), 3–12.

[27] J. M. Wills, *Zwei Sätze über inhomogene diophantische Approximation von Irrationalzahlen*, Monatsh. Math. **71** (1967), 263–269.

[28] X. Zhu, *Circular chromatic number: A survey*, Discrete Math. **229** (2001), no. 1-3, 371–410.

[29] ———, *Circular chromatic number of distance graphs with distance sets of cardinality* 3, J. Graph Theory **41** (2002), no. 3, 195–207.

Ram Krishna Pandey and Neha Rai
Department of Mathematics, Indian Institute of Technology Roorkee–247667,
Uttarakhand, India
E-mail address: ram.pandey@ma.iitr.ac.in, neha.rai0184@gmail.com