

Topological Sensitivity Analysis and Kohn-Vogelius Formulation for Detecting a Rigid Inclusion in an Elastic Body

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Abstract. Our main interest in this work is to detect a rigid inclusion immersed in an isotropic elastic body Ω from a single pair of Cauchy data on $\partial\Omega$ in two dimensions. We want to completely characterize the unknown rigid inclusion, namely, the shape and the location of inclusion. The idea is to rewrite the inverse problem as an optimization problem, where an energy like functional is minimized with respect to the presence of a small inclusion. A topological sensitivity analysis is derived for an energy like functional. We proposed a non-iterative reconstruction algorithm based on the topological gradient concept. The unknown rigid inclusion is defined by a level curve of a scalar function. The proposed numerical approach is very robust with respect to noisy data. Finally, in order to show the efficiency and accuracy of the proposed algorithm, we present some numerical results.

1. Introduction

The detection of inclusions and/or cavities using the over-determined boundary data is a classical inverse problem that arises in nondestructive testing for damage assessment of mechanical specimens which are possible defective due to the presence of interior void (rigid inclusions and cavities) induced during the manufacturing process. There have been numerous methods for void detection [3, 8, 10, 15, 20, 21, 37, 41, 43, 44, 52].

In this paper, we address the problem of nondestructing testing: to reconstruct the location and shape of a rigid inclusion (defined by a Dirichlet boundary condition) immersed in an elastic body Ω (represented by a bounded domain in \mathbb{R}^2) by applying a traction field at the boundary $\partial\Omega$ and by measuring the induced displacement field on the boundary $\partial\Omega$. The displacement field satisfies the elasticity framework. This inverse problem is known to be severely ill-posed and nonlinear. Alves and Martins in [9] solved this inverse problem and proposed an approach based on the method of fundamental solutions (MFS). They established a numerical scheme that connects the MFS. While Karageorghis et al. [39, 40] proposed a regularization approach relies on the minimization of a nonlinear least-squares

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functional, penalized with respect to both the MFS and the derivative of the radial polar coordinates describing the position of the rigid inclusions or cavities. Then Shifrin and Shushpannikov in [55] determined a spheroidal cavity or rigid inclusion in an elastic solid by applying an analytic approach based on the reciprocity gap functional (RGF). Moreover, the reconstruction of inclusion can also be analyzed in a different framework in terms of a change in the elastic material properties [4, 48, 50].

Here, to reconstruct a rigid inclusion inside an elastic body, we propose an alternative reconstruction approach combining the advantages of the Kohn-Vogelius formulation [42] and the topological sensitivity analysis method [51]. The idea is to reformulate the inverse problem into a shape optimization one, where the rigid inclusion is the unknown variable. The method relies on the minimization of the so-called Kohn-Vogelius type functional, which measures the difference between the solutions of two auxiliary problems. To minimize this functional, we apply the topological sensitivity analysis. The main advantage of this reconstruction method is that it provides fast and accurate results for detection.

The main contribution of this paper concerns the theoretical and numerical aspects. In the theoretical part, we have derived a topological sensitivity analysis for the linear elasticity problem with respect to the insertion of a small inclusion in the domain Ω with a Dirichlet condition on the boundary of the inclusion. The obtained results are based on a rigorous and simplified mathematical analysis valid for a large class of shape functions. This method allows us to perform the topological sensitivity analysis without using the truncation technique proposed in [29]. In the numerical part, we propose a non-iterative algorithm for reconstructing an inclusion. The efficiency of the proposed algorithm is illustrated by some numerical examples. Particularly, we test the influence of some parameters in our algorithm such as the shape, location, size and the number of the inclusions.

This paper is summarized and rounded up as follows. Section 2 contains some notation and presents the direct and inverse problems. The proposed approach to solve the considered inverse problem is described in Section 3. In Section 4, we derive a topological asymptotic analysis with respect to the creation of a small rigid inclusion. Then, we propose a non-iterative algorithm. The efficiency and accuracy of the proposed algorithm are illustrated by some numerical results provided in Section 5. The sixth section is devoted to a conclusion, whereas the closing section records some definitions and preliminary results.

2. The problem setting

2.1. Notations

Let us introduce some notations which will be useful in what follows. For an open and bounded domain $\Omega \subset \mathbb{R}^2$, we denote by $W^{m,p}(\Omega)$ and $H^s(\Omega) := [H^s(\Omega)]^2$ the usual

Lebesgue and Sobolev spaces. Moreover, we denote by $\|\cdot\|_{1,\Omega}$ the norm $\|\cdot\|_{H^1(\Omega)}$ and $|\cdot|_{1,\Omega}$ the semi-norm $\|\cdot\|_{H^1(\Omega)}$ defined in (7.1). We also denote by $\|\cdot\|_{1/2,\partial\Omega}$ the norm $\|\cdot\|_{H^{1/2}(\partial\Omega)}$ and $\|\cdot\|_{-1/2,\partial\Omega}$ the norm $\|\cdot\|_{H^{-1/2}(\partial\Omega)}$. We represent the duality product between $H^{-1/2}(\partial\Omega)$ and $H^{1/2}(\partial\Omega)$ by the notation $\langle \cdot, \cdot \rangle_{-1/2,1/2,\partial\Omega}$. Finally, we define an inner product for matrices by $M : N = \sum_{i,j=1}^2 M_{ij}N_{i,j}$ for $M, N \in \mathbb{R}^{2 \times 2}$; the associated norm is $|M| = \sqrt{M : M}$. The corresponding inner product on $L^2(\Omega)^{2 \times 2}$ is

$$\langle M, N \rangle_{L^2(\Omega)^{2 \times 2}} = \int_{\Omega} M(x) : N(x) \, dx \quad \text{for } M, N \in L^2(\Omega)^{2 \times 2}.$$

2.2. The direct problem

Let Ω be an open and bounded domain of \mathbb{R}^2 with smooth boundary $\partial\Omega$, occupied by a linear elastic material. Inside the domain Ω , we assume the existence of a simply connected subdomain $\mathcal{B}^* \subset \Omega$ (rigid inclusion) with boundary $\partial\mathcal{B}^*$ of class C^1 such that $\text{dist}(\partial\mathcal{B}^*, \partial\Omega) > 0$ (see Figure 2.1).

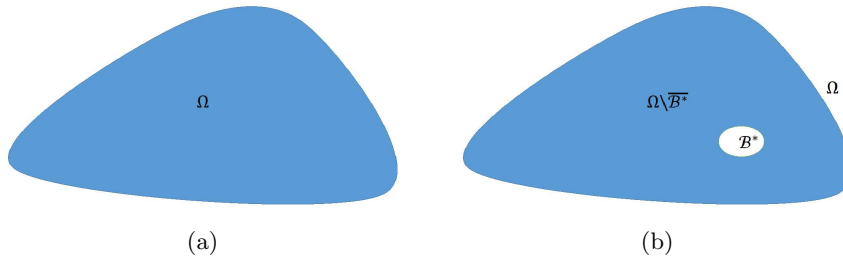


Figure 2.1: (a) Domain Ω without inclusion and (b) Domain Ω with inclusion \mathcal{B}^* .

The forward linear elastic problem is, therefore, given by

$$(2.1) \quad \begin{cases} -\text{div } \sigma(\psi) = 0 & \text{in } \Omega \setminus \overline{\mathcal{B}^*}, \\ \sigma(\psi)\mathbf{n} = \varphi & \text{on } \partial\Omega, \\ \psi = 0 & \text{on } \partial\mathcal{B}^*, \end{cases}$$

where $\varphi \neq 0$ is a given traction acting on the boundary $\partial\Omega$ and \mathbf{n} denotes the outward unit normal vector to the boundary $\partial\Omega$. The vector ψ denotes the displacement field, $\sigma = (\sigma_{ij})_{1 \leq i,j \leq 2}$ is the associated Cauchy stress tensor and $e = (e_{ij})_{1 \leq i,j \leq 2}$ is the linearized strain given by

$$e_{ij}(\psi) = \frac{1}{2} \left(\frac{\partial\psi_i}{\partial x_j} + \frac{\partial\psi_j}{\partial x_i} \right), \quad 1 \leq i, j \leq 2.$$

Note that σ and e are related by the Hooke constitutive law [26], the medium being

assumed to be homogeneous and isotropic. Therefore,

$$\sigma_{ij}(\psi) = 2\mu e_{ij}(\psi) + \lambda \delta_{ij} \sum_{k=1}^2 e_{kk}(\psi), \quad 1 \leq i, j \leq 2.$$

Above, δ_{ij} is the Kronecker symbol and μ, λ are the Lamé coefficients related to Young’s modulus E and the Poisson’s ratio ν via

$$\mu = \frac{E}{2(1 + \nu)} \quad \text{and} \quad \lambda = \frac{E\nu}{(1 + \nu)(1 - 2\nu)}.$$

For the physical meaning of these parameters, see [38].

2.3. The inverse problem

We define the class of admissible geometries

$$V_{\text{ad}} := \left\{ \mathcal{B} \subset\subset \Omega : \mathcal{B} \text{ is a simply connected open set, } \partial\mathcal{B} \text{ is of class } C^1, \right. \\ \left. \text{dist}(x, \partial\Omega) \geq \rho_0 > 0, \forall x \in \mathcal{B} \text{ and } \Omega \setminus \overline{\mathcal{B}} \text{ is connected} \right\}$$

with ρ_0 being a small fixed parameter.

In this work, we focus on the reconstruction of the location and the shape of an unknown rigid inclusion $\mathcal{B}^* \in V_{\text{ad}}$ immersed in the elastic body Ω from a single pair of displacement and surface traction data on the boundary of Ω . Therefore, the inverse problem that we consider here can be formulated as follows:

- Assume having an over-determined data on $\partial\Omega$ such as
 - $\varphi \in H^{-1/2}(\partial\Omega)$ is a given traction field:

$$\sigma(\psi)\mathbf{n} = \varphi \quad \text{on } \partial\Omega.$$

- $\mathcal{U}_d \in H^{1/2}(\partial\Omega)$ is a measured displacement field:

$$\psi = \mathcal{U}_d \quad \text{on } \partial\Omega.$$

- Reconstruct the unknown rigid inclusion $\mathcal{B}^* \in V_{\text{ad}}$ such that the displacement field ψ in the presence of the rigid inclusion \mathcal{B}^* satisfies the following over-determined boundary value problem

$$(2.2) \quad \begin{cases} -\operatorname{div} \sigma(\psi) = 0 & \text{in } \Omega \setminus \overline{\mathcal{B}^*}, \\ \sigma(\psi)\mathbf{n} = \varphi & \text{on } \partial\Omega, \\ \psi = \mathcal{U}_d & \text{on } \partial\Omega, \\ \psi = 0 & \text{on } \partial\mathcal{B}^*. \end{cases}$$

The data \mathcal{U}_d for which this problem has a solution ψ is said to be *compatible*.

The theoretical aspect of the over-determined problem (2.2) has been the subject of various researcher’s works. Particularly, one can consult [9, 49] for identifiability and stability results. Alves and Martins in [9] discussed the question of the identification of inclusions/cavities in an elastic body, using a single boundary measurement. Therefore, we have the following uniqueness theorem from [9, Theorem 3]:

Theorem 2.1 (Uniqueness). *Let $\overline{\mathcal{B}^*} \subset \Omega \subset \mathbb{R}^2$ be bounded and simply connected domains with \mathcal{C}^1 -smooth boundaries such that the domain $\Omega \setminus \overline{\mathcal{B}^*}$ is connected. Let also the Neumann and Dirichlet data $\varphi \in H^{-1/2}(\partial\Omega)$ and $\mathcal{U}_d \in H^{1/2}(\partial\Omega)$. If $\mathcal{U}_d \not\equiv 0$ then a single pair of Cauchy data (φ, \mathcal{U}_d) determines uniquely the displacement field $\psi^* \in H^1(\Omega \setminus \overline{\mathcal{B}^*})$ and the rigid inclusion \mathcal{B}^* satisfying the inverse problem is given by (2.2).*

On the other hand, Morassi and Rosset in [49] established the uniqueness and a conditional stability estimate of log-log type in determining a rigid inclusion inside an isotropic elastic body, we have the following stability theorem from [49, Theorem 2.5]:

Theorem 2.2 (Stability). *Let $\mathcal{B}_i, i = 1, 2$, be two open subsets in the class of admissible geometries V_{ad} . Moreover, let Σ be an open portion of $\partial\Omega$. Let $\psi_i \in H^1(\Omega \setminus \overline{\mathcal{B}_i})$ be the solution to (2.1), and let $\varphi \in H^{1/2}(\partial\Omega)$ when $\mathcal{B} = \mathcal{B}_i, i = 1, 2$. If for a given $\varepsilon > 0$ we have*

$$\min_{r \in \mathcal{R}} \|(\psi_1 - \psi_2) - r\|_{L^2(\Sigma)} \leq \rho_0^{1/2} \varepsilon,$$

then

$$d_{\mathcal{H}}(\partial\mathcal{B}_1, \partial\mathcal{B}_2) \leq \rho_0 \omega \left(\frac{\varepsilon}{\rho_0^{1/2} \|\varphi\|_{H^{-1/2}(\partial\Omega)}} \right),$$

where \mathcal{R} denotes the linear space of the infinitesimal rigid displacements and ω is an increasing continuous function on $[0, \infty)$ which satisfies

$$\omega(t) \leq C(\log |\log t|)^{-\iota} \quad \text{for all } t, 0 < t < e^{-1},$$

and $C, \iota, C > 0, 0 < \iota \leq 1$, are constants only depending on the a priori data.

From the above Theorems 2.1 and 2.2, we deduce that the solution of the inverse problem (2.2) is unique and stable.

In order to solve the reconstruction problem (2.2), we rephrase the inverse problem under consideration into an optimal design one. We propose a reconstruction approach based on the Kohn-Vogelius formulation and the topological sensitivity analysis method.

3. The reconstruction method

Our reconstruction approach is based on two main steps. The first one consists in determining the location and the shape of the unknown rigid inclusion \mathcal{B}^* using the Kohn-Vogelius formulation. The idea is to minimize a misfit functional between the solutions of two forward problems. One of them is associated to the Neumann data φ whereas the other one contains information on the Dirichlet data \mathcal{U}_d . In the second part, to minimize this misfit functional, we resort to the topological sensitivity analysis.

3.1. The Kohn-Vogelius formulation

The Kohn-Vogelius formulation is a self regularization technique and reformulate the inverse problem into a shape optimization one. For an admissible rigid inclusion $\mathcal{B} \in V_{ad}$; the Kohn-Vogelius formulation consists in splitting the over-determined boundary value problem (2.2) in two auxiliary problems. The first one is called the *Neumann problem* and associated to the traction field φ :

$$(\mathcal{P}_N) \quad \text{Find } \psi^N \in H^1(\Omega \setminus \overline{\mathcal{B}}) \text{ solving } \begin{cases} -\operatorname{div} \sigma(\psi^N) = 0 & \text{in } \Omega \setminus \overline{\mathcal{B}}, \\ \sigma(\psi^N) \mathbf{n} = \varphi & \text{on } \partial\Omega, \\ \psi^N = 0 & \text{on } \partial\mathcal{B}. \end{cases}$$

The second one is associated to the measured displacement \mathcal{U}_d :

$$(\mathcal{P}_D) \quad \text{Find } \psi^D \in H^1(\Omega \setminus \overline{\mathcal{B}}) \text{ solving } \begin{cases} -\operatorname{div} \sigma(\psi^D) = 0 & \text{in } \Omega \setminus \overline{\mathcal{B}}, \\ \psi^D = \mathcal{U}_d & \text{on } \partial\Omega, \\ \psi^D = 0 & \text{on } \partial\mathcal{B}. \end{cases}$$

For some results concerning the existence and uniqueness of solutions of problems (\mathcal{P}_N) and (\mathcal{P}_D) , we refer to [16].

We remark here that if the admissible rigid inclusion \mathcal{B} coincides with the actual one \mathcal{B}^* then the misfit between the solutions vanishes $\psi^N = \psi^D$ in $\Omega \setminus \overline{\mathcal{B}^*}$. According to this observation, the given inverse problem can be formulated as a shape optimization one where the unknown inclusion \mathcal{B}^* can be characterized as a solution to the following shape optimization problem:

$$(\mathcal{P}_{op}) \quad \text{Find } \mathcal{B}^* \subset \Omega \text{ such that } \mathcal{H}(\Omega \setminus \overline{\mathcal{B}^*}) = \min_{\mathcal{B} \in V_{ad}} \mathcal{H}(\Omega \setminus \overline{\mathcal{B}}),$$

where \mathcal{H} is a shape function defined as

$$\mathcal{H}(\Omega \setminus \overline{\mathcal{B}}) = \mathcal{J}(\psi^N, \psi^D)$$

with \mathcal{J} a given cost functional defined on $H^1(\Omega \setminus \overline{\mathcal{B}})$, measuring the difference between the Dirichlet and Neumann problem solutions.

In this work, we will use the following Kohn-Vogelius type cost functional,

$$\mathcal{J}(\psi^N, \psi^D) = \int_{\Omega \setminus \overline{\mathcal{B}}} \sigma(\psi^N - \psi^D) : e(\psi^N - \psi^D) \, dx, \quad \forall \mathcal{B} \in V_{\text{ad}}.$$

This function is positive and vanishes only if $\psi^N = \psi^D$, which is the case when the parameter \mathcal{B} fits the inclusion. We can notice that, integrating by parts the expression of $\mathcal{J}(\psi^N, \psi^D)$, we have:

$$\mathcal{J}(\psi^N, \psi^D) = \int_{\partial\Omega} (\psi^N - \mathcal{U}_d) \cdot (\varphi - \sigma(\psi^D)\mathbf{n}) \, ds.$$

The above equality give a boundary expression of \mathcal{K} .

This kind of cost function was originally introduced by Wexler et al. in [57] proposed a procedure to reconstruct the unknown impedance from the knowledge of solution on the boundary of the domain. Then Khon and Vogelius in [42] suggested a modification of Wexler’s procedure to make it an alternating direction one by proposing a new misfit cost functional. Since then, this formulation has been proved to be extremely useful in the treatment of various inverse problems [13, 18, 22, 28, 35], and so on.

Regarding the minimizer of $(\mathcal{P}_{\text{op}})$, we now show the following uniqueness result.

Proposition 3.1. *Let $(\varphi, \mathcal{U}_d) \in H^{-1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)$ be given nontrivial Cauchy data. We assume that there exists $\mathcal{B}^* \in V_{\text{ad}}$ such that the problem (2.2) has a solution then \mathcal{B}^* is the unique minimum of the shape functional \mathcal{K} :*

$$\mathcal{B}^* = \arg \min_{\mathcal{B} \in V_{\text{ad}}} \mathcal{K}(\Omega \setminus \overline{\mathcal{B}}).$$

Proof. If $\mathcal{B}^* \in V_{\text{ad}}$ is the solution of the inverse problem (2.2), then $\psi^N(\mathcal{B}^*) = \psi^D(\mathcal{B}^*)$ and \mathcal{B}^* is the minimum of the functional \mathcal{K} with $\mathcal{K}(\Omega \setminus \overline{\mathcal{B}^*}) = 0$. Let $\mathcal{B} \in V_{\text{ad}}$ be another minimizer of \mathcal{K} . Then $\psi^N(\mathcal{B})$ verifies the problem (2.1) with $\psi^N(\mathcal{B}) = \mathcal{U}_d$ on $\partial\Omega$. By using the uniqueness Theorem 2.1, we obtain $\mathcal{B} = \mathcal{B}^*$. \square

Next, to solve the shape optimization problem $(\mathcal{P}_{\text{op}})$, we introduce the topological sensitivity analysis method.

3.2. Topological sensitivity analysis

The main idea of the topological sensitivity consists in studying the variation of a shape functional with respect to an infinitesimal singular domain perturbation, such as the insertion of cavities, inclusions, source-terms or even cracks [5, 6, 12, 14, 17, 19, 23, 24, 34].

Let us briefly browse the history of this method. Its main idea was first introduced by Schumacher [54] in the context of compliance minimization. In the same context, Sokolowski and Zochowski [56] gave some mathematical justifications in the plane stress case and generalized it to various cost functions. A topological sensitivity framework using an adaptation of the adjoint method and truncation technique was then introduced by Masmoudi [45] for the Laplace equation. It was generalized in [29] to the elasticity equations in the case of arbitrarily shaped holes. By using the adjoint method and the domain truncation technique, the topological gradient has been calculated for several equations such as Laplace [31], Stokes [32], quasi-Stokes [33], Navier-Stokes [11], elastodynamic [27], Maxwell [46] and Helmholtz [53] and many other equations.

To present the idea of this method, let us consider a perturbation of Ω confined in a small region $\mathcal{B}_{z,\varepsilon}$ of size ε and has the form $\mathcal{B}_{z,\varepsilon} = z + \varepsilon\mathcal{C}$, where $z \in \Omega$ and $\mathcal{C} \subset \mathbb{R}^2$ is a fixed open and bounded set containing the origin, whose boundary $\partial\mathcal{C}$ is connected and piecewise of class C^1 . Then the topological sensitivity analysis leads to an asymptotic expansion of the shape functional \mathcal{K} on the form:

$$\mathcal{K}(\Omega \setminus \overline{\mathcal{B}_{z,\varepsilon}}) = \mathcal{K}(\Omega) + \xi(\varepsilon)\delta\mathcal{K}(z) + o(\xi(\varepsilon)), \quad \forall z \in \Omega,$$

where

- $\varepsilon \mapsto \xi(\varepsilon)$ is a scalar positive function verifying $\lim_{\varepsilon \rightarrow 0} \xi(\varepsilon) = 0$ and describes the behavior of the variation $\mathcal{K}(\Omega \setminus \overline{\mathcal{B}_{z,\varepsilon}}) - \mathcal{K}(\Omega)$ with respect to ε .

- The function $z \mapsto \delta\mathcal{K}(z)$ measures the sensitivity of the shape functional \mathcal{K} with respect to a geometry perturbation around the point z . The function $\delta\mathcal{K}$ is called the “*topological gradient*”. Mathematically, $\delta\mathcal{K}$ can be expressed as

$$\delta\mathcal{K}(z) := \lim_{\varepsilon \rightarrow 0} \frac{\mathcal{K}(\Omega \setminus \overline{\mathcal{B}_{z,\varepsilon}}) - \mathcal{K}(\Omega)}{\xi(\varepsilon)}.$$

- The shape functional \mathcal{K} associated with the topologically perturbed domain $\Omega \setminus \overline{\mathcal{B}_{z,\varepsilon}}$ is written as

$$\mathcal{K}(\Omega \setminus \overline{\mathcal{B}_{z,\varepsilon}}) = \int_{\Omega \setminus \overline{\mathcal{B}_{z,\varepsilon}}} \sigma(\psi_\varepsilon^D - \psi_\varepsilon^N) : e(\psi_\varepsilon^D - \psi_\varepsilon^N) \, dx,$$

with ψ_ε^N and ψ_ε^D solve respectively problems $(\mathcal{P}_N^\varepsilon)$ and $(\mathcal{P}_D^\varepsilon)$.

$$(\mathcal{P}_N^\varepsilon) \quad \text{Find } \psi_\varepsilon^N \in H^1(\Omega \setminus \overline{\mathcal{B}_{z,\varepsilon}}) \text{ solving } \begin{cases} -\operatorname{div} \sigma(\psi_\varepsilon^N) = 0 & \text{in } \Omega \setminus \overline{\mathcal{B}_{z,\varepsilon}}, \\ \sigma(\psi_\varepsilon^N) \mathbf{n} = \varphi & \text{on } \partial\Omega, \\ \psi_\varepsilon^N = 0 & \text{on } \partial\mathcal{B}_{z,\varepsilon} \end{cases}$$

and

$$(\mathcal{P}_D^\varepsilon) \quad \text{Find } \psi_\varepsilon^D \in H^1(\Omega \setminus \overline{\mathcal{B}_{z,\varepsilon}}) \text{ solving } \begin{cases} -\operatorname{div} \sigma(\psi_\varepsilon^D) = 0 & \text{in } \Omega \setminus \overline{\mathcal{B}_{z,\varepsilon}}, \\ \psi_\varepsilon^D = \mathcal{U}_d & \text{on } \partial\Omega, \\ \psi_\varepsilon^D = 0 & \text{on } \partial\mathcal{B}_{z,\varepsilon}. \end{cases}$$

In the particular case, if $\varepsilon = 0$, we will consider as a convention that $\mathcal{B}_{z,0} = \emptyset$, and consequently $\Omega \setminus \overline{\mathcal{B}_{z,0}} = \Omega$. Therefore, we denote by ψ_0^N and ψ_0^D respectively the solutions to the following unperturbed problems:

$$(\mathcal{P}_N^0) \quad \text{Find } \psi_0^N \in H^1(\Omega) \text{ solving } \begin{cases} -\operatorname{div} \sigma(\psi_0^N) = 0 & \text{in } \Omega, \\ \sigma(\psi_0^N) \mathbf{n} = \varphi & \text{on } \partial\Omega, \end{cases}$$

$$(\mathcal{P}_D^0) \quad \text{Find } \psi_0^D \in H^1(\Omega) \text{ solving } \begin{cases} -\operatorname{div} \sigma(\psi_0^D) = 0 & \text{in } \Omega, \\ \psi_0^D = \mathcal{U}_d & \text{on } \partial\Omega. \end{cases}$$

Then the shape functional \mathcal{K} associated to the unperturbed domain Ω is written as

$$(3.1) \quad \mathcal{K}(\Omega) = \int_{\Omega} \sigma(\psi_0^N - \psi_0^D) : e(\psi_0^N - \psi_0^D) \, dx.$$

In order to minimize the shape functional \mathcal{K} , the best location to insert a small inclusion $\mathcal{B}_{z,\varepsilon}$ in Ω is where $\delta\mathcal{K}$ is the most negative. In fact, if $\delta\mathcal{K}(z) < 0$ we have $\mathcal{K}(\Omega \setminus \overline{\mathcal{B}_{z,\varepsilon}}) \leq \mathcal{K}(\Omega)$ for small $\varepsilon > 0$. Particularly, the solution of the optimization problem

$$\min_{\mathcal{B}_{z,\varepsilon} \in \mathcal{V}_{\text{ad}}} \mathcal{K}(\Omega \setminus \overline{\mathcal{B}_{z,\varepsilon}})$$

is given by $\mathcal{B}_{z^*,\varepsilon} = z^* + \varepsilon\mathcal{C}$ where $z^* \in \Omega$ such that $\delta\mathcal{K}(z^*) < 0$ and $\delta\mathcal{K}(z^*) < \delta\mathcal{K}(z)$, $\forall z \in \Omega$.

Remark 3.2. Notice that the solution ψ_0^N of the unperturbed Neumann problem (\mathcal{P}_N^0) has a unique solution ψ_0^N up to an additive constant. In order to ensure uniqueness we add the following normalization condition:

$$(3.2) \quad \int_{\partial\Omega} \varphi \cdot w \, ds = 0, \quad \forall w \in \mathcal{V},$$

where \mathcal{V} denotes the rigid motions space of the domain Ω defined as

$$\mathcal{V} = \{v \in H^1(\Omega), e(v) = 0\}.$$

In practice, to avoid the normalization condition (3.2), one can solve the following variational problem:

$$\int_{\Omega} \sigma(\psi_0^N) : e(w) \, dx + \int_{\Omega} \varsigma \psi_0^N \cdot w \, dx = \int_{\partial\Omega} \varphi \cdot w \, ds, \quad \forall w \in H^1(\Omega)$$

where ς is a small parameter ($\sim 10^{-10}$).

Remark 3.3. For the sake of simplicity in what follows we will work with an origin-centered inclusion (i.e., $z = 0$). The general case comes from a linear change of coordinates. Then, we denote by $\mathcal{B}_{z,\varepsilon} = \mathcal{B}_{0,\varepsilon} =: \mathcal{B}_\varepsilon$ also consider $\Omega_\varepsilon := \Omega \setminus \overline{\mathcal{B}_{0,\varepsilon}}$.

In the next section, we present the theoretical aspect of the proposed method. We derive a topological sensitivity analysis for the linear elasticity operator. The obtained results are valid for a Dirichlet and/or Neumann boundary condition on $\partial\Omega$ (see Proposition 4.10). In Section 4.2, from the obtained results, we will deduce an asymptotic expansion for the Kohn-Vogelius functional \mathcal{K} (see Theorem 4.13).

4. Asymptotic analysis

4.1. General case

This section is devoted to the topological sensitivity analysis for the linear elasticity operator with respect to the presence of a small inclusion $\mathcal{B}_\varepsilon = \varepsilon\mathcal{C}$ inside the background domain Ω . This question has been investigated in the past by Garreau et al. [29] and Giusti et al. [30]. In the first reference they considered a Neumann boundary condition on the small inclusion obtaining general results in two and three dimensional cases. They established a topological sensitivity analysis based on the truncation technique and the sensitivity analysis of the Dirichlet-to-Neumann operator. In the work of Giusti et al. they obtained a topological derivative of inclusion in two dimensions in linear elasticity problem using the Topological-Shape Sensitivity Method.

Here, we do not use the truncation technique and the Topological-Shape Sensitivity Method, but we follow the simplified approach presented in [1] (see also [2, 25]). However, contrary to the problem studied in [29], we impose Dirichlet boundary conditions on the interior boundary of \mathcal{B}_ε . We propose a simple and rigorous mathematical analysis based on a preliminary estimate describing the influence of the geometric perturbation \mathcal{B}_ε on the elasticity problem solution. In order to derive a topological sensitivity analysis valid for the problems $(\mathcal{P}_N^\varepsilon)$ and $(\mathcal{P}_D^\varepsilon)$, we introduce the following auxiliary problem:

$$(4.1) \quad \begin{cases} -\operatorname{div} \sigma(\psi_\varepsilon) = 0 & \text{in } \Omega_\varepsilon, \\ \sigma(\psi_\varepsilon)\mathbf{n} = \varphi_n & \text{on } \Sigma, \\ \psi_\varepsilon = 0 & \text{on } \Gamma, \\ \psi_\varepsilon = 0 & \text{on } \partial\mathcal{B}_\varepsilon, \end{cases}$$

where $\varphi_n \in H^{-1/2}(\Sigma)$ is a given data, with Σ and Γ form a non-overlapping decomposition of the boundary $\partial\Omega$: Σ and Γ are relatively open, $\partial\Omega = \overline{\Sigma} \cup \overline{\Gamma}$, and $\Sigma \cap \Gamma = \emptyset$.

In the absence of a rigid inclusion (i.e., $\varepsilon = 0$) $\Omega_0 = \Omega$ and ψ_0 solves

$$(4.2) \quad \begin{cases} -\operatorname{div} \sigma(\psi_0) = 0 & \text{in } \Omega, \\ \sigma(\psi_0)\mathbf{n} = \varphi_n & \text{on } \Sigma, \\ \psi_0 = 0 & \text{on } \Gamma. \end{cases}$$

Remark 4.1. In the particular case $\Sigma = \partial\Omega$ ($\Gamma = \emptyset$), the problem (4.2) has a unique solution ψ_0 , up to an additive constant. To ensure uniqueness, the boundary force φ_n should satisfy the compatibility condition:

$$\int_{\Sigma} \varphi_n \cdot w \, ds = 0, \quad \forall w \in \mathcal{V}.$$

Consider now the shape function

$$\mathcal{J}(\Omega_\varepsilon) = \int_{\Omega_\varepsilon} \sigma(\psi_\varepsilon) : e(\psi_\varepsilon) \, dx.$$

The aim is to derive an asymptotic expansion of the shape function \mathcal{J} with respect to the presence of a small rigid inclusion \mathcal{B}_ε in Ω . More precisely, we study the variation $\mathcal{J}(\Omega_\varepsilon) - \mathcal{J}(\Omega)$ with respect to ε and establish an asymptotic formula of the form

$$\mathcal{J}(\Omega_\varepsilon) - \mathcal{J}(\Omega) = \xi(\varepsilon)\delta \mathcal{J}(0) + o(\xi(\varepsilon)).$$

We recall that we will determine the above asymptotic expansion of \mathcal{J} only for the case of an origin-centered inclusion (see Remark 3.3).

The variation of the shape function \mathcal{J} reads

$$(4.3) \quad \begin{aligned} \mathcal{J}(\Omega_\varepsilon) - \mathcal{J}(\Omega) &= \int_{\Omega_\varepsilon} \sigma(\psi_\varepsilon) : e(\psi_\varepsilon) \, dx - \int_{\Omega} \sigma(\psi_0) : e(\psi_0) \, dx \\ &= \int_{\mathcal{B}_\varepsilon} \sigma(\psi_0) : e(\psi_0) \, dx + \int_{\Omega_\varepsilon} \sigma(\psi_\varepsilon - \psi_0) : e(\psi_\varepsilon - \psi_0) \, dx \\ &\quad + 2 \int_{\Omega} \sigma(\psi_0) : e(\psi_\varepsilon - \psi_0) \, dx. \end{aligned}$$

Remark 4.2. In the equality (4.3), the solution ψ_ε of problem (4.1) is extended by zero inside the region \mathcal{B}_ε . Its extension will be denoted by ψ_ε throughout the rest of the paper.

Let us introduce an adjoint state ϑ_0 solution to the following auxiliary variational problem

$$(4.4) \quad \begin{aligned} &\text{Find } \vartheta_0 \in H^1_\Gamma(\Omega), \text{ such that} \\ &\int_{\Omega} \sigma(w) : e(\vartheta_0) \, dx = -2 \int_{\Omega} \sigma(\psi_0) : e(w) \, dx, \quad \forall w \in H^1_\Gamma(\Omega). \end{aligned}$$

The space $H^1_\Gamma(\Omega)$ is defined as

$$H^1_\Gamma(\Omega) = \{w \in H^1(\Omega); w = 0 \text{ on } \Gamma\}.$$

Then ϑ_0 it can be interpreted as the solution to the following adjoint problem

$$\begin{cases} -\operatorname{div} \sigma(\vartheta_0) = -DJ_0(\psi_0) & \text{in } \Omega, \\ \sigma(\vartheta_0)\mathbf{n} = 0 & \text{on } \Sigma, \\ \vartheta_0 = 0 & \text{on } \Gamma, \end{cases}$$

associated to the minimization problem

$$\min_{w \in H^1_\Gamma(\Omega)} J_0(w),$$

with J_0 being the cost functional defined by

$$J_0(w) = \int_\Omega \sigma(w) : e(w) \, dx, \quad \forall w \in H^1_\Gamma(\Omega).$$

By taking $w = \psi_\varepsilon - \psi_0$ as a test function in (4.4) and using Remark 4.2, we deduce that the last term of (4.3) can be rewritten as

$$\begin{aligned} 2 \int_\Omega \sigma(\psi_0) : e(\psi_\varepsilon - \psi_0) \, dx &= - \int_\Omega \sigma(\psi_\varepsilon - \psi_0) : e(\vartheta_0) \, dx \\ &= \int_{\mathcal{B}_\varepsilon} \sigma(\psi_0) : e(\vartheta_0) \, dx + \int_{\Omega_\varepsilon} \sigma(\psi_0 - \psi_\varepsilon) : e(\vartheta_0) \, dx. \end{aligned}$$

Consequently, the variation $\mathcal{J}(\Omega_\varepsilon) - \mathcal{J}(\Omega)$ can be decomposed as:

$$\begin{aligned} \mathcal{J}(\Omega_\varepsilon) - \mathcal{J}(\Omega) &= \int_{\mathcal{B}_\varepsilon} \sigma(\psi_0) : e(\psi_0) \, dx + \int_{\mathcal{B}_\varepsilon} \sigma(\psi_0) : e(\vartheta_0) \, dx \\ (4.5) \quad &+ \int_{\Omega_\varepsilon} \sigma(\psi_0 - \psi_\varepsilon) : e(\vartheta_0) \, dx + \int_{\Omega_\varepsilon} \sigma(\psi_\varepsilon - \psi_0) : e(\psi_\varepsilon - \psi_0) \, dx. \end{aligned}$$

In the next section, we will derive an estimate for each term on the right-hand side of (4.5).

4.1.1. Preliminary estimates

The following lemma gives an estimate for the first term of (4.5).

Lemma 4.3. *We have*

$$\int_{\mathcal{B}_\varepsilon} \sigma(\psi_0) : e(\psi_0) \, dx = O(\varepsilon^2).$$

Proof. We can write

$$\begin{aligned} \int_{\mathcal{B}_\varepsilon} \sigma(\psi_0) : e(\psi_0) \, dx &= \varepsilon^2 |\mathcal{S}| \sigma(\psi_0)(0) : e(\psi_0)(0) \\ &\quad + \int_{\mathcal{B}_\varepsilon} \{ \sigma(\psi_0) : e(\psi_0) - \sigma(\psi_0)(0) : e(\psi_0)(0) \} \, dx. \end{aligned}$$

We obtain by a change of variable $x = \varepsilon y$ that

$$\begin{aligned} &\int_{\mathcal{B}_\varepsilon} \{ \sigma(\psi_0) : e(\psi_0) - \sigma(\psi_0)(0) : e(\psi_0)(0) \} \, dx \\ &= \varepsilon^2 \int_{\mathcal{C}} \{ \sigma(\psi_0)(\varepsilon y) : e(\psi_0)(\varepsilon y) - \sigma(\psi_0)(0) : e(\psi_0)(0) \} \, dy. \end{aligned}$$

Since ψ_0 is of C^1 in a neighborhood of the origin, we obtain immediately with the help of a Taylor expansion that

$$\int_{\mathcal{C}} \{ \sigma(\psi_0)(\varepsilon y) : e(\psi_0)(\varepsilon y) - \sigma(\psi_0)(0) : e(\psi_0)(0) \} \, dy = O(\varepsilon).$$

Hence,

$$\int_{\mathcal{B}_\varepsilon} \sigma(\psi_0) : e(\psi_0) \, dx = O(\varepsilon^2). \quad \square$$

In a similar manner, we derive the following lemma. It describes an estimate of the second term of right-hand side of (4.5).

Lemma 4.4. *We have*

$$\int_{\mathcal{B}_\varepsilon} \sigma(\psi_0) : e(\vartheta_0) \, dx = O(\varepsilon^2).$$

To estimate the third term of (4.5), we will need two preliminary lemmas. The first one concerns an estimate of the norm $\|\cdot\|_{1/2, \partial \mathcal{B}_\varepsilon}$ of a uniformly bounded function. In the second lemma, we give an asymptotic expansion of the solution of the perturbed elasticity problem (4.1).

Lemma 4.5. *Let $\varepsilon \in (0, 1/2)$. If $\psi \in H^1(\Omega)$ is such that its restriction to $\overline{\mathcal{B}_1} = \overline{\mathcal{C}}$ is C^1 , then there exists a constant $c > 0$ independent of ε such that*

$$\|\psi\|_{1/2, \partial \mathcal{B}_\varepsilon} \leq \frac{c}{\sqrt{-\log(\varepsilon)}}.$$

Proof. Thanks to Theorem 7.3, there exists a constant $c > 0$ (independent of ε) such that

$$\|\psi\|_{1/2, \partial \mathcal{B}_\varepsilon} \leq \frac{c\varepsilon^{-1/2}}{\sqrt{-\log(\varepsilon)}} \|\psi\|_{L^2(\partial \mathcal{B}_\varepsilon)} + c \left(\iint_{\partial \mathcal{B}_\varepsilon \times \partial \mathcal{B}_\varepsilon} \frac{|\psi(x) - \psi(y)|^2}{|x - y|^2} \, ds(x) ds(y) \right)^{1/2}.$$

Using the changes of variables $x = \varepsilon x_1$ and $y = \varepsilon y_1$ and Taylor formula, we have

$$\begin{aligned} \psi(\varepsilon x_1) &= \psi(0) + \varepsilon \nabla \psi(\xi_{x_1}) x_1 \quad \text{with } \xi_{x_1} \in \mathcal{B}_\varepsilon, \\ \psi(\varepsilon y_1) &= \psi(0) + \varepsilon \nabla \psi(\xi_{y_1}) y_1 \quad \text{with } \xi_{y_1} \in \mathcal{B}_\varepsilon. \end{aligned}$$

Then there exists a constant $c > 0$ independent of ε such that

$$\begin{aligned} &\left(\iint_{\partial \mathcal{B}_\varepsilon \times \partial \mathcal{B}_\varepsilon} \frac{|\psi(x) - \psi(y)|^2}{|x - y|^2} \, ds(x) ds(y) \right)^{1/2} \\ &= \left(\iint_{\partial \mathcal{C} \times \partial \mathcal{C}} \varepsilon^2 \frac{|\varepsilon(\nabla \psi(\xi_{x_1}) x_1 - \nabla \psi(\xi_{y_1}) y_1)|^2}{\varepsilon^2 |x_1 - y_1|^2} \, ds(x) ds(y) \right)^{1/2} \leq c\varepsilon. \end{aligned}$$

Moreover, using the change of variable $x = \varepsilon y$ and that ψ is uniformly bounded on $\partial \mathcal{B}_\varepsilon$, then there exists a constant $c > 0$ independent of ε such that

$$\|\psi\|_{L^2(\partial \mathcal{B}_\varepsilon)} \leq c\varepsilon^{1/2}.$$

Consequently, we get

$$\|\psi\|_{1/2, \partial \mathcal{B}_\varepsilon} \leq \frac{c\varepsilon^{-1/2}}{\sqrt{-\log(\varepsilon)}} \varepsilon^{1/2} + c\varepsilon \leq \frac{c}{\sqrt{-\log(\varepsilon)}}. \quad \square$$

Before establishing the asymptotic expansion of ψ_ε (solution of problem (4.1)) let us recall the expression of the fundamental solution G to the linear elasticity system in \mathbb{R}^2

$$(4.6) \quad G(x) = \beta \log(\|x\|) \mathbf{I} + \gamma e_r^t e_r,$$

with $e_r = x/\|x\|$; that is $-\operatorname{div} \sigma(G_j) = \delta e_j$, where G_j denotes the j^{th} column of G , $(e_j)_{j=1}^2$ is the canonical basis of \mathbb{R}^2 and δ is the Dirac distribution. The constants β and γ are defined as

$$(4.7) \quad \beta = -\frac{\lambda + 3\mu}{4\pi\mu(\lambda + 2\mu)} \quad \text{and} \quad \gamma = \frac{\lambda + \mu}{4\pi\mu(\lambda + 2\mu)} \quad (\text{plane strain}).$$

For plane stress, $\lambda^* = 2\mu\lambda/(\lambda + 2\mu)$ must be substituted for λ .

Lemma 4.6. *The solution ψ_ε of problem (4.1) admits the following asymptotic expansion:*

$$\psi_\varepsilon(x) = \psi_0(x) + \mathbf{T}_\varepsilon(\mathcal{H}(x) - \zeta(x)) + O_{H^1(\Omega_\varepsilon)}\left(\frac{1}{-\log(\varepsilon)}\right),$$

where $\zeta \in H^1(\Omega)$ solves the following linear elasticity problem

$$(4.8) \quad \begin{cases} -\operatorname{div} \sigma(\zeta) = 0 & \text{in } \Omega, \\ \zeta = \mathcal{H} & \text{on } \Gamma, \end{cases}$$

with $T_\varepsilon = \frac{1}{-\log(\varepsilon)}$ and

$$\mathcal{H}(x) = \frac{-1}{\beta}G(x)\psi_0(0).$$

The notation $O_{H^1(\Omega_\varepsilon)}(\frac{1}{-\log(\varepsilon)})$ means that there exists a constant $c > 0$ (independent of ε) and $\varepsilon_1 > 0$ such that for all $0 < \varepsilon < \varepsilon_1$

$$\|\psi_\varepsilon(x) - \psi_0(x) - T_\varepsilon(\mathcal{H}(x) - \zeta(x))\|_{1,\Omega_\varepsilon} \leq \frac{c}{-\log(\varepsilon)}.$$

To prove Lemma 4.6, we need to establish the following lemma. We postpone its technical proof in Section 7.

Lemma 4.7. *Let $\varepsilon > 0$. For $\theta_N \in H^{-1/2}(\Sigma)$, $\theta_D \in H^{1/2}(\Gamma)$ and $\theta \in H^{1/2}(\partial\mathcal{B}_{z,\varepsilon})$, let $v_\varepsilon \in H^1(\Omega \setminus \overline{\mathcal{B}_{z,\varepsilon}})$ be the solution of the linear elasticity problem*

$$\begin{cases} -\operatorname{div} \sigma(v_\varepsilon) = 0 & \text{in } \Omega \setminus \overline{\mathcal{B}_{z,\varepsilon}}, \\ \sigma(v_\varepsilon)\mathbf{n} = \theta_N & \text{on } \Sigma, \\ v_\varepsilon = \theta_D & \text{on } \Gamma, \\ v_\varepsilon = \theta & \text{on } \partial\mathcal{B}_{z,\varepsilon}. \end{cases}$$

Then there exists a constant $c > 0$ (independent of ε) such that

$$\|v_\varepsilon\|_{1,\Omega \setminus \overline{\mathcal{B}_{z,\varepsilon}}} \leq c(\|\theta_N\|_{-1/2,\Sigma} + \|\theta_D\|_{1/2,\Gamma} + \|\theta\|_{1/2,\partial\mathcal{B}_{z,\varepsilon}}).$$

Now, we turn to prove Lemma 4.6.

Proof of Lemma 4.6. In order to simplify the notations, let us define

$$r_\varepsilon = \psi_\varepsilon - \psi_0 - T_\varepsilon(\mathcal{H} - \zeta),$$

where ψ_ε and ψ_0 solve respectively problem (4.1) and (4.2). One can easily remark that r_ε is solution to the system

$$\begin{cases} -\operatorname{div} \sigma(r_\varepsilon) = 0 & \text{in } \Omega_\varepsilon, \\ \sigma(r_\varepsilon)\mathbf{n} = \frac{1}{\log(\varepsilon)}(\sigma(\mathcal{H})\mathbf{n} - \sigma(\zeta)\mathbf{n}) & \text{on } \Sigma, \\ r_\varepsilon = \frac{1}{\log(\varepsilon)}(\mathcal{H} - \zeta) & \text{on } \Gamma, \\ r_\varepsilon = -\psi_0 - T_\varepsilon(\mathcal{H} - \zeta) & \text{on } \partial\mathcal{B}_\varepsilon. \end{cases}$$

Thanks to Lemma 4.7 that there exists a constant $c > 0$, independent of ε such that

$$(4.9) \quad \|r_\varepsilon\|_{1,\Omega_\varepsilon} \leq c \left\{ \frac{1}{-\log(\varepsilon)} (\|\sigma(\mathcal{H})\mathbf{n}\|_{-1/2,\Sigma} + \|\sigma(\zeta)\mathbf{n}\|_{-1/2,\Sigma} + \|\mathcal{H}\|_{1/2,\Gamma} + \|\zeta\|_{1/2,\Gamma}) + \|\psi_0 + T_\varepsilon(\mathcal{H} - \zeta)\|_{1/2,\partial\mathcal{B}_\varepsilon} \right\}.$$

Next we will derive an estimate for each term on the right-hand side of (4.9).

- Estimate of the imposed boundary data on Σ . We have

$$(4.10) \quad \|\sigma(\mathcal{H})\mathbf{n}\|_{-1/2,\Sigma} \leq c|\mathcal{H}|_{1,\Omega \setminus \mathbf{B}(0,1)} \quad \text{and} \quad \|\sigma(\zeta)\mathbf{n}\|_{-1/2,\Sigma} \leq c|\zeta|_{1,\Omega_\varepsilon}.$$

Here and in the sequel, $c > 0$ is used to denote any constant (independent of ε), that may have different values.

Let us first focus on the first estimate on the right-hand side of (4.10). For all $\vartheta \in H^{1/2}(\Sigma)$ and all $\phi \in H^1(\Omega \setminus \mathbf{B}(0,1))$, extension of ϑ such that $\phi|_\Gamma = 0$, we have

$$\begin{aligned} \langle \sigma(\mathcal{H})\mathbf{n}, \vartheta \rangle_{-1/2,1/2,\Sigma} &= \int_{\Omega \setminus \mathbf{B}(0,1)} \sigma(\mathcal{H}) : e(\phi) \, dx \\ &= \int_{\Omega \setminus \mathbf{B}(0,1)} 2\mu e(\mathcal{H}) : e(\vartheta) + \lambda \operatorname{div}(\mathcal{H}) \operatorname{div}(\vartheta) \, dx \\ &\leq c\|e(\mathcal{H})\|_{0,\Omega \setminus \mathbf{B}(0,1)} \|\phi\|_{1,\Omega \setminus \mathbf{B}(0,1)} \\ &\leq c|\mathcal{H}|_{1,\Omega \setminus \mathbf{B}(0,1)} \|\phi\|_{1,\Omega \setminus \mathbf{B}(0,1)}. \end{aligned}$$

We recall that the semi-norm $|\cdot|_{1,\Omega}$ is defined by (7.1) and the dual space $H^{-1/2}(\Sigma)$ is equipped with the natural norm

$$\|w\|_{-1/2,\Sigma} = \sup \{ \langle w, v \rangle_{-1/2,1/2,\Sigma}; v \in H^{1/2}(\Sigma), \|v\|_{1/2,\Sigma} = 1 \}.$$

Hence, choosing ϕ such that $\|\phi\|_{1,\Omega \setminus \mathbf{B}(0,1)} = \|\vartheta\|_{1/2,\Sigma}$, we get

$$(4.11) \quad \|\sigma(\mathcal{H})\mathbf{n}\|_{-1/2,\Sigma} \leq c|\mathcal{H}|_{1,\Omega \setminus \mathbf{B}(0,1)}.$$

For the second estimate in (4.10) we adopt the same analysis just by taking Ω_ε , instead of $\Omega \setminus \mathbf{B}(0,1)$ that we will use.

Now we need an estimate for the functions \mathcal{H} and ζ . For \mathcal{H} we will need a bound for the term $|\mathcal{H}|_{1,\Omega \setminus \mathbf{B}(0,1)}$. To this end, notice that $|\nabla \mathcal{H}| = O(1/\|x\|)$ and let $R > 0$ such that the domain $\Omega \subset \mathbf{B}(0, R)$, then

$$\begin{aligned} |\mathcal{H}|_{1,\Omega \setminus \mathbf{B}(0,1)} &\leq |\mathcal{H}|_{1,\mathbf{B}(0,R) \setminus \mathbf{B}(0,1)} \\ &\leq c \left(\int_{\mathbf{B}(0,R) \setminus \mathbf{B}(0,1)} \frac{1}{\|x\|^2} \, dx \right)^{1/2} = c(2\pi \log(R))^{1/2}. \end{aligned}$$

For ζ , from (4.8), we have

$$\|\zeta\|_{1,\Omega} \leq c\|\mathcal{H}\|_{1/2,\partial\Omega}.$$

Let $r > 0$ be such that the closed ball $\overline{\mathbf{B}(0, r)}$ is included in Ω and $\mathcal{B}_\varepsilon \subset \mathbf{B}(0, r)$. By trace theorem we deduce that

$$\|\zeta\|_{1,\Omega} \leq c\|\mathcal{H}\|_{1,\Omega \setminus \overline{\mathbf{B}(0,r)}}.$$

Since $\mathcal{H}(x) = \frac{-1}{\beta}G(x)\psi_0(0)$ which is bounded if x is away from zero. Then

$$(4.12) \quad \|\zeta\|_{1,\Omega} \leq c.$$

Consequently, from (4.10),

$$(4.13) \quad \|\sigma(\mathcal{H})\mathbf{n}\|_{-1/2,\Sigma} \leq c \quad \text{and} \quad \|\sigma(\zeta)\mathbf{n}\|_{-1/2,\Sigma} \leq c.$$

Using the same argument as the one used in the deduction of (4.12), we obtain

$$(4.14) \quad \begin{aligned} \|\mathcal{H}\|_{1/2,\Gamma} &\leq c, \\ \|\zeta\|_{1/2,\Gamma} &\leq c. \end{aligned}$$

• Estimate of the imposed boundary data on $\partial\mathcal{B}_\varepsilon$. Using the change of variable $x = \varepsilon y$, we have

$$\begin{aligned} &\psi_0(x) + \mathbf{T}_\varepsilon(\mathcal{H}(x) - \zeta(x)) \\ &= \psi_0(\varepsilon y) + \frac{1}{-\log(\varepsilon)} \left[\left(\log(\varepsilon\|y\|)\mathbf{I} - \frac{\gamma}{\beta}e_r^t e_r \right) \cdot \psi_0(0) - \zeta(\varepsilon y) \right] \\ &= \psi_0(\varepsilon y) - \psi_0(0) + \frac{1}{-\log(\varepsilon)} \left[\left(\log(\|y\|)\mathbf{I} - \frac{\gamma}{\beta}e_r^t e_r \right) \cdot \psi_0(0) - \zeta(\varepsilon y) \right]. \end{aligned}$$

Using a Taylor expansion, we obtain $\psi_0(\varepsilon y) = \psi_0(0) + \varepsilon\nabla\psi_0(s_y)y$ with $s_y \in \mathcal{B}_\varepsilon$. Then

$$\begin{aligned} &\psi_0(x) + \mathbf{T}_\varepsilon(\mathcal{H}(x) - \zeta(x)) \\ &= \varepsilon\nabla\psi_0(s_y) + \frac{1}{-\log(\varepsilon)} \left[\left(\log(\|y\|) - \frac{\gamma}{\beta}e_r^t e_r \right) \cdot \psi_0(0) - \zeta(\varepsilon y) \right]. \end{aligned}$$

Since $\nabla\psi_0$ is uniformly bounded on \mathcal{B}_ε and from the boundness of ζ , we get that

$$(4.15) \quad \|\psi_0 + \mathbf{T}_\varepsilon(\mathcal{H} - \zeta)\|_{1/2,\partial\mathcal{B}_\varepsilon} \leq c_1\varepsilon + \frac{c_2}{-\log(\varepsilon)} \leq \frac{c}{-\log(\varepsilon)}.$$

Gathering (4.9), (4.13), (4.14) and (4.15), we have

$$\|\psi_\varepsilon - \psi_0 - \mathbf{T}_\varepsilon(\mathcal{H} - \zeta)\|_{1,\Omega_\varepsilon} \leq \frac{c}{-\log(\varepsilon)}. \quad \square$$

Resorting to Lemmas 4.5 and 4.6, we deduce the following estimate of the third term of (4.5).

Lemma 4.8. *We have*

$$\int_{\partial\mathcal{B}_\varepsilon} \sigma(\psi_0 - \psi_\varepsilon)\mathbf{n} \cdot \vartheta_0 \, ds = \frac{1}{-\log(\varepsilon)} \frac{4\pi\mu(\mu + \eta)}{2\mu + \eta} \psi_0(0) \cdot \vartheta_0(0) + o\left(\frac{1}{-\log(\varepsilon)}\right),$$

where the constant η is defined by

$$(4.16) \quad \eta = \begin{cases} \frac{\mu(3\lambda+2\mu)}{\lambda+\mu} & \text{plane stress,} \\ \lambda + \mu & \text{plane strain.} \end{cases}$$

Proof. We have

$$\begin{aligned}
 (4.17) \quad & \int_{\partial\mathcal{B}_\varepsilon} (\sigma(\psi_0 - \psi_\varepsilon)\mathbf{n}) \cdot \vartheta_0 \, ds \\
 &= \int_{\partial\mathcal{B}_\varepsilon} (\sigma(\psi_0 - \psi_\varepsilon + \mathbf{T}_\varepsilon(\mathcal{H} - \zeta))\mathbf{n}) \cdot \vartheta_0 \, ds - \mathbf{T}_\varepsilon \int_{\partial\mathcal{B}_\varepsilon} (\sigma(\mathcal{H} - \zeta)\mathbf{n}) \cdot \vartheta_0 \, ds.
 \end{aligned}$$

Let us first focus on the estimate of the first term on the right-hand side of (4.17). We have

$$\left| \int_{\partial\mathcal{B}_\varepsilon} (\sigma(\psi_0 - \psi_\varepsilon + \mathbf{T}_\varepsilon(\mathcal{H} - \zeta))\mathbf{n}) \cdot \vartheta_0 \, ds \right| \leq \|\sigma(\psi_0 - \psi_\varepsilon + \mathbf{T}_\varepsilon(\mathcal{H} - \zeta))\mathbf{n}\|_{-1/2, \partial\mathcal{B}_\varepsilon} \|\vartheta_0\|_{1/2, \partial\mathcal{B}_\varepsilon}.$$

Using the same argument as the one used in the deduction of (4.10), we get

$$\|\sigma(\psi_0 - \psi_\varepsilon + \mathbf{T}_\varepsilon(\mathcal{H} - \zeta))\mathbf{n}\|_{-1/2, \partial\mathcal{B}_\varepsilon} \leq c \|\psi_0 - \psi_\varepsilon + \mathbf{T}_\varepsilon(\mathcal{H} - \zeta)\|_{1, \Omega_\varepsilon}.$$

Thanks to Lemma 4.6, one obtains

$$\|\sigma(\psi_0 - \psi_\varepsilon + \mathbf{T}_\varepsilon(\mathcal{H} - \zeta))\mathbf{n}\|_{-1/2, \partial\mathcal{B}_\varepsilon} \leq \frac{c}{-\log(\varepsilon)}.$$

Thus, from Lemma 4.5, we deduce that

$$\begin{aligned}
 (4.18) \quad & \left| \int_{\partial\mathcal{B}_\varepsilon} (\sigma(\psi_0 - \psi_\varepsilon + \mathbf{T}_\varepsilon(\mathcal{H} - \zeta))\mathbf{n}) \cdot \vartheta_0 \, ds \right| \leq \frac{c}{-\log(\varepsilon)} \|\vartheta_0\|_{1/2, \partial\mathcal{B}_\varepsilon} \\
 & \leq \frac{c}{(-\log(\varepsilon))^{3/2}}.
 \end{aligned}$$

Now, we estimate the last term on the right-hand side of (4.17). We have

$$(4.19) \quad \int_{\partial\mathcal{B}_\varepsilon} (\sigma(\mathcal{H} - \zeta)\mathbf{n}) \cdot \vartheta_0 \, ds = \int_{\partial\mathcal{B}_\varepsilon} (\sigma(\mathcal{H})\mathbf{n}) \cdot \vartheta_0 \, ds - \int_{\partial\mathcal{B}_\varepsilon} (\sigma(\zeta)\mathbf{n}) \cdot \vartheta_0 \, ds.$$

Firstly, we start by studying the integral $\int_{\partial\mathcal{B}_\varepsilon} (\sigma(\mathcal{H})\mathbf{n}) \cdot \vartheta_0 \, ds$. Expanding

$$\vartheta_0(\varepsilon y) = \vartheta_0(0) + \varepsilon \nabla \vartheta_0(\tau_y) \quad \text{with } \tau_y \in \mathcal{B}_\varepsilon,$$

we obtain

$$\int_{\partial\mathcal{B}_\varepsilon} (\sigma(\mathcal{H})\mathbf{n}) \cdot \vartheta_0 \, ds = \varepsilon \int_{\partial\mathcal{B}_\varepsilon} (\sigma(\mathcal{H})\mathbf{n}) \cdot \nabla \vartheta_0(\tau_y) \, ds + \int_{\partial\mathcal{B}_\varepsilon} (\sigma(\mathcal{H})\mathbf{n}) \cdot \vartheta_0(0) \, ds.$$

Since $\nabla \vartheta_0$ is uniformly bounded in \mathcal{B}_ε , we see that

$$\int_{\partial\mathcal{B}_\varepsilon} (\sigma(\mathcal{H})\mathbf{n}) \cdot \vartheta_0 \, ds = \left(\int_{\partial\mathcal{B}_\varepsilon} (\sigma(\mathcal{H})\mathbf{n}) \, ds \right) \cdot \vartheta_0(0) + O(\varepsilon).$$

On the other hand, we have

$$\int_{\partial\mathcal{B}_\varepsilon} (\sigma(\mathcal{H})\mathbf{n}) \, ds = - \int_{\mathcal{B}_\varepsilon} -\operatorname{div} \sigma(\mathcal{H}) \, dx = \frac{-1}{\beta} \psi_0(0).$$

Consequently,

$$(4.20) \quad \int_{\partial\mathcal{B}_\varepsilon} (\sigma(\mathcal{H})\mathbf{n}) \cdot \vartheta_0 \, ds = \frac{-1}{\beta} \psi_0(0) \cdot \vartheta_0(0) + O(\varepsilon).$$

Similarly, we obtain

$$\int_{\partial\mathcal{B}_\varepsilon} (\sigma(\zeta)\mathbf{n}) \cdot \vartheta_0 \, ds = \left(\int_{\partial\mathcal{B}_\varepsilon} (\sigma(\zeta)\mathbf{n}) \, ds \right) \cdot \vartheta_0(0) + O(\varepsilon).$$

From (4.8), we deduce that $-\operatorname{div} \sigma(\zeta) = 0$ in $\mathcal{B}_{z,\varepsilon}$ and using the Green's formula, we obtain

$$\int_{\partial\mathcal{B}_\varepsilon} \sigma(\zeta)\mathbf{n} \, ds = \int_{\mathcal{B}_\varepsilon} \operatorname{div} \sigma(\zeta) \, dx = 0.$$

Therefore,

$$(4.21) \quad \int_{\partial\mathcal{B}_\varepsilon} (\sigma(\zeta)\mathbf{n}) \cdot \vartheta_0 \, ds = O(\varepsilon).$$

Inserting (4.20) and (4.21) into (4.19), we obtain

$$(4.22) \quad \int_{\partial\mathcal{B}_\varepsilon} (\sigma(\mathcal{H} - \zeta)\mathbf{n}) \cdot \vartheta_0 \, ds = \frac{-1}{\beta} \psi_0(0) \cdot \vartheta_0(0) + O(\varepsilon).$$

Gathering (4.17), (4.18) and (4.22) also from the definition of β (see (4.7)), we deduce the claimed expansion:

$$\int_{\partial\mathcal{B}_\varepsilon} \sigma(\psi_0 - \psi_\varepsilon)\mathbf{n} \cdot \vartheta_0 \, ds = \frac{1}{-\log(\varepsilon)} \frac{4\pi\mu(\mu + \eta)}{2\mu + \eta} \psi_0(0) \cdot \vartheta_0(0) + o\left(\frac{1}{-\log(\varepsilon)}\right). \quad \square$$

We present in the following lemma the asymptotic expansion of the last term on the right-hand side of (4.5).

Lemma 4.9. *The fourth term on the right-hand side of (4.5) admits the asymptotic expansion*

$$\int_{\Omega_\varepsilon} \sigma(\psi_\varepsilon - \psi_0) : e(\psi_\varepsilon - \psi_0) \, dx = \frac{1}{-\log(\varepsilon)} \frac{4\pi\mu(\mu + \eta)}{2\mu + \eta} |\psi_0(0)|^2 + o\left(\frac{1}{-\log(\varepsilon)}\right)$$

where the constant η is defined in (4.16).

Proof. One can easily remark that $\psi_\varepsilon - \psi_0$ is the solution to the following boundary value problem:

$$\begin{cases} -\operatorname{div} \sigma(\psi_\varepsilon - \psi_0) = 0 & \text{in } \Omega_\varepsilon, \\ \psi_\varepsilon - \psi_0 = 0 & \text{on } \Gamma, \\ \sigma(\psi_\varepsilon - \psi_0)\mathbf{n} = 0 & \text{on } \Sigma, \\ \psi_\varepsilon - \psi_0 = -\psi_0 & \text{on } \partial\mathcal{B}_\varepsilon. \end{cases}$$

Using Green formula, we have

$$\int_{\Omega_\varepsilon} \sigma(\psi_\varepsilon - \psi_0) : e(\psi_\varepsilon - \psi_0) \, dx = \int_{\partial\mathcal{B}_\varepsilon} (\sigma(\psi_0 - \psi_\varepsilon)\mathbf{n}) \cdot \psi_0 \, dx.$$

Applying the same technique developed in the proof of Lemma 4.8, one can derive

$$\int_{\partial\mathcal{B}_\varepsilon} (\sigma(\psi_0 - \psi_\varepsilon)\mathbf{n}) \cdot \psi_0 \, dx = \frac{1}{-\log(\varepsilon)} \frac{4\pi\mu(\mu + \eta)}{2\mu + \eta} |\psi_0(0)|^2 + o\left(\frac{1}{-\log(\varepsilon)}\right).$$

Therefore,

$$\int_{\Omega_\varepsilon} \sigma(\psi_\varepsilon - \psi_0) : e(\psi_\varepsilon - \psi_0) \, dx = \frac{1}{-\log(\varepsilon)} \frac{4\pi\mu(\mu + \eta)}{2\mu + \eta} |\psi_0(0)|^2 + o\left(\frac{1}{-\log(\varepsilon)}\right). \quad \square$$

4.1.2. Asymptotic expansion

We are now ready to present the main results of this section. Based on the estimate Lemmas 4.3, 4.4, 4.8 and 4.9, we deduce the following topological asymptotic expansion of the shape functional \mathcal{J} .

Proposition 4.10. *We have*

$$\mathcal{J}(\Omega_\varepsilon) = \mathcal{J}(\Omega) + \frac{1}{-\log(\varepsilon)} \frac{4\pi\mu(\mu + \eta)}{2\mu + \eta} (\psi_0(0) \cdot \vartheta_0(0) + |\psi_0(0)|^2) + o\left(\frac{1}{-\log(\varepsilon)}\right).$$

4.2. Sensitivity analysis of the Kohn-Vogelius functional

In this section, we consider the Kohn-Vogelius functional \mathcal{K} . Recall that \mathcal{K} measures the difference between the Dirichlet and Neumann solutions. In the presence of a small inclusion $\mathcal{B}_{z,\varepsilon}$ in Ω , \mathcal{K} is defined by

$$\mathcal{K}(\Omega \setminus \overline{\mathcal{B}_{z,\varepsilon}}) = \int_{\Omega \setminus \overline{\mathcal{B}_{z,\varepsilon}}} \sigma(\psi_\varepsilon^D - \psi_\varepsilon^N) : e(\psi_\varepsilon^D - \psi_\varepsilon^N) \, dx,$$

where $\psi_\varepsilon^N \in H^1(\Omega)/\mathcal{V}$ and $\psi_\varepsilon^D \in H^1(\Omega)$ solve respectively the problems $(\mathcal{P}_N^\varepsilon)$ and $(\mathcal{P}_D^\varepsilon)$.

The aim here is to derive an asymptotic expansion for \mathcal{K} with respect to the presence of a small inclusion $\mathcal{B}_{z,\varepsilon}$ in Ω . In order to derive a topological asymptotic expansion for the functional \mathcal{K} , we start our analysis by the following simplified variation of \mathcal{K} .

4.2.1. Variation of the Kohn-Vogelius functional

We present in this section the variation of the functional \mathcal{K} with respect to the presence of a small inclusion $\mathcal{B}_{z,\varepsilon}$ inside the domain Ω . We will derive a simplified expression of the variation $\mathcal{K}(\Omega \setminus \overline{\mathcal{B}_{z,\varepsilon}}) - \mathcal{K}(\Omega)$. The obtained result is presented in the following theorem.

Theorem 4.11. *We have*

$$\mathcal{K}(\Omega \setminus \overline{\mathcal{B}_{z,\varepsilon}}) - \mathcal{K}(\Omega) = \mathbf{A}_N(\varepsilon) + \mathbf{A}_D(\varepsilon),$$

where

$$\mathbf{A}_N(\varepsilon) := \int_{\partial\mathcal{B}_{z,\varepsilon}} (\sigma(\psi_\varepsilon^N - \psi_0^N)\mathbf{n}) \cdot \psi_0^N \, ds - \int_{\mathcal{B}_{z,\varepsilon}} \sigma(\psi_0^N) : e(\psi_0^N) \, dx$$

and

$$\mathbf{A}_D(\varepsilon) := -3 \int_{\mathcal{B}_{z,\varepsilon}} \sigma(\psi_0^D) : e(\psi_0^D) \, dx - \int_{\partial\mathcal{B}_{z,\varepsilon}} (\sigma(\psi_\varepsilon^D - \psi_0^D)\mathbf{n}) \cdot \psi_0^D \, ds.$$

Proof. The Kohn-Vogelius functional \mathcal{K} can be splitted into three terms

$$\mathcal{K}(\Omega \setminus \overline{\mathcal{B}_{z,\varepsilon}}) = \mathcal{K}_{DD}(\Omega \setminus \overline{\mathcal{B}_{z,\varepsilon}}) - 2\mathcal{K}_{ND}(\Omega \setminus \overline{\mathcal{B}_{z,\varepsilon}}) + \mathcal{K}_{NN}(\Omega \setminus \overline{\mathcal{B}_{z,\varepsilon}}),$$

where

$$\mathcal{K}_{DD}(\Omega \setminus \overline{\mathcal{B}_{z,\varepsilon}}) = \int_{\Omega \setminus \overline{\mathcal{B}_{z,\varepsilon}}} \sigma(\psi_\varepsilon^D) : e(\psi_\varepsilon^D) \, dx,$$

$$\mathcal{K}_{ND}(\Omega \setminus \overline{\mathcal{B}_{z,\varepsilon}}) = \int_{\Omega \setminus \overline{\mathcal{B}_{z,\varepsilon}}} \sigma(\psi_\varepsilon^N) : e(\psi_\varepsilon^D) \, dx,$$

$$\mathcal{K}_{NN}(\Omega \setminus \overline{\mathcal{B}_{z,\varepsilon}}) = \int_{\Omega \setminus \overline{\mathcal{B}_{z,\varepsilon}}} \sigma(\psi_\varepsilon^N) : e(\psi_\varepsilon^N) \, dx.$$

- Variation of \mathcal{K}_{DD} . We have

$$\begin{aligned} & \mathcal{K}_{DD}(\Omega \setminus \overline{\mathcal{B}_{z,\varepsilon}}) - \mathcal{K}_{DD}(\Omega) \\ &= \int_{\Omega \setminus \overline{\mathcal{B}_{z,\varepsilon}}} \sigma(\psi_\varepsilon^D) : e(\psi_\varepsilon^D) \, dx - \int_{\Omega} \sigma(\psi_0^D) : e(\psi_0^D) \, dx \\ &= \int_{\Omega \setminus \overline{\mathcal{B}_{z,\varepsilon}}} \sigma(\psi_\varepsilon^D - \psi_0^D) : e(\psi_\varepsilon^D - \psi_0^D) \, dx + 2 \int_{\Omega \setminus \overline{\mathcal{B}_{z,\varepsilon}}} \sigma(\psi_\varepsilon^D - \psi_0^D) : e(\psi_0^D) \, dx \\ & \quad - \int_{\mathcal{B}_{z,\varepsilon}} \sigma(\psi_0^D) : e(\psi_0^D) \, dx. \end{aligned}$$

Using Green's formula in the two first integrals, we obtain

$$\begin{aligned} \int_{\Omega \setminus \overline{\mathcal{B}_{z,\varepsilon}}} \sigma(\psi_\varepsilon^D - \psi_0^D) : e(\psi_\varepsilon^D - \psi_0^D) \, dx &= - \int_{\partial\mathcal{B}_{z,\varepsilon}} (\sigma(\psi_\varepsilon^D - \psi_0^D)\mathbf{n}) \cdot \psi_0^D \, ds, \\ \int_{\Omega \setminus \overline{\mathcal{B}_{z,\varepsilon}}} \sigma(\psi_\varepsilon^D - \psi_0^D) : e(\psi_0^D) \, dx &= \int_{\partial\mathcal{B}_{z,\varepsilon}} (\sigma(\psi_0^D)\mathbf{n}) \cdot (\psi_\varepsilon^D - \psi_0^D) \, ds \\ &= - \int_{\partial\mathcal{B}_{z,\varepsilon}} (\sigma(\psi_0^D)\mathbf{n}) \cdot \psi_0^D \, ds \\ &= - \int_{\mathcal{B}_{z,\varepsilon}} \sigma(\psi_0^D) : e(\psi_0^D) \, dx. \end{aligned}$$

Then, it follows that

$$\mathcal{H}_{DD}(\Omega \setminus \overline{\mathcal{B}_{z,\varepsilon}}) - \mathcal{H}_{DD}(\Omega) = - \int_{\partial \mathcal{B}_{z,\varepsilon}} (\sigma(\psi_\varepsilon^D - \psi_0^D) \mathbf{n}) \cdot \psi_0^D \, ds - 3 \int_{\mathcal{B}_{z,\varepsilon}} \sigma(\psi_0^D) : e(\psi_0^D) \, dx.$$

- Variation of \mathcal{H}_{ND} . Again using Green formula on $(\mathcal{P}_N^\varepsilon)$ and (\mathcal{P}_N^0) , we get

$$\begin{aligned} \mathcal{H}_{ND}(\Omega \setminus \overline{\mathcal{B}_{z,\varepsilon}}) - \mathcal{H}_{ND}(\Omega) &= \int_{\Omega \setminus \overline{\mathcal{B}_{z,\varepsilon}}} \sigma(\psi_\varepsilon^N) : e(\psi_\varepsilon^D) \, dx - \int_{\Omega} \sigma(\psi_0^N) : e(\psi_0^D) \, dx \\ &= \int_{\partial \Omega} (\sigma(\psi_\varepsilon^N) \mathbf{n}) \cdot \psi_\varepsilon^D \, ds - \int_{\partial \Omega} (\sigma(\psi_0^N) \mathbf{n}) \cdot \psi_0^D \, ds. \end{aligned}$$

- Variation of \mathcal{H}_{NN} . We have

$$\begin{aligned} &\mathcal{H}_{NN}(\Omega \setminus \overline{\mathcal{B}_{z,\varepsilon}}) - \mathcal{H}_{NN}(\Omega) \\ &= \int_{\Omega \setminus \overline{\mathcal{B}_{z,\varepsilon}}} \sigma(\psi_\varepsilon^N) : e(\psi_\varepsilon^N) \, dx - \int_{\Omega} \sigma(\psi_0^N) : e(\psi_0^N) \, dx \\ &= \int_{\Omega \setminus \overline{\mathcal{B}_{z,\varepsilon}}} \sigma(\psi_\varepsilon^N - \psi_0^N) : e(\psi_\varepsilon^N) \, dx + \int_{\Omega_{z,\varepsilon}} \sigma(\psi_\varepsilon^N - \psi_0^N) : e(\psi_\varepsilon^N) \, dx \\ &\quad - \int_{\mathcal{B}_{z,\varepsilon}} \sigma(\psi_0^N) : e(\psi_0^N) \, dx. \end{aligned}$$

Moreover, using Green formula, we have

$$\begin{aligned} \mathcal{H}_{NN}(\Omega \setminus \overline{\mathcal{B}_{z,\varepsilon}}) - \mathcal{H}_{NN}(\Omega) &= \int_{\partial \Omega} (\sigma(\psi_\varepsilon^N - \psi_0^N) \mathbf{n}) \cdot \psi_\varepsilon^N \, ds + \int_{\partial \Omega} (\sigma(\psi_\varepsilon^N - \psi_0^N) \mathbf{n}) \cdot \psi_0^N \, ds \\ &\quad + \int_{\partial \mathcal{B}_{z,\varepsilon}} (\sigma(\psi_\varepsilon^N - \psi_0^N) \mathbf{n}) \cdot \psi_0^N \, ds - \int_{\mathcal{B}_{z,\varepsilon}} \sigma(\psi_0^N) : e(\psi_0^N) \, dx. \end{aligned}$$

Combining the variations of \mathcal{H}_{DD} , \mathcal{H}_{ND} , and \mathcal{H}_{NN} the Kohn-Vogelius functional \mathcal{H} has the following variation

$$\begin{aligned} &\mathcal{H}(\Omega \setminus \overline{\mathcal{B}_{z,\varepsilon}}) - \mathcal{H}(\Omega) \\ &= \int_{\partial \mathcal{B}_{z,\varepsilon}} (\sigma(\psi_\varepsilon^N - \psi_0^N) \mathbf{n}) \cdot \psi_0^N \, ds - \int_{\partial \mathcal{B}_{z,\varepsilon}} (\sigma(\psi_\varepsilon^D - \psi_0^D) \mathbf{n}) \cdot \psi_0^D \, ds \\ &\quad - 3 \int_{\mathcal{B}_{z,\varepsilon}} \sigma(\psi_0^D) : e(\psi_0^D) \, dx - \int_{\mathcal{B}_{z,\varepsilon}} \sigma(\psi_0^N) : e(\psi_0^N) \, dx \\ &\quad - 2 \int_{\partial \Omega} (\sigma(\psi_\varepsilon^N) \mathbf{n}) \cdot \psi_\varepsilon^D \, ds + 2 \int_{\partial \Omega} (\sigma(\psi_0^N) \mathbf{n}) \cdot \psi_0^D \, ds \\ &\quad + \int_{\partial \Omega} (\sigma(\psi_\varepsilon^N - \psi_0^N) \mathbf{n}) \cdot \psi_\varepsilon^N \, ds + \int_{\partial \Omega} (\sigma(\psi_\varepsilon^N - \psi_0^N) \mathbf{n}) \cdot \psi_0^N \, ds. \end{aligned}$$

Since $\psi_\varepsilon^D = \psi_0^D = \mathcal{U}_d$ on $\partial \Omega$ and $\sigma(\psi_\varepsilon^N) \mathbf{n} = \sigma(\psi_0^N) \mathbf{n} = \varphi$ on $\partial \Omega$, we prove that

$$\int_{\partial \Omega} (\sigma(\psi_\varepsilon^N) \mathbf{n}) \cdot \psi_\varepsilon^D \, ds - \int_{\partial \Omega} (\sigma(\psi_0^N) \mathbf{n}) \cdot \psi_0^D \, ds = \int_{\partial \Omega} \varphi \cdot \mathcal{U}_d \, ds - \int_{\partial \Omega} \varphi \cdot \mathcal{U}_d \, ds = 0$$

and

$$\int_{\partial\Omega} (\sigma(\psi_\varepsilon^N - \psi_0^N)\mathbf{n}) \cdot \psi_\varepsilon^N \, ds = 0 \quad \text{and} \quad \int_{\partial\Omega} (\sigma(\psi_\varepsilon^N - \psi_0^N)\mathbf{n}) \cdot \psi_0^N \, ds = 0.$$

Then by combining the above equalities we obtain the desired result. \square

Remark 4.12. The variation of the terms \mathcal{K}_{DD} , \mathcal{K}_{ND} and \mathcal{K}_{NN} can be calculated using the established results in Section 4.1. But in this case the topological gradient expression will be dependent on the solution of the adjoint state.

4.2.2. Asymptotic expansion of the Kohn-Vogelius functional

We are now in position to state the main result of this section. Based on the previous variation of \mathcal{K} , we derive a topological asymptotic expansion which is summarized in the following theorem.

Theorem 4.13. *For $z \in \Omega$, the Kohn-Vogelius functional \mathcal{K} admits the following asymptotic expansion:*

$$(4.23) \quad \mathcal{K}(\Omega \setminus \overline{\mathcal{B}_{z,\varepsilon}}) = \mathcal{K}(\Omega) + \frac{1}{-\log(\varepsilon)} \delta\mathcal{K}(z) + o\left(\frac{1}{-\log(\varepsilon)}\right)$$

with $\delta\mathcal{K}$ being the topological gradient given by

$$\delta\mathcal{K}(x) = \frac{4\pi\mu(\mu + \eta)}{2\mu + \eta} (|\psi_0^D(x)|^2 - |\psi_0^N(x)|^2), \quad \forall x \in \Omega,$$

where ψ_0^N and ψ_0^D solve the problems (\mathcal{P}_N^0) and (\mathcal{P}_D^0) respectively and the constant η is defined in (4.16).

Proof. We recall that we will detail the proof only for the case of an origin-centered inclusion (see Remark 3.3).

From Theorem 4.11, we have

$$\mathcal{K}(\Omega \setminus \overline{\mathcal{B}_\varepsilon}) - \mathcal{K}(\Omega) = A_N(\varepsilon) + A_D(\varepsilon).$$

Next, we will derive an asymptotic expansion of $A_N(\varepsilon)$ and $A_D(\varepsilon)$.

Asymptotic expansion of $A_N(\varepsilon)$: We have

$$A_N(\varepsilon) = \int_{\partial\mathcal{B}_\varepsilon} (\sigma(\psi_\varepsilon^N - \psi_0^N)\mathbf{n}) \cdot \psi_0^N \, ds - \int_{\mathcal{B}_\varepsilon} \sigma(\psi_0^N) : e(\psi_0^N) \, dx.$$

Now, we study each term of $A_N(\varepsilon)$ separately. Let us first focus on the first term of $A_N(\varepsilon)$.

We have

$$\int_{\partial\mathcal{B}_\varepsilon} (\sigma(\psi_0^N - \psi_\varepsilon^N)\mathbf{n}) \cdot \psi_0^N \, ds = \int_{\partial\mathcal{B}_\varepsilon} (\sigma(r_\varepsilon^N)\mathbf{n}) \cdot \psi_0^N \, ds - T_\varepsilon \int_{\partial\mathcal{B}_\varepsilon} (\sigma(\mathcal{H}^N - \zeta^N)\mathbf{n}) \cdot \psi_0^N \, ds$$

with $r_\varepsilon^N = \psi_0^N - \psi_\varepsilon^N + \mathbb{T}_\varepsilon(\mathcal{H}^N - \zeta^N)$ where $\mathcal{H}^N(x) = \frac{-1}{\beta}G(x)\psi_0^N(0)$ and ζ^N solve the following value problem

$$\begin{cases} -\operatorname{div} \sigma(\zeta^N) = 0 & \text{in } \Omega, \\ \zeta^N = \mathcal{H}^N & \text{on } \partial\Omega. \end{cases}$$

Using the same technique described in the proof of Lemma 4.8, one can prove that

$$\int_{\partial\mathcal{B}_\varepsilon} (\sigma(\psi_0^N - \psi_\varepsilon^N)\mathbf{n}) \cdot \psi_0^N \, ds = \frac{1}{-\log(\varepsilon)} \frac{4\pi\mu(\mu + \eta)}{2\mu + \eta} |\psi_0^N(0)|^2 + o\left(\frac{1}{-\log(\varepsilon)}\right).$$

For the other term, we know using elliptic regularity that $\nabla\psi_0^N$ is uniformly bounded in \mathcal{B}_ε . Thus,

$$\int_{\mathcal{B}_\varepsilon} \sigma(\psi_0^N) : e(\psi_0^N) \, dx \leq c \int_{\mathcal{C}} \varepsilon^2 = O(\varepsilon^2).$$

Therefore,

$$A_N(\varepsilon) = \frac{1}{-\log(\varepsilon)} \frac{4\pi\mu(\mu + \eta)}{2\mu + \eta} |\psi_0^N(0)|^2 + o\left(\frac{1}{-\log(\varepsilon)}\right).$$

In a similar manner, we prove the following asymptotic expansion:

$$A_D(\varepsilon) = \frac{1}{-\log(\varepsilon)} \frac{4\pi\mu(\mu + \eta)}{2\mu + \eta} |\psi_0^D(0)|^2 + o\left(\frac{1}{-\log(\varepsilon)}\right).$$

By combining the above equalities, we have

$$\mathcal{H}(\Omega_\varepsilon) - \mathcal{H}(\Omega) = \frac{1}{-\log(\varepsilon)} \frac{4\pi\mu(\mu + \eta)}{2\mu + \eta} (|\psi_0^D(0)|^2 - |\psi_0^N(0)|^2) + o\left(\frac{1}{-\log(\varepsilon)}\right). \quad \square$$

5. Numerical implementation

This section is concerned with some numerical investigations. We start this section by a numerical validation of the topological asymptotic expansion established in Theorem 4.13. After that, we give some reconstruction results obtained by a one-iteration algorithm showing the efficiency and accuracy of our approach: only one iteration is needed to reconstruct an unknown rigid inclusion immersed in an elastic material from over-determined boundary measurements.

In our numerical examples, we use the following setting and parameters:

- The computational domain Ω is defined by the square $\Omega = (0, 1) \times (0, 1)$.
- The measurement \mathcal{U}_d is reconstructed using a synthetic data: we fix a shape \mathcal{B}^* , solve the linear elasticity problem (2.1) in $\Omega \setminus \overline{\mathcal{B}^*}$ and extract the measurement \mathcal{U}_d by computing ψ on $\partial\Omega$.
- Concerning the mesh, we impose a fixed number of discretization points for the boundary $\partial\Omega$, that is 100 mesh points in each direction (i.e., $h = 1/100$).

The numerical simulations is implemented using the free software FreeFem++.

5.1. Numerical validation

It is proved in Section 5.1 that the variation of the Kohn-Vogelius function \mathcal{K} , with respect to the presence of a small arbitrary shape of a rigid inclusion, admits the following asymptotic expansion:

$$\mathcal{K}(\Omega_{z,\varepsilon}) - \mathcal{K}(\Omega) = \frac{1}{-\log(\varepsilon)} \delta\mathcal{K}(z) + o\left(\frac{1}{-\log(\varepsilon)}\right),$$

where $\delta\mathcal{K}$ is the topological gradient defined by

$$(5.1) \quad \delta\mathcal{K}(z) = \frac{4\pi\mu(2\mu + \lambda)}{3\mu + \lambda} (|\psi_0^D(z)|^2 - |\psi_0^N(z)|^2), \quad \forall z \in \Omega.$$

Here, we give a numerical validation of this asymptotic behavior. To this end, we simulate the variation of the following function:

$$(5.2) \quad \Delta_z(\varepsilon) = \mathcal{K}(\Omega_{z,\varepsilon}) - \mathcal{K}(\Omega) + \frac{1}{\log(\varepsilon)} \delta\mathcal{K}(z), \quad \forall z \in \Omega$$

with respect to $-1/\log(\varepsilon)$ for some spherical inclusions $\mathcal{B}_{z_i,\varepsilon} = z_i + \varepsilon B(0, 1)$, $i = 1, \dots, 4$, created in the unit square Ω . The locations z_i of the considered rigid inclusions $\mathcal{B}_{z_i,\varepsilon}$ are described in Table 5.1. We expect to prove numerically that $\Delta_{z_i}(\varepsilon)$ satisfies the obtained theoretical estimate

$$\Delta_{z_i}(\varepsilon) = o\left(\frac{1}{-\log(\varepsilon)}\right).$$

Inclusion $\mathcal{B}_{z_i,\varepsilon}$	$\mathcal{B}_{z_1,\varepsilon}$	$\mathcal{B}_{z_2,\varepsilon}$	$\mathcal{B}_{z_3,\varepsilon}$	$\mathcal{B}_{z_4,\varepsilon}$
Location z_i	(0.4, 0.4)	(0.5, 0.1)	(0.7, 0.3)	(0.9, 0.2)

Table 5.1: Coordinates the inclusions centers z_i , $i = 1, \dots, 4$.

From the expression of $\Delta_z(\varepsilon)$, we adopt the following numerical procedure.

Numerical validation algorithm:

- Step 1:
 - Solve the problems (\mathcal{P}_D^0) and (\mathcal{P}_N^0) in Ω .
 - Compute $\mathcal{K}(\Omega)$ (see (3.1)).
- Step 2: For each inclusion $\mathcal{B}_{z_i,\varepsilon} = z_i + \varepsilon B(0, 1)$, $i = 1, \dots, 4$:
 - Determine the variation $\delta\mathcal{K}(z_i)$ defined in (5.2).
 - Choose $\varepsilon_0^i = \max\{\varepsilon > 0, \text{ such that } z_i + \varepsilon_0^i B(0, 1) \subset \Omega\}$.
 - Compute an approximation of the function $\varepsilon \mapsto \mathcal{K}(\Omega \setminus \overline{\mathcal{B}_{z_i,\varepsilon}})$, $\varepsilon \in]0, \varepsilon_0^i]$.

- Step 3: Deduce a numerical approximation of the function $\varepsilon \mapsto \log(|\Delta_{z_i}(\varepsilon)|)$ with respect to $\log(-\log(\varepsilon))$.

The obtained results are illustrated in Figure 5.1. For each inclusions $\mathcal{B}_{z_i, \varepsilon}$, $i = 1, \dots, 4$, we numerically compute the variation of the function $\varepsilon \mapsto \log(|\Delta_{z_i}(\varepsilon)|)$ with respect to $\log(-\log(\varepsilon))$.

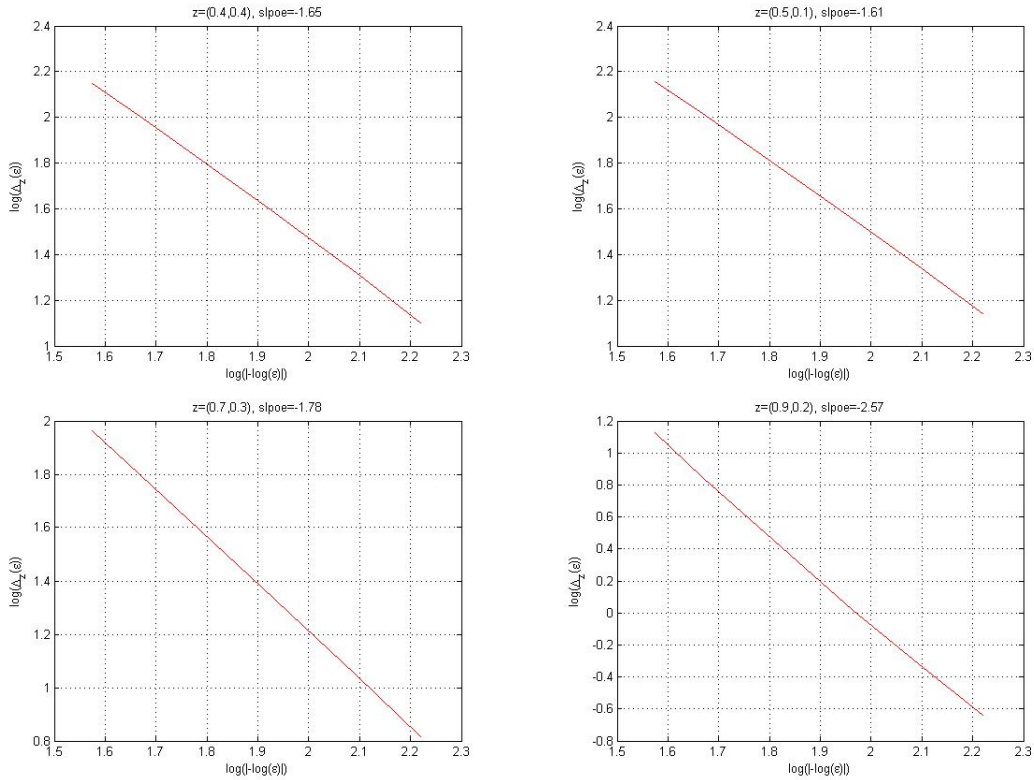


Figure 5.1: Variation of $\log(|\Delta_{z_i}(\varepsilon)|)$ with respect to $\log(-\log(\varepsilon))$, $i = 1, \dots, 4$.

Denoting by α_i the unknown parameter describing the behavior of the function $\varepsilon \mapsto \Delta_{z_i}(\varepsilon)$ with respect to $-\log(\varepsilon)$, i.e.,

$$|\Delta_{z_i}(\varepsilon)| = O((-\log(\varepsilon))^{\alpha_i}).$$

Then, one can observe that α_i can be characterized as the slope of the line approximating the variation $\varepsilon \mapsto \log(|\Delta_{z_i}(\varepsilon)|)$ with respect to $\log(-\log(\varepsilon))$.

From the presented curves in Figure 5.1, we deduce the slopes α_i , $i = 1, \dots, 4$ (see Table 5.2) describing the behavior of the function $\varepsilon \mapsto \Delta_z(\varepsilon)$ with respect to $-\log(\varepsilon)$.

Inclusion $\mathcal{B}_{z_i,\varepsilon}$	$\mathcal{B}_{z_1,\varepsilon}$	$\mathcal{B}_{z_2,\varepsilon}$	$\mathcal{B}_{z_3,\varepsilon}$	$\mathcal{B}_{z_4,\varepsilon}$
The slope α_i	$\alpha_1 = -1.65$	$\alpha_2 = -1.61$	$\alpha_3 = -1.78$	$\alpha_4 = -2.57$

Table 5.2: The slopes α_i , $i = 1, \dots, 4$ of the line approximating the variation of the function $\log(|\Delta_{z_i}(\varepsilon)|)$ with respect to $\log(-\log(\varepsilon))$.

From Table 5.2, for each considered inclusion $\mathcal{B}_{z_i,\varepsilon}$, one can observe that the obtained slope α_i satisfies the inequality: $\alpha_i < -1$, $i = 1, \dots, 4$, which confirm the behavior predicted by the theoretical result

$$\Delta_{z_i}(\varepsilon) = o\left(\frac{1}{-\log(\varepsilon)}\right), \quad i = 1, \dots, 4.$$

5.2. Reconstruction results

In this section, we propose an accurate and efficient non-iterative algorithm for reconstructing an unknown rigid inclusion. The proposed procedure based on the asymptotic expansion (4.23). The rigid inclusion \mathcal{B}^* is located and reconstructed using a level-set curve of the topological gradient $\delta\mathcal{K}$. The main steps of the proposed procedure are described in the following algorithm.

The one-shot Algorithm.

1. Solve the unperturbed problems (\mathcal{P}_N^0) and (\mathcal{P}_D^0) in Ω .
2. Compute the topological gradient $\delta\mathcal{K}$ defined in (5.1).
3. Determine the rigid inclusion \mathcal{B}^* .

The location of the rigid inclusion \mathcal{B}^* is given by the point $z^* \in \Omega$ where the topological gradient $\delta\mathcal{K}$ is most negative. The size of \mathcal{B}^* is approximated using numerical simulation. Let $\delta_{\min} = \delta\mathcal{K}(z^*) \leq \delta\mathcal{K}(x)$, $\forall x \in \Omega$, the rigid inclusion \mathcal{B}^* is approximated as follows

$$\mathcal{B}^* = \{x \in \Omega, \delta\mathcal{K}(x) \leq c^* \delta_{\min}\},$$

where $c^* \in (0, 1)$ such that

$$\mathcal{H}(\Omega \setminus \overline{\mathcal{B}_{c^*}}) \leq \mathcal{H}(\Omega \setminus \overline{\mathcal{B}_c}), \quad \forall c \in (0, 1)$$

with $\mathcal{B}_c = \{x \in \Omega, \delta\mathcal{K}(x) \leq c\delta_{\min}\}$.

Remark 5.1. The idea to determine the constant c is to calculate the minimum of the Hausdorff distance d_c between \mathcal{B}^* and \mathcal{B}_c

$$d_c(\mathcal{B}^*, \mathcal{B}_c) = \frac{\text{meas}(\mathcal{B}^* \cup \mathcal{B}_c) - \text{meas}(\mathcal{B}^* \cap \mathcal{B}_c)}{\text{meas}(\mathcal{B}^*)},$$

where $\text{meas}(E)$ denotes the Lebesgue measure of the set $E \subset \Omega$.

Next, we apply the proposed algorithm for solving some examples to demonstrate the efficiency of the proposed algorithm. Examples 5.2–5.5 are free of noise, while in Example 5.6 the measured data \mathcal{U}_d is corrupted with noise.

Example 5.2 (Reconstruction of circular-shaped inclusion). In this example, we test our reconstruction algorithm to reconstruct an inclusion having a circular shape centered at $(0.5, 0.5)$ with radius $r^* = 0.1$. The obtained reconstruction results are presented in Figure 5.2.

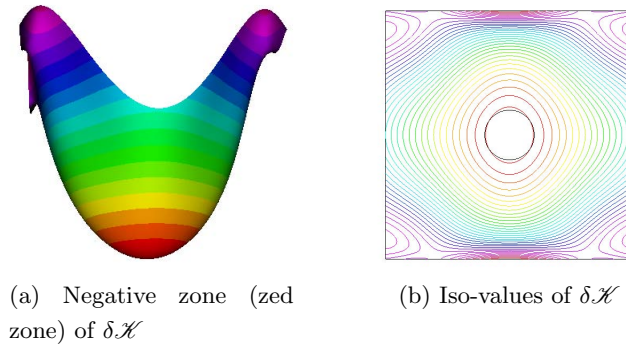


Figure 5.2: Topological gradient $\delta\mathcal{K}$ in the presence of a circle shape.

In this first simulation, we show:

- The negative zone (red zone) described by the function $x \mapsto (x, \delta\mathcal{K}(x))$, $\forall x \in \Omega$, see Figure 5.2(a).
- The iso-values of the topological gradient function $\delta\mathcal{K}$ in the presence of the unknown boundary $\partial\mathcal{B}^*$ (circle black line), see Figure 5.2(b).

As one can observe in Figure 5.2, the unknown boundary $\partial\mathcal{B}^*$ (circle centered at $(0.5, 0.5)$ with radius $r^* = 0.1$) is located in the region where the topological gradient is the most negative and it is approximated by a level set curve of the topological gradient. The result is efficient and the reconstruction of circular shape is very close to the actual rigid inclusion.

To reconstruct the exact geometry of the unknown inclusion in Figure 5.2(b) (disc centered at $(0.5, 0.5)$ with shared radius $r^* = 0.1$), we minimize the error function of the Hausdorff distance d_c . In order to compute numerically an approximation of the minimum of the error function d_c , we divide the interval $[0, 1]$ into M equal subintervals (i.e., of size $1/M$). We denote by $c_i = i/M$, $1 \leq i \leq M$ the $(M + 1)$ endpoints of these intervals and we take $c^* = \arg \min_{c \in \{c_1, \dots, c_M\}} d_c$. The reconstruction results are illustrated in Figures 5.3 and 5.4.

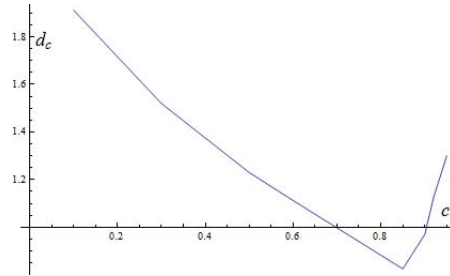


Figure 5.3: Variation of the error function d_c with respect to c .

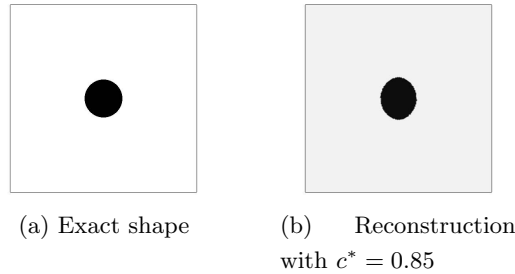


Figure 5.4: Reconstruction of circular shaped inclusion.

Example 5.3 (Reconstruction of ellipse-shaped inclusion). In this example, we examine the numerical reconstruction of unknown inclusion described by an ellipse centered at $(0.5, 0.5)$. We represent the reconstruction result in Figure 5.5 where we see that the reconstruction is again efficient.

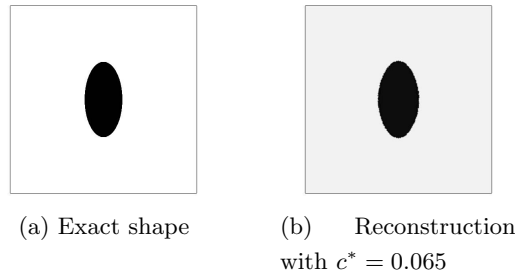


Figure 5.5: Reconstruction of an elliptical shaped.

In the next example, we prove that the computation of the topological gradient does not depend on the number of inclusions inside the domain Ω .

Example 5.4 (Reconstruction of multiple inclusions). In this example, we want to reconstruct three discs \mathcal{B}_1^* , \mathcal{B}_2^* and \mathcal{B}_3^* centered respectively at $(0.3, 0.7)$, $(0.2, 0.2)$ and $(0.8, 0.8)$ with shared radius $r^* = 0.02$. The obtained results are illustrated in Figure 5.6.

Here again, as one can see in Figure 5.6, the one-shot algorithm gives quite efficient reconstruction result for location, size and number of rigid inclusions.

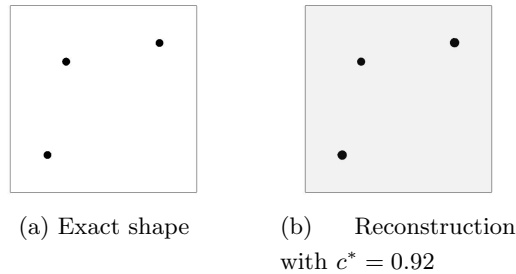


Figure 5.6: Reconstruction of multi-inclusions.

In conclusion of these first simulations, this approach permits to give us acceptable knowledge of the number, location and shape of inclusions. Moreover, it is efficient to detect different types of shapes such as circle and ellipse.

In the previous examples, we reconstructed an inclusion with a regular shape (circle and ellipse). Next we test our algorithm to reconstruct more complicated geometry.

Example 5.5 (Shape with corners). In the previous example, the reconstruction seems to be efficient to reconstruct simple rigid inclusions. We apply, now, the proposed algorithm to detect more complicated geometry. Our aim is to reconstruct geometry containing straight lines and corners from over-determined boundary data. More precisely, we want to detect a small square $\mathcal{B}^* = (0.4, 0.6) \times (0.4, 0.6)$. We present the reconstruction results in Figure 5.7. One can remark here, that the unknown rigid inclusion (square) is located but the boundary $\partial\mathcal{B}^*$ cannot be well approximated by any iso-value curve. Therefore, the obtained result can serve as a good initial guess in level-sets-based methods, for instance.

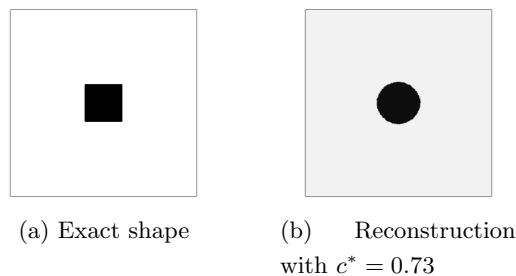


Figure 5.7: Reconstruction of an inclusion with corner.

Example 5.6 (Reconstruction with noisy data). Now the synthetic boundary measure-

ment \mathcal{U}_d is corrupted with noise. The noisy measurement is generated by

$$\mathcal{U}_d^\rho(x) = (1 + \rho \times \text{rand}) \times \mathcal{U}_d(x), \quad x \in \partial\Omega,$$

where ρ indicates the noise level of the measurement and the function rand generates a random number with the uniform distribution over $(-1, 1)$. The idea is to test the stability of the one-shot algorithm with respect to noisy data.

For this test, we reconstruct two ellipses \mathcal{B}_1^* and \mathcal{B}_2^* centered respectively at $(0.15, 0.5)$ and $(0.85, 0.5)$. The obtained results are illustrated in Figure 5.8. From the reconstruction results in Figure 5.8, we can notice that if the noise level ρ is no more than 10%, that our algorithm is able to detect, with precision, the location and the shape of rigid inclusion, whereas for a noise level larger than 15%, fictitious inclusions show up.

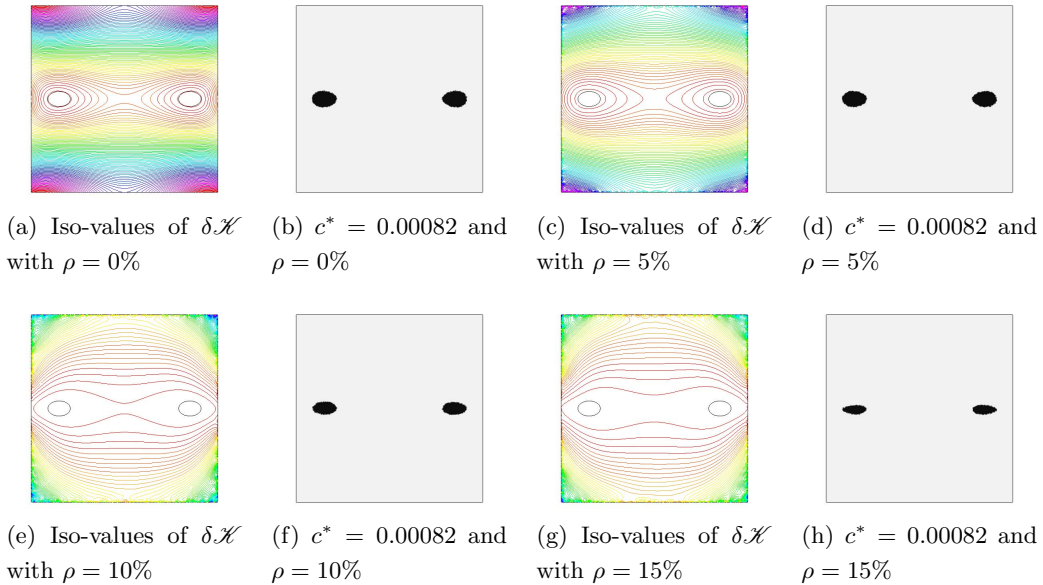


Figure 5.8: Reconstruction with noise data.

6. Conclusion

The presented paper concerns the detection of location and shape of a rigid inclusions inside an elastic body from a single pair Cauchy data in two dimensional case. We have proposed a non-iterative reconstruction algorithm based on the topological sensitivity analysis method combined with the so-called Kohn-Vogelius type functional. The main idea consists in reformulate the inverse problem as an optimization problem, where a Kohn-Vogelius functional is minimized in the class of admissible geometries inclusions. A topological sensitivity analysis is derived for the Kohn-Vogelius functional. The obtained

an asymptotic formula valid for any arbitrary geometric perturbations. From these results, a new non-iterative reconstruction algorithm has been devised. Particularly, we test the efficiency and accuracy of the proposed procedure by some parameters such as the number, the shape, the location, and the size of the inclusions.

7. Proofs

In this section, we prove Lemma 4.7. We start by presenting some definitions and preliminary lemmas used to prove this lemma.

7.1. Definitions and notations

Definition 7.1. Let \mathcal{O} be a bounded, connected, and open subset in \mathbb{R}^d ($d \in \{2, 3\}$) with a Lipschitz continuous boundary $\partial\mathcal{O}$.

- The Sobolev space $H^1(\mathcal{O})$ is equipped with the norm

$$\|\psi\|_{1,\mathcal{O}}^2 = \int_{\mathcal{O}} \sigma(\psi) : e(\psi) + \psi \cdot \psi \, dx,$$

which, due to the Korn’s inequality [36, Lemma 5.4.4], is equivalent to the usual norm. We will also need the semi-norm defined by

$$(7.1) \quad |\psi|_{1,\mathcal{O}}^2 = \int_{\mathcal{O}} \sigma(\psi) : e(\psi) \, dx.$$

Moreover, for a given function $\psi \in H^1(\mathcal{O})$, we define the function $\tilde{\psi}$ on $\tilde{\mathcal{O}} := \mathcal{O}/\varepsilon$ by

$$\tilde{\psi}(y) = \psi(x), \quad y = x/\varepsilon.$$

Using that $e_{ij}(\psi) = \frac{1}{2}(\frac{\partial\psi_i}{\partial x_j} + \frac{\partial\psi_j}{\partial x_i})$ and $\nabla_x \psi(x) = (\nabla_y \psi(y))/\varepsilon$, we obtain

$$|\psi|_{1,\mathcal{O}}^2 = \int_{\mathcal{O}} \sigma(\psi) : e(\psi) \, dx = \frac{1}{\varepsilon^2} \int_{\tilde{\mathcal{O}}} \sigma(\tilde{\psi}) : e(\tilde{\psi}) \varepsilon^2 \, dy.$$

Hence,

$$(7.2) \quad |\psi|_{1,\mathcal{O}} = |\tilde{\psi}|_{1,\tilde{\mathcal{O}}}.$$

Similarly, we have

$$(7.3) \quad \|\psi\|_{0,\mathcal{O}} = \varepsilon \|\tilde{\psi}\|_{0,\tilde{\mathcal{O}}}.$$

- The trace norm $\|\cdot\|_{1/2,\partial\mathcal{B}_{z,\varepsilon}}$ is defined by

$$\|g\|_{1/2,\partial\mathcal{B}_{z,\varepsilon}} = \inf \{ \|\psi\|_{H^1(\Omega \setminus \overline{\mathcal{B}_{z,\varepsilon}})}; \psi \in H^1(\mathcal{O} \setminus \overline{\mathcal{B}_{z,\varepsilon}}), \psi = g \text{ on } \partial\mathcal{B}_{z,\varepsilon} \}.$$

In the following definition, we recall the weighted Sobolev spaces and we introduce the weight function $\kappa(x) := (2 + |x|^2)^{1/2}$.

Definition 7.2. Let $1 < p < \infty$. For each real number ρ and each open set $\mathcal{O} \subset \mathbb{R}^d$, we set

$$L^p_\rho(\mathcal{O}) := \{\psi \in \mathcal{D}'(\mathcal{O}), \kappa^\rho \psi \in L^p(\mathcal{O})\},$$

$$W^{1,p}_\rho(\mathcal{O}) := \begin{cases} \{\psi \in \mathcal{D}'(\mathcal{O}), \psi \in L^p_{\rho-1}(\mathcal{O}), \nabla \psi \in L^p_\rho(\mathcal{O})\} & \text{if } \frac{d}{p} + \rho \neq 1, \\ \{\psi \in \mathcal{D}'(\mathcal{O}), \frac{\psi}{\log(\varepsilon)} \in L^p_{\rho-1}(\mathcal{O}), \nabla \psi \in L^p_\rho(\mathcal{O})\} & \text{if } \frac{d}{p} + \rho = 1. \end{cases}$$

We define now the space $\overset{\circ}{W}^{1,p}_\rho(\mathcal{O}) = \{\psi \in W^{1,p}_\rho(\mathcal{O}), \psi = 0 \text{ on } \partial\mathcal{O}\}$. The dual space of $\overset{\circ}{W}^{1,p}_\rho(\mathcal{O})$ is denoted by $W^{-1,p'}_\rho(\mathcal{O})$, where p' is such that $1/p + 1/p' = 1$ (it is a subspace of $\mathcal{D}'(\mathcal{O})$). Notice that $L^p_\rho(\mathcal{O})$ and $W^{1,p}_\rho(\mathcal{O})$ are reflexive Banach spaces with respect to the norms:

$$\|\psi\|_{L^p_\rho(\mathcal{O})} := \|\kappa^\rho \psi\|_{L^p(\mathcal{O})},$$

$$\|\psi\|_{W^{1,p}_\rho(\mathcal{O})} := \begin{cases} (\|\psi\|_{L^p_{\rho-1}(\mathcal{O})}^p + \|\nabla \psi\|_{L^p_\rho(\mathcal{O})}^p)^{1/p} & \text{if } \frac{d}{p} + \rho \neq 1, \\ (\|\frac{\psi}{\log(\kappa)}\|_{L^p_{\rho-1}(\mathcal{O})}^p + \|\nabla \psi\|_{L^p_\rho(\mathcal{O})}^p)^{1/p} & \text{if } \frac{d}{p} + \rho = 1. \end{cases}$$

We refer to [7] for more details about the previous Sobolev spaces.

Next, we recall a result concerning domains depending on a parameter.

7.2. Preliminary lemmas

Theorem 7.3. [47, Section 4.1.3, p. 214] *Let Ω and \mathcal{C} be two bounded simply connected domains of \mathbb{R}^d ($d \in \{2, 3\}$) of class $C^{0,1}$. Let $p \in (1, +\infty)$, $\varepsilon \in (0, 1/2)$ and $\mathcal{B}_{z,\varepsilon} = z + \varepsilon\mathcal{C}$. Let us assume that $\overline{\mathcal{B}_{z,\varepsilon}} \subset \Omega$ and that there exists a constant $c > 0$ depending only of d , p , \mathcal{C} and Ω such that $\text{dist}(\mathcal{B}_{z,\varepsilon}, \partial\Omega) > c\varepsilon$. Then*

$$\langle \cdot \rangle_{p, \partial\mathcal{B}_{z,\varepsilon}} \sim b(\varepsilon) \|\cdot\|_{L^p(\partial\mathcal{B}_{z,\varepsilon})} + [\cdot]_{p, \partial\mathcal{B}_{z,\varepsilon}}$$

where

$$\langle g \rangle_{p, \partial\mathcal{B}_{z,\varepsilon}} = \inf \left\{ \|\psi\|_{W^{1,p}(\Omega \setminus \overline{\mathcal{B}_{z,\varepsilon}})}; \psi \in W^{1,p}(\Omega \setminus \overline{\mathcal{B}_{z,\varepsilon}}), \psi = g \text{ on } \partial\mathcal{B}_{z,\varepsilon} \right\},$$

$$b(\varepsilon) = \begin{cases} \varepsilon^{(1-d)/p} \min(1, \varepsilon^{d/p-1}) & \text{if } p < d, \\ \varepsilon^{(1-d)/p} \min(1, |\log(\varepsilon)|^{(1-p)/p}) & \text{if } p = d, \\ \varepsilon^{(1-d)/p} & \text{if } p > d, \end{cases}$$

and

$$\begin{aligned}
 [g]_{1, \partial \mathcal{B}_{z, \varepsilon}} &= \frac{1}{|\partial \mathcal{B}_{z, \varepsilon}|} \iint_{\partial \mathcal{B}_{z, \varepsilon} \times \partial \mathcal{B}_{z, \varepsilon}} |g(x) - g(y)| \, ds(x) ds(y), \\
 [g]_{p, \partial \mathcal{B}_{z, \varepsilon}} &= \left(\iint_{\partial \mathcal{B}_{z, \varepsilon} \times \partial \mathcal{B}_{z, \varepsilon}} \frac{|g(x) - g(y)|^p}{|x - y|^{d+p-2}} \, ds(x) ds(y) \right)^{1/p} \quad \text{for } p \in (1, +\infty).
 \end{aligned}$$

The next lemma concerns the linear elasticity problem in the whole space \mathbb{R}^2 .

Lemma 7.4. *Let ψ be a solution of*

$$(7.4) \quad -\operatorname{div} \sigma(\psi) = 0 \quad \text{in } \mathbb{R}^2.$$

Then every solution which is a tempered distribution should be a polynomial.

Proof. Applying Fourier transform to (7.4) we immediately notice that the support of $\widehat{\psi}$ is contained in $\{0\}$. Therefore, those distributions should be a finite sum of Dirac deltas. By the inverse Fourier transform we deduce that ψ is polynomial. □

The next lemma gives a decomposition of the exterior linear elasticity problem.

Lemma 7.5. *Let \mathcal{C} be a Lipschitz open set of \mathbb{R}^2 and let $\gamma \in H^{1/2}(\partial \mathcal{C})$. We consider $\psi \in W_0^{1,2}(\mathbb{R}^2 \setminus \overline{\mathcal{C}})$ the solution of the exterior elasticity problem*

$$(7.5) \quad \begin{cases} -\operatorname{div} \sigma(\psi) = 0 & \text{in } \mathbb{R}^2 \setminus \overline{\mathcal{C}}, \\ \psi = \gamma & \text{on } \partial \mathcal{C}. \end{cases}$$

Then there exists a constant $\varrho \in \mathbb{R}^2$ such that $\psi = \varrho + W_{\mathcal{C}}$ with $W_{\mathcal{C}}(y) = O(1/r)$ and

$$|\varrho| \leq c \|\gamma\|_{1/2, \partial \mathcal{C}}$$

with $c > 0$ is a constant.

Proof. Let $\phi \in \mathcal{D}(\mathbb{R}^2)$, we have

$$\langle -\operatorname{div} \sigma(\psi), \phi \rangle_{\mathcal{D}, \mathcal{D}', \mathbb{R}^2} = \int_{\partial \mathcal{C}} [\sigma(\psi) \mathbf{n}] \cdot \phi \, ds.$$

Then,

$$-\operatorname{div} \sigma(\psi) = [\sigma(\psi) \mathbf{n}] \delta_{\partial \mathcal{C}} := T \quad \text{in } \mathcal{D}'(\mathbb{R}^2).$$

Now, let us define

$$u = G * T,$$

where $*$ denotes the convolution product and G is the fundamental solution of the elasticity problem given by (4.6). Therefore,

$$-\operatorname{div} \sigma(u) = T \quad \text{in } \mathcal{D}'(\mathbb{R}^2).$$

Now notice that the variable $w = \psi - u$ solve the problem (7.5), then by Lemma 7.4 the solution w of (7.5) should be a polynomial. Therefore,

$$\psi = G * T + \varrho = \int_{\partial\mathcal{E}} l(x)G(y - x) \, ds(x) + P,$$

where P is polynomial and $l(x) = \sigma(v)\mathbf{n}(x)$.

Using a Taylor development of G , we obtain

$$G(y - x) = G(y) - \nabla G(\xi(y, x))x,$$

where $\xi(y, x) = y - vx$ with $v \in (0, 1)$, then

$$\psi = G(y) \int_{\partial\mathcal{E}} l(x) \, ds(x) - \int_{\partial\mathcal{E}} l(x)\nabla G(\xi(y, x)) \, ds(x) + P.$$

On the other hand, we know that $\log \notin W_0^{1,2}(\mathbb{R}^2 \setminus \overline{\mathcal{E}})$, it follows that

$$\int_{\partial\mathcal{E}} l(x) \, ds(x) = 0.$$

Also, due to $P \in W_0^{1,2}(\mathbb{R}^2 \setminus \overline{\mathcal{E}})$, we must have

$$P = \varrho \text{ with } \varrho \text{ is a constant in } \mathbb{R}^2.$$

Therefore,

$$\psi(y) = \varrho - \int_{\partial\mathcal{E}} l(x)\nabla G(\xi(y, x)) \, ds(x) = \varrho + W_{\mathcal{E}}(y)$$

and ψ is bounded at infinity. Moreover, we have $W_{\mathcal{E}}(y) = O(1/r)$ (see [26]) and there exist $c > 0$ such that

$$|\varrho| \leq c\|\gamma\|_{1/2, \partial\mathcal{E}}. \quad \square$$

Lemma 7.6. *Let $\theta \in H^{1/2}(\partial\mathcal{E})$ such that $\int_{\partial\mathcal{E}} \theta \, ds = 0$, let $\psi \in W_0^{1,2}(\mathbb{R}^2 \setminus \overline{\mathcal{E}})$ be the solution to the problem*

$$\begin{cases} -\operatorname{div} \sigma(\psi) = 0 & \text{in } \mathbb{R}^2 \setminus \overline{\mathcal{E}}, \\ \psi = \theta & \text{on } \partial\mathcal{E}. \end{cases}$$

From the previous lemma ψ decomposed as follow $\psi = \rho + W_{\mathcal{E}}$. Then there exist a constant $c > 0$ and $\varepsilon_1 > 0$ such that for all $0 < \varepsilon < \varepsilon_1$:

$$(7.6) \quad \|W_{\mathcal{E}}\|_{0, \Omega_{R/\varepsilon}} \leq c\|\theta\|_{1/2, \partial\mathcal{E}} \quad \text{and} \quad |W_{\mathcal{E}}|_{1, \Omega_{R/\varepsilon}} \leq c\varepsilon^2\|\theta\|_{1/2, \partial\mathcal{E}}.$$

Proof. Following the same lines as the proof in [29, Lemma 4.1] we can prove (7.6). □

The next lemma is a particular case of Lemma 4.7.

Lemma 7.7. *Let $\varepsilon > 0$. For $\theta_N \in H^{-1/2}(\Sigma)$, $\theta_D \in H^{1/2}(\Gamma)$ and $\varrho \in \mathbb{R}^2$, let $v_\varepsilon \in H^1(\Omega \setminus \overline{\mathcal{B}_{z,\varepsilon}})$ be the solution of the linear elasticity problem*

$$(7.7) \quad \begin{cases} -\operatorname{div} \sigma(v_\varepsilon) = 0 & \text{in } \Omega \setminus \overline{\mathcal{B}_{z,\varepsilon}}, \\ \sigma(v_\varepsilon)\mathbf{n} = \theta_N & \text{on } \Sigma, \\ v_\varepsilon = \theta_D & \text{on } \Gamma, \\ v_\varepsilon = \varrho & \text{on } \partial\mathcal{B}_{z,\varepsilon}. \end{cases}$$

There exists a constant $c > 0$ (independent of ε) such that

$$\|v_\varepsilon\|_{1,\Omega \setminus \overline{\mathcal{B}_{z,\varepsilon}}} \leq c(\|\theta_N\|_{-1/2,\Sigma} + \|\theta_D\|_{1/2,\Gamma} + |\varrho|).$$

Proof. For simplicity, we denote by $\Omega_{z,\varepsilon} = \Omega \setminus \overline{\mathcal{B}_{z,\varepsilon}}$. Let $\varepsilon > 0$ and $v_\varepsilon \in H^1(\Omega_{z,\varepsilon})$ be the solution of the value problem (7.7). Let $Z_\varepsilon \in H^1(\Omega_{z,\varepsilon})$ be the solution of

$$(7.8) \quad \begin{cases} -\operatorname{div} \sigma(Z_\varepsilon) = 0 & \text{in } \Omega_{z,\varepsilon}, \\ \sigma(Z_\varepsilon)\mathbf{n} = 0 & \text{on } \Sigma, \\ Z_\varepsilon = \theta_D & \text{on } \Gamma, \\ Z_\varepsilon = \varrho & \text{on } \partial\mathcal{B}_{z,\varepsilon}. \end{cases}$$

Let \tilde{v}_ε and \tilde{Z}_ε be the respective extensions of v_ε and Z_ε to Ω by ϱ . From (7.7) and (7.8), we have

$$\int_{\Omega_{z,\varepsilon}} \sigma(v_\varepsilon - Z_\varepsilon) : e(\Phi) \, dx = \langle \theta_N, \Phi \rangle_{-1/2,1/2,\Sigma} \quad \text{for all } \Phi \in H^1_\Gamma(\Omega_{z,\varepsilon}),$$

where

$$H^1_\Gamma(\Omega_{z,\varepsilon}) = \{w \in H^1(\Omega_{z,\varepsilon}), w = 0 \text{ on } \Gamma \text{ and } w = \varrho \text{ on } \partial\mathcal{B}_{z,\varepsilon}\}.$$

By taking $\Phi = v_\varepsilon - Z_\varepsilon$ as test function, we obtain

$$|\tilde{v}_\varepsilon - \tilde{Z}_\varepsilon|_{1,\Omega}^2 = \langle \theta_N, v_\varepsilon - Z_\varepsilon \rangle_{-1/2,1/2,\Sigma}.$$

Then, from the trace theorem there exists a constant $c > 0$ (independent of ε) such that

$$|\tilde{v}_\varepsilon - \tilde{Z}_\varepsilon|_{1,\Omega}^2 \leq c\|\theta_N\|_{-1/2,\Sigma}\|v_\varepsilon - Z_\varepsilon\|_{1/2,\Sigma} \leq c\|\theta_N\|_{-1/2,\Sigma}\|\tilde{v}_\varepsilon - \tilde{Z}_\varepsilon\|_{1,\Omega}.$$

Moreover, using that $v_\varepsilon - Z_\varepsilon = 0$ on Γ and Korn's inequality we get

$$\|\tilde{v}_\varepsilon - \tilde{Z}_\varepsilon\|_{1,\Omega} \leq c|\tilde{v}_\varepsilon - \tilde{Z}_\varepsilon|_{1,\Omega}.$$

Hence,

$$\begin{aligned} \|v_\varepsilon - Z_\varepsilon\|_{1,\Omega_{z,\varepsilon}}^2 &= \|\tilde{v}_\varepsilon - \tilde{Z}_\varepsilon\|_{1,\Omega}^2 \leq c\|\theta_N\|_{-1/2,\Sigma}\|\tilde{v}_\varepsilon - \tilde{Z}_\varepsilon\|_{1,\Omega} \\ &\leq c\|\theta_N\|_{-1/2,\Sigma}\|v_\varepsilon - Z_\varepsilon\|_{1,\Omega_{z,\varepsilon}}. \end{aligned}$$

Therefore, we have

$$\|v_\varepsilon - Z_\varepsilon\|_{1,\Omega_{z,\varepsilon}} \leq c\|\theta_N\|_{-1/2,\Sigma}.$$

Now, we prove $\|Z_\varepsilon\|_{1,\Omega_{z,\varepsilon}} \leq c(\|\theta_D\|_{1/2,\Gamma} + |\varrho|)$. For a fixed $\varepsilon_0 > 0$, the problem (7.7) is well-posed and admits a unique solution $Z_\varepsilon \in H^1(\Omega_{z,\varepsilon_0})$ and there exists a constant $c > 0$ such that

$$\|Z_\varepsilon\|_{1,\Omega_{z,\varepsilon_0}} \leq c(\|\theta_D\|_{1/2,\Gamma} + \|\varrho\|_{1/2,\partial\mathcal{B}_{z,\varepsilon_0}}).$$

Using Theorem 7.3, we get that

$$\|\varrho\|_{1/2,\partial\mathcal{B}_{z,\varepsilon_0}} \sim \frac{1}{(\varepsilon_0(-\log(\varepsilon_0)))^{1/2}} \|\varrho\|_{L^2(\partial\mathcal{B}_{z,\varepsilon_0})} + [\varrho]_{1/2,\partial\mathcal{B}_{z,\varepsilon_0}}.$$

Using that ϱ is constant and by a change of variables, we have

$$\|\varrho\|_{1/2,\partial\mathcal{B}_{z,\varepsilon_0}} \sim \frac{1}{(\varepsilon_0(-\log(\varepsilon_0)))^{1/2}} \|\varrho\|_{L^2(\partial\mathcal{B}_{z,\varepsilon_0})} = \frac{1}{(-\log(\varepsilon_0))^{1/2}} \|\varrho\|_{L^2(\partial\mathcal{C})} = c|\varrho|$$

with c being a constant depend of ε_0 and $\partial\mathcal{C}$.

Then, one can deduce

$$(7.9) \quad \|Z_\varepsilon\|_{1,\Omega_{z,\varepsilon_0}} \leq c(\|\theta_D\|_{1/2,\Gamma} + |\varrho|).$$

Let $0 < \varepsilon_1 < \varepsilon_0$ such that $\Omega_{z,\varepsilon_0} \subset \Omega_{z,\varepsilon}$ for all $0 < \varepsilon < \varepsilon_1$. We denote by $\tilde{Z}_{\varepsilon_0}$ the extension of Z_{ε_0} to Ω by ϱ . Notice that the solution Z_ε of the problem (7.7) can be considered as the solution of the following minimization problem:

$$\min_{Z \in \Theta} \{ |Z|_{1,\Omega_{z,\varepsilon}} \},$$

where the functional space Θ is defined as

$$\Theta = \{ Z \in H^1(\Omega_{z,\varepsilon}), Z = \varrho \text{ on } \partial\mathcal{B}_{z,\varepsilon}, Z = \theta_D \text{ on } \Gamma \}.$$

Therefore, for all $0 < \varepsilon < \varepsilon_1$, we have

$$(7.10) \quad |Z_\varepsilon|_{1,\Omega_{z,\varepsilon}} \leq c|\tilde{Z}_{\varepsilon_0}|_{1,\Omega_{z,\varepsilon}} = c|Z_{\varepsilon_0}|_{1,\Omega_{z,\varepsilon_0}} \leq c\|Z_{\varepsilon_0}\|_{1,\Omega_{z,\varepsilon_0}}.$$

Inserting (7.9) into (7.10), we deduce

$$(7.11) \quad |Z_\varepsilon|_{1,\Omega_{z,\varepsilon}} \leq c(\|\theta_D\|_{1/2,\Gamma} + |\varrho|).$$

Using the Korn's inequality, it follows that

$$(7.12) \quad \begin{aligned} \|Z_\varepsilon\|_{0,\Omega_{z,\varepsilon}} &= \|\tilde{Z}_\varepsilon\|_{0,\Omega} \leq \|\tilde{Z}_\varepsilon - Z_0\|_{0,\Omega} + \|Z_0\|_{0,\Omega} \leq c\|\tilde{Z}_\varepsilon - Z_0\|_{1,\Omega} + \|Z_0\|_{0,\Omega} \\ &\leq c|\tilde{Z}_\varepsilon|_{1,\Omega} + c\|Z_0\|_{1,\Omega} \leq c(|Z_\varepsilon|_{1,\Omega_{z,\varepsilon}} + \|Z_0\|_{1,\Omega}). \end{aligned}$$

Notice that Z_0 satisfies

$$(7.13) \quad \begin{cases} -\operatorname{div} \sigma(Z_0) = 0 & \text{in } \Omega, \\ \sigma(Z_0)\mathbf{n} = 0 & \text{on } \Sigma, \\ Z_0 = \theta_D & \text{on } \Gamma. \end{cases}$$

From the weak formulation of the value problem (7.13) and we take Z_0 as test function, we deduce

$$(7.14) \quad \|Z_0\|_{1,\Omega} \leq c\|\theta_D\|_{1/2,\Gamma}.$$

Inserting (7.11) and (7.14) into (7.12), we obtain

$$(7.15) \quad \|Z_\varepsilon\|_{0,\Omega_{z,\varepsilon}} \leq c(\|\theta_D\|_{1/2,\Gamma} + |\varrho|).$$

Then, by combining the equalities (7.11) and (7.15), we get

$$\|Z_\varepsilon\|_{1,\Omega_{z,\varepsilon}} \leq c(\|\theta_D\|_{1/2,\Gamma} + |\varrho|).$$

Therefore, we finally get

$$\|v_\varepsilon\|_{1,\Omega_{z,\varepsilon}} \leq \|v_\varepsilon - Z_\varepsilon\|_{1,\Omega_{z,\varepsilon}} + \|Z_\varepsilon\|_{1,\Omega_{z,\varepsilon}} \leq c(\|\theta_N\|_{-1/2,\Sigma} + \|\theta_D\|_{1/2,\Gamma} + |\varrho|). \quad \square$$

Now, we are ready to prove Lemma 4.7.

7.3. Proof of Lemma 4.7

The proof of Lemma 4.7 is decomposed in the following two cases.

First case if θ is constant on $\partial\mathcal{B}_{z,\varepsilon}$, the previous lemma gives the desired result.

Second case if θ is not constant. Let Z the solution of

$$\begin{cases} -\operatorname{div} \sigma(Z) = 0 & \text{in } \mathbb{R}^2 \setminus \overline{\mathcal{B}_{z,\varepsilon}}, \\ Z = \theta & \text{on } \partial\mathcal{B}_{z,\varepsilon}. \end{cases}$$

From Lemma 7.5, we have $Z = \varrho + W_\varphi$ with $\varrho \in \mathbb{R}^2$ and $W_\varphi(y) = O(1/r)$. We define $w_\varepsilon = v_\varepsilon - W_\varphi\left(\frac{x-z}{\varepsilon}\right)$ with $y = (x-z)/\varepsilon$. Notice that w_ε satisfies

$$\begin{cases} -\operatorname{div} \sigma(w_\varepsilon) = 0 & \text{in } \Omega_{z,\varepsilon}, \\ \sigma(w_\varepsilon)\mathbf{n} = \theta_N - \sigma\left(W_\varphi\left(\frac{x-z}{\varepsilon}\right)\right)\mathbf{n} & \text{on } \Sigma, \\ w_\varepsilon = \theta_D - W_\varphi\left(\frac{x-z}{\varepsilon}\right) & \text{on } \Gamma, \\ w_\varepsilon = \varrho & \text{on } \partial\mathcal{B}_{z,\varepsilon}, \end{cases}$$

where $\Omega_{z,\varepsilon} = \Omega \setminus \overline{\mathcal{B}_{z,\varepsilon}}$. Using the previous lemma there exists a constant $c > 0$ independent of ε such that

$$\begin{aligned}
 & \|w_\varepsilon\|_{1,\Omega_{z,\varepsilon}} \\
 & \leq c \left(\left\| \theta_N - \sigma \left(W_\mathcal{G} \left(\frac{x-z}{\varepsilon} \right) \right) \mathbf{n} \right\|_{-1/2,\Sigma} + \left\| \theta_D - W_\mathcal{G} \left(\frac{x-z}{\varepsilon} \right) \right\|_{1/2,\Gamma} + |\varrho| \right) \\
 (7.16) \quad & \leq c \left(\|\theta_N\|_{-1/2,\Sigma} + \left\| \sigma \left(W_\mathcal{G} \left(\frac{x-z}{\varepsilon} \right) \right) \mathbf{n} \right\|_{-1/2,\Sigma} + \|\theta_D\|_{1/2,\Gamma} \right. \\
 & \quad \left. + \left\| W_\mathcal{G} \left(\frac{x-z}{\varepsilon} \right) \right\|_{1/2,\Gamma} + \|\theta\|_{1/2,\partial\mathcal{B}_{z,\varepsilon}} \right).
 \end{aligned}$$

Notice that we obtain, by an adaptation of the same technique used in (4.11):

$$\left\| \sigma \left(W_\mathcal{G} \left(\frac{x-z}{\varepsilon} \right) \right) \mathbf{n} \right\|_{-1/2,\Sigma} \leq c \left\| e(W_\mathcal{G}) \left(\frac{x-z}{\varepsilon} \right) \right\|_{0,\Omega_R},$$

where $\Omega_R = \Omega \setminus \overline{\mathcal{B}(z, R)}$ with $R > 0$ such that $\overline{\mathcal{B}_{z,\varepsilon}} \subset \overline{\mathcal{B}(z, R)} \subset \Omega$.

Using the change of variables $y = (x-z)/\varepsilon$, we get

$$\begin{aligned}
 \left\| e(W_\mathcal{G}) \left(\frac{x-z}{\varepsilon} \right) \right\|_{0,\Omega_R} &= \varepsilon \left\| e \left(W_\mathcal{G} \left(\frac{x-z}{\varepsilon} \right) \right) \right\|_{0,\Omega_R} \\
 &= \varepsilon^2 \|e(W_\mathcal{G})\|_{0,\Omega_R/\varepsilon} \leq c\varepsilon^2 |W_\mathcal{G}|_{1,\Omega_R/\varepsilon},
 \end{aligned}$$

and from Lemma 7.6, we have

$$(7.17) \quad \left\| \sigma \left(W_\mathcal{G} \left(\frac{x-z}{\varepsilon} \right) \right) \mathbf{n} \right\|_{-1/2,\Sigma} \leq c\varepsilon^4 \|\theta\|_{1/2,\partial\mathcal{G}}.$$

Now we estimate $\|W_\mathcal{G}(\frac{x-z}{\varepsilon})\|_{1/2,\Gamma}$. By trace theorem, we obtain

$$\begin{aligned}
 \left\| W_\mathcal{G} \left(\frac{x-z}{\varepsilon} \right) \right\|_{1/2,\Gamma} &\leq c \left\| W_\mathcal{G} \left(\frac{x-z}{\varepsilon} \right) \right\|_{1,\Omega_R} \\
 &\leq c \left(\left\| W_\mathcal{G} \left(\frac{x-z}{\varepsilon} \right) \right\|_{0,\Omega_R} + \left\| e(W_\mathcal{G}) \left(\frac{x-z}{\varepsilon} \right) \right\|_{0,\Omega_R} \right).
 \end{aligned}$$

Using the same argument as in (7.17) and we apply (7.2) and (7.3) we deduce that

$$(7.18) \quad \left\| W_\mathcal{G} \left(\frac{x-z}{\varepsilon} \right) \right\|_{1/2,\Gamma} \leq c(\varepsilon \|\theta(z + \varepsilon y)\|_{1/2,\partial\mathcal{G}} + \varepsilon^4 \|\theta(z + \varepsilon y)\|_{1/2,\partial\mathcal{G}}).$$

Combining (7.16), (7.17) and (7.18), we have

$$\begin{aligned}
 & \|w_\varepsilon\|_{1,\Omega_{z,\varepsilon}} \\
 & \leq c(\|\theta_N\|_{-1/2,\Sigma} + \|\theta_D\|_{1/2,\Gamma} + \varepsilon \|\theta(z + \varepsilon y)\|_{1/2,\partial\mathcal{G}} + \varepsilon^4 \|\theta(z + \varepsilon y)\|_{1/2,\partial\mathcal{G}} + \|\theta\|_{1/2,\partial\mathcal{B}_{z,\varepsilon}}).
 \end{aligned}$$

Then, for a small enough ε , we get

$$\|w_\varepsilon\|_{1,\Omega_{z,\varepsilon}} \leq c(\|\theta_N\|_{-1/2,\Sigma} + \|\theta_D\|_{1/2,\Gamma} + \|\theta\|_{1/2,\partial\mathcal{B}_{z,\varepsilon}}).$$

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