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Products of Composition, Multiplication and Iterated Differentiation Operators Between Banach Spaces of Holomorphic Functions

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Abstract. Let $H(\mathbb{D})$ denote the space of holomorphic functions on the unit disk $\mathbb{D}$ of $\mathbb{C}$, $\psi, \varphi \in H(\mathbb{D})$, $\varphi(\mathbb{D}) \subset \mathbb{D}$ and $n \in \mathbb{N} \cup \{0\}$. Let C_φ, M_ψ and D^n denote the composition, multiplication and iterated differentiation operators, respectively. To treat the operators induced by products of these operators in a unified manner, we introduce a sum operator $\sum_{j=0}^{n} M_{\psi_j} C_\varphi D^j$. We characterize the boundedness and compactness of this sum operator mapping from a large class of Banach spaces of holomorphic functions into the kth weighted-type space $\mathcal{W}_\mu^{(k)}$ (or $\mathcal{W}_{\mu,0}^{(k)}$), $k \in \mathbb{N} \cup \{0\}$, and give its estimates of norm and essential norm. Our results show that the boundedness and compactness of the sum operator depend only on the symbols and the norm of the point-evaluation functionals on the domain space. Our results cover many known results in the literature. Moreover, we introduce the order boundedness of the sum operator and turn its study into that of the boundedness and compactness.

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$, $H(\mathbb{D})$ the class of holomorphic functions on $\mathbb{D}$, $S(\mathbb{D})$ the class of holomorphic self-maps of $\mathbb{D}$, $\mathrm{Aut}(\mathbb{D})$ the group of disk automorphisms, $\mathbb{N}$ the set of all positive integers, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. A Banach space X contained in $H(\mathbb{D})$ is called a Banach space of holomorphic functions provided that the point-evaluation functional on X is continuous. We denote by B_X the closed unit ball of X.

Each $\varphi \in S(\mathbb{D})$ induces a composition operator C_φ defined by

$$C_\varphi f = f \circ \varphi, \quad f \in H(\mathbb{D}).$$

Each $\psi \in H(\mathbb{D})$ induces a multiplication operator M_ψ defined by

$$M_\psi f = \psi \cdot f, \quad f \in H(\mathbb{D}).$$

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The product $M_\psi C_\varphi$ of these two operators is known as the weighted composition operator which becomes the composition operator C_φ for $\psi \equiv 1$ and the multiplication operator M_ψ for $\varphi(z) = z$ for $z \in \mathbb{D}$. The weighted composition operators play an important role in the isometry theory of Banach spaces. The relationship between the operator-theoretic properties of C_φ or M_ψ and the function-theoretic properties of φ or ψ has been extensively studied over the past several decades. We refer to a standard reference [3] for various aspects on the theory of (weighted) composition operators acting on several spaces of holomorphic functions.

For $n \in \mathbb{N}_0$, the nth differentiation operator D^n is defined by

$$D^n f = f^{(n)}, \quad f \in H(\mathbb{D}),$$

where $f^{(0)} = f$. If $n = 1$, it is the classical differentiation operator D and typically unbounded on many familiar spaces of holomorphic functions. Denote by $D^n_{\psi,\varphi}$ the weighted differentiation composition operator $M_\psi C_\varphi D^n$, i.e.,

$$D^n_{\psi,\varphi} f = \psi \cdot f^{(n)} \circ \varphi, \quad f \in H(\mathbb{D}),$$

where $\psi \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. Note that we have: $M_\psi C_\varphi = D^0_{\psi,\varphi}$, $C_\varphi M_\psi = D^0_{\psi\circ\varphi,\varphi}$, $DC_\varphi = D^1_{\varphi',\varphi}$, $C_\varphi D = D^1_{1,\varphi}$ and $M_\psi D = D^1_{\psi,\mathrm{id}}$, where $\mathrm{id}(z) = z$, $z \in \mathbb{D}$. For more about the weighted differentiation composition operators, see [9, 16, 22].

The products of C_φ, M_ψ and D^n can be obtained in six ways, i.e., $M_\psi C_\varphi D^n$, $C_\varphi M_\psi D^n$, $M_\psi D^n C_\varphi$, $C_\varphi D^n M_\psi$, $D^n M_\psi C_\varphi$ and $D^n C_\varphi M_\psi$. Many authors studied these product-type operators separately, see, e.g., [7, 8]. In order to treat these operators in a unified manner, we introduce a sum operator $\sum_{j=0}^n D^j_{\psi_j,\varphi}$, denoted by $T_{\psi_{(n)},\varphi}$, i.e.,

$$T_{\psi_{(n)},\varphi} f = \sum_{j=0}^n \psi_j \cdot f^{(j)} \circ \varphi, \quad f \in H(\mathbb{D}),$$

where $\varphi \in S(\mathbb{D})$, $\psi_j \in H(\mathbb{D})$ and $\psi_{(n)}$ denotes the sequence $\{\psi_0, \psi_1, \ldots, \psi_n\}$. The sum operator for $n = 1$ has been studied in several papers, see [10–12, 17–20]. Recall that the Bell polynomial for $n, k \in \mathbb{N}_0$ is defined as

$$B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) = \sum \frac{n!}{\prod_{i=1}^{n-k+1} j_i!} \prod_{i=1}^{n-k+1} \left(\frac{x_i}{i!}\right)^{j_i},$$

where the sum is taken over all sequences $j_1, j_2, \ldots, j_{n-k+1}$ of non-negative integers satisfying $\sum_{i=1}^{n-k+1} j_i = k$ and $\sum_{i=1}^{n-k+1} i j_i = n$. For $\varphi \in S(\mathbb{D})$, we set

$$B^\varphi_{n,j} := B_{n,j}(\varphi', \varphi'', \ldots, \varphi^{(n-j+1)}).$$

Due to the following classical Faà di Bruno's formula

$$(f\circ\varphi)^{(n)}=\sum_{j=0}^{n}B_{n,j}^{\varphi}\cdot f^{(j)}\circ\varphi,\quad f\in H(\mathbb{D}),\ \varphi\in S(\mathbb{D}) \tag{1.1}$$

for $l,m,n\in\mathbb{N}_0$, we have that

$$D^lM_\psi D^mC_\varphi D^nf=\sum_{j=n}^{l+m+n}\left[\sum_{i=\max\{0,j-m-n\}}^{l}\binom{l}{i}\psi^{(l-i)}B_{m+i,j-n}^{\varphi}\right]f^{(j)}\circ\varphi$$

and

$$D^lC_\varphi D^mM_\psi D^nf=\sum_{j=n}^{l+m+n}\left[\sum_{i=\max\{0,j-m-n\}}^{l}\binom{m+i}{j-n}\psi^{(m+n+i-j)}\circ\varphi B_{l,i}^{\varphi}\right]f^{(j)}\circ\varphi.$$

As special cases of the above two identities or by direct calculations, we have

$$M_\psi C_\varphi D^nf=\psi\cdot f^{(n)}\circ\varphi,\quad C_\varphi M_\psi D^nf=\psi\circ\varphi\cdot f^{(n)}\circ\varphi,$$

$$M_\psi D^nC_\varphi f=\sum_{j=0}^{n}\psi B_{n,j}^{\varphi}\cdot f^{(j)}\circ\varphi,\quad C_\varphi D^nM_\psi f=\sum_{j=0}^{n}\binom{n}{j}\psi^{(n-j)}\circ\varphi\cdot f^{(j)}\circ\varphi,$$

$$D^nM_\psi C_\varphi f=\sum_{j=0}^{n}\left[\sum_{i=j}^{n}\binom{n}{i}\psi^{(n-i)}B_{i,j}^{\varphi}\right]f^{(j)}\circ\varphi,$$

$$D^nC_\varphi M_\psi f=\sum_{j=0}^{n}\left[\sum_{i=j}^{n}\binom{n}{i}(\psi\circ\varphi)^{(n-i)}B_{i,j}^{\varphi}\right]f^{(j)}\circ\varphi.$$

Recall that a positive continuous function on $\mathbb{D}$ is called a weight. Let μ be a weight and $k\in\mathbb{N}_0$. The kth weighted-type space on $\mathbb{D}$ (see [16]), denoted by $\mathcal{W}_\mu^{(k)}$, consists of all $f\in H(\mathbb{D})$ such that

$$b_{\mathcal{W}_\mu^{(k)}}(f):=\sup_{z\in\mathbb{D}}\mu(z)|f^{(k)}(z)|<\infty.$$

The quantity $b_{\mathcal{W}_\mu^{(k)}}(f)$ is a semi-norm on $\mathcal{W}_\mu^{(k)}$ and a norm on $\mathcal{W}_\mu^{(k)}/\mathcal{P}_{k-1}$, where $\mathcal{P}_{k-1}$ is the set of all polynomials whose degrees are less than or equal to $k-1$. Here, we identify the quotient spaces $\mathcal{W}_\mu^{(k)}/\mathcal{P}_{k-1}$ with the subspace of $\mathcal{W}_\mu^{(k)}$ which satisfies the condition that $f^{(i)}(0)=0$ for $i=0,1,\dots,k-1$. A natural norm on $\mathcal{W}_\mu^{(k)}$ is

$$\|f\|_{\mathcal{W}_\mu^{(k)}}=\sum_{i=0}^{k-1}|f^{(i)}(0)|+b_{\mathcal{W}_\mu^{(k)}}(f).$$

$\mathcal{W}_\mu^{(k)}$ becomes a Banach space with the norm above. The corresponding little kth weighted-type space, denoted by $\mathcal{W}_{\mu,0}^{(k)}$, is a closed subspace of $\mathcal{W}_\mu^{(k)}$ consisting of those f for which

$$\lim_{|z|\to1}\mu(z)|f^{(k)}(z)|=0.$$

It is well known that $\mathcal{W}_\mu^{(k)}$ and $\mathcal{W}_{\mu,0}^{(k)}$ are the weighted-type space H_μ^∞ and the little weighted-type space $H_{\mu,0}^\infty$, the Bloch-type space $\mathcal{B}_\mu$ and the little Bloch-type space $\mathcal{B}_{\mu,0}$, the Zygmund-type space $\mathcal{Z}_\mu$ and the little Zygmund-type space $\mathcal{Z}_{\mu,0}$ for $k=0$, $k=1$, $k=2$, respectively. For $\mu(z)=(1-|z|^2)^\alpha$, $\alpha>0$, the space H_μ^∞ is the growth space $\mathcal{A}^{-\alpha}$ (see [5]), $H_{\mu,0}^\infty$ is the closure $\mathcal{A}_0^{-\alpha}$ of the polynomials in $\mathcal{A}^{-\alpha}$, $\mathcal{B}_\mu$ is the α-Bloch space $\mathcal{B}_\alpha$, and $\mathcal{B}_{\mu,0}$ is the little α-Bloch space $\mathcal{B}_{\alpha,0}$ which is the closure of the polynomials in $\mathcal{B}_\alpha$ (see [21]). In particular, if $\alpha=1$, $\mathcal{B}_\alpha$ and $\mathcal{B}_{\alpha,0}$ are the classical Bloch space $\mathcal{B}$ and the little Bloch space $\mathcal{B}_0$. For $\mu(z)=(1-|z|^2)\log\frac{2}{1-|z|}$, $\mathcal{B}_\mu$ is the logarithmic Bloch space $\mathcal{B}_{\log}$ and $\mathcal{B}_{\mu,0}$ is its little version $\mathcal{B}_{\log,0}$. For $\mu\equiv 1$, H_μ^∞ becomes the space H^∞ of bounded holomorphic functions on $\mathbb{D}$ with norm usually denoted by $\|\cdot\|_\infty$.

Recall that for $1\le p<\infty$ and $-1<\alpha<\infty$, the Hardy space H^p and the weighted Bergman space A_α^p are Banach spaces defined as

$$H^p=\left\{f\in H(\mathbb{D}):\|f\|_{H^p}:=\left(\sup_{0<r<1}\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^p\,\mathrm{d}\theta\right)^{1/p}<\infty\right\}$$

and

$$A_\alpha^p=\left\{f\in H(\mathbb{D}):\|f\|_{A_\alpha^p}:=\left((\alpha+1)\int_{\mathbb{D}}|f(z)|^p(1-|z|^2)^\alpha\,\mathrm{d}A(z)\right)^{1/p}<\infty\right\},$$

respectively, where $\mathrm{d}A(z)$ denotes the normalized area measure on $\mathbb{D}$.

There is a vast literature on the multiplication, composition, differentiation, integration or weighted composition operators between specific holomorphic function spaces. Recently, much attention has been paid to the study of these operators acting from general classes of Banach spaces of holomorphic functions mapping into weighted-type or Bloch-type spaces (see [1,2,4,23]). Motivated by these work, we provide the following framework that unified the settings studied in several papers.

Let X be a Banach space of holomorphic functions on $\mathbb{D}$. For each $z\in\mathbb{D}$, denote by $K(z)$ (more precisely, $K_X(z)$) the norm of the point-evaluation functional at z on X, i.e.,

$$K(z):=\sup_{f\in B_X}|f(z)|.$$

Thus, for any function $f\in X$ and $z\in\mathbb{D}$,

$$|f(z)|\le K(z)\|f\|_X,$$

and K is bounded on compact subsets of $\mathbb{D}$ by the Uniform Boundedness Principle. The space X is *admissible* provided it satisfies the following conditions:

(I) X contains the polynomials;

(II) B_X is compact with respect to the compact-open topology *co*;

(III) there is a constant C such that for $S \in \mathrm{Aut}(\mathbb{D})$ and $f \in X$,

$$\|S \cdot f\|_X \leq C\|f\|_X;$$

(IV) for $j \in \mathbb{N}$, there exists a constant $C_j > 0$, such that for $f \in X$, $z \in \mathbb{D}$,

$$(1 - |z|^2)^j |f^{(j)}(z)| \leq C_j K(z) \|f\|_X.$$

An admissible space X is said to be *polynomial dense* if the set of polynomials is dense in X.

For example, the spaces H^∞, $\mathcal{A}^{-\alpha}$ $(\alpha > 0)$, H^p and A^p_α $(1 \leq p < \infty, -1 < \alpha < \infty)$ are admissible. Indeed, it is obvious that these spaces fulfill the conditions (I) and (III). Condition (II) is valid for these spaces using Montel's theorem and Fatou's lemma. It is well known that $K_{H^\infty} \equiv 1$, $K_{\mathcal{A}^{-\alpha}}(z) = (1 - |z|^2)^{-\alpha}$, $K_{H^p}(z) = (1 - |z|^2)^{-1/p}$ and $K_{A^p_\alpha}(z) = (1 - |z|^2)^{-(\alpha+2)/p}$ (see [4]). By [5,21] and Schwarz's lemma, (IV) holds for these spaces. Furthermore, H^p and A^p_α are also polynomial dense (see [5]).

Recall that the essential norm of an operator is its distance from the compact operators in the operator norm. More precisely, assume that X_1 and X_2 are Banach spaces and $T\colon X_1 \to X_2$ is a bounded operator, then the essential norm of T, denoted by $\|T\|_{e,X_1\to X_2}$, is defined as

$$\|T\|_{e,X_1\to X_2} = \inf_{K\colon X_1\to X_2 \text{ is compact}} \|T - K\|_{X_1\to X_2},$$

where $\|\cdot\|_{X_1\to X_2}$ denotes the operator norm. Obviously, T is compact if and only if $\|T\|_{e,X_1\to X_2} = 0$. For more about the essential norm, see [1,3,4,9,14].

In this paper, we characterize the boundedness and compactness of the operator $\sum_{j=0}^n D^j_{\psi_j,\varphi}$ mapping from an admissible space X into the space $\mathcal{W}^{(k)}_\mu$ (or $\mathcal{W}^{(k)}_{\mu,0}$) $(k \in \mathbb{N}_0)$. Our results show that the boundedness and compactness of the sum operator depend only on the symbols and the norm of the point-evaluation functionals on X. As a corollary, we obtain that the boundedness and compactness of $\sum_{j=0}^n D^j_{\psi_j,\varphi}$ is equivalent to that of all the $D^j_{\psi_j,\varphi}$, $j = 0, \ldots, n$ for $k = 0$. Moreover, we construct an explicit example showing that this equivalence is not expected for $k \in \mathbb{N}$. We also obtain the estimates of norm and essential norm of the sum operator. Since the sum operators and its domain and range spaces are very general, our main results cover many known results in the literature.

Recall that the order boundedness is a property of operators which is closely related to the notion of boundedness (see [6, 15]). For the notion of the order boundedness of $T\colon X \to \mathcal{W}^{(k)}_\mu$ (or $\mathcal{W}^{(k)}_{\mu,0}$), $k \in \mathbb{N}_0$, we introduce the following spaces. Let μ be a weight. We denote by BC_μ the space of all continuous functions on $\mathbb{D}$ such that

$$\|f\|_\mu = \sup_{z\in\mathbb{D}} \mu(z)|f(z)| < \infty.$$

BC_μ is a Banach space with the above norm. Moreover, we denote by $\mathrm{BC}_{\mu,0}$ the closed subspace of BC_μ consisting of those f for which

$$\lim_{|z|\to 1} \mu(z)|f(z)| = 0.$$

It is obvious that $H_\mu^\infty = \mathrm{BC}_\mu \cap H(\mathbb{D})$ and $H_{\mu,0}^\infty = \mathrm{BC}_{\mu,0} \cap H(\mathbb{D})$. Furthermore, $\|f\|_{H_\mu^\infty} = \|f\|_\mu$ for $f \in H_\mu^\infty$. Now we give the definition of order boundedness of $T\colon X \to \mathcal{W}_\mu^{(k)}$ and $T\colon X \to \mathcal{W}_{\mu,0}^{(k)}$ respectively. We say $T\colon X \to \mathcal{W}_\mu^{(k)}$ is order bounded if there exists an $h \in \mathrm{BC}_\mu$ such that $|(Tf)^{(k)}| \le h$ for all $f \in B_X$. Here and below the notation $f \le g$ means that $f(z) \le g(z)$ for all $z \in \mathbb{D}$. Notice that there are two kinds of order boundedness of $T\colon X \to \mathcal{W}_{\mu,0}^{(k)}$. We say that $T\colon X \to \mathcal{W}_{\mu,0}^{(k)}$ is (big) order bounded if there exists an $h \in \mathrm{BC}_\mu$ such that $|(Tf)^{(k)}| \le h$ for all $f \in B_X$ and that $T\colon X \to \mathcal{W}_{\mu,0}^{(k)}$ is (little) order bounded if there exists an $h \in \mathrm{BC}_{\mu,0}$ such that $|(Tf)^{(k)}| \le h$ for all $f \in B_X$. Our main result shows that the study of the order boundedness of the sum operator can be turned into that of the boundedness and compactness.

This paper is organized as follows. In Section 2, we give some notation and auxiliary results to be used in the sequel. In Section 3, we characterize the boundedness of the operator $T_{\psi_{(n)},\varphi}\colon X \to \mathcal{W}_\mu^{(k)}$ (or $\mathcal{W}_{\mu,0}^{(k)}$) for $k \in \mathbb{N}_0$ and give its norm estimate. In Section 4, we characterize its compactness and give its essential norm estimate. In Section 5, we study its order boundedness.

Constants. In the rest of the paper the letter C will be used to denote various positive constants which may vary at each occurrence but do not depend on the essential parameters. We use the notion $X \asymp Y$ for nonnegative quantities X and Y to mean $Y/C \le X \le CY$ for some inessential constant $C > 0$.

2. Preliminaries

2.1. Test functions

In most of the literature on characterizing the boundedness and compactness of the sum operators for $n = 1$, the choice of test functions is separate and needs a large amount of computation. Below we provide a systematical and simple method to do this. In this subsection, we suppose that $n, k \in \mathbb{N}_0$ and X satisfies (I) and (III).

Fix $\varepsilon > 0$. For $w \in \mathbb{D}$, choose $f_w \in B_X$ such that

$$|f_w(w)| > K(w) - \varepsilon. \tag{2.1}$$

For $w \in \mathbb{D}$ and $j \in \{0, 1, \ldots, n+k\}$, let

$$f_{w,j}(z) := \sum_{i=0}^{n+k} c_i^{(j)} \left(\frac{1-|w|^2}{1-\overline{w}z} \right)^{i+1} \tag{2.2}$$

and

$$g_{w,j} := f_{w,j} \cdot f_w, \tag{2.3}$$

where $\{c_0^{(j)}, c_1^{(j)}, \ldots, c_{n+k}^{(j)}\}$ is the unique solution of the $n+k+1$ linear equations

$$\sum_{i=0}^{n+k} \left(\prod_{m=1}^{l} (m+i) \right) c_i^{(j)} = \begin{cases} 1 & \text{if } l = j, \\ 0 & \text{if } l \in \{0, 1, \ldots, n+k\} \setminus \{j\} \end{cases}$$

whose determinant of coefficient matrix equals $\prod_{i=1}^{n+k} i! \neq 0$. Since for $i, l \in \{0, 1, \ldots, n+k\}$

$$\left(\left(\frac{1-|w|^2}{1-\overline{w}z} \right)^{i+1} \right)^{(l)} (w) = \frac{\overline{w}^l \prod_{m=1}^{l} (m+i)}{(1-|w|^2)^l},$$

it follows that for $j \in \{0, 1, \ldots, n+k\}$,

$$f_{w,j}^{(l)}(w) = \begin{cases} \frac{\overline{w}^j}{(1-|w|^2)^j} & \text{if } l = j, \\ 0 & \text{if } l \in \{0, 1, \ldots, n+k\} \setminus \{j\} \end{cases}$$

and

$$g_{w,j}^{(l)}(w) = \begin{cases} \binom{l}{j} \frac{\overline{w}^j}{(1-|w|^2)^j} f_w^{(l-j)}(w) & \text{if } l \in \{j, \ldots, n+k\}, \\ 0 & \text{if } l \in \{0, \ldots, j-1\}. \end{cases}$$

By (III) and the fact that $\frac{z-w}{1-\overline{w}z} \in \mathrm{Aut}(\mathbb{D})$, we have that for $f \in X$,

$$\begin{aligned} \left\| \frac{1-|w|^2}{1-\overline{w}z} \cdot f \right\|_X &= \left\| \left(1 + \overline{w} \frac{z-w}{1-\overline{w}z} \right) \cdot f \right\|_X \\ &\leq \|f\|_X + \left\| \left(\frac{z-w}{1-\overline{w}z} \right) \cdot f \right\|_X \leq C\|f\|_X. \end{aligned}$$

By induction, we have that for $i \in \{0, 1, \ldots, n+k\}$ and $f \in X$,

$$\left\| \left(\frac{1-|w|^2}{1-\overline{w}z} \right)^{i+1} \cdot f \right\|_X \leq C\|f\|_X.$$

Since each $c_i^{(j)}$ is independent of w, letting $f \equiv 1$ and $f = f_w$ in the above inequality respectively, we have that

$$\sup_{w \in \mathbb{D}} \|f_{w,j}\|_X < \infty \quad \text{and} \quad \sup_{w \in \mathbb{D}} \|g_{w,j}\|_X < \infty.$$

The test functions $f_{w,j}$ and $g_{w,j}$ will be used in Sections 3 and 4.

2.2. Compactness criterion

The following criterion for compactness follows easily from the standard arguments (see, e.g., [3, Proposition 3.11]). We omit its proof.

Lemma 2.1. *Suppose X and Y are Banach spaces of holomorphic functions such that X satisfies* (II) *and the identity map $I\colon (Y,\|\cdot\|_Y)\to (Y,co)$ is continuous. Then the operator $T_{\psi_{(n)},\varphi}\colon X\to Y$ is compact if and only if $T_{\psi_{(n)},\varphi}\colon X\to Y$ is bounded and for any bounded sequence $\{f_i\}_{i\in\mathbb{N}}$ in X which converges to zero uniformly on compact subsets of $\mathbb{D}$, we have $\|T_{\psi_{(n)},\varphi}f_i\|_Y\to 0$ as $i\to\infty$.*

It is obvious that $\mathcal{W}_\mu^{(k)}$, $\mathcal{W}_{\mu,0}^{(k)}$ and all the admissible spaces satisfy the above condition on Y.

The next lemma can be proved similarly as in [13, Lemma 1].

Lemma 2.2. *Assume μ is a weight and $k\in\mathbb{N}_0$. A set K in $\mathcal{W}_{\mu,0}^{(k)}$ is relatively compact if and only if it is bounded and satisfies*

$$\lim_{|z|\to 1}\sup_{f\in K}\mu(z)|f^{(k)}(z)|=0.$$

2.3. Notations

We also need the following notations. For any fixed $n,k\in\mathbb{N}_0$, $\psi_{(n)}=\{\psi_j\}_{j=0}^n$ and $\varphi\in S(\mathbb{D})$, set $\psi_{(n)}^{[k]}:=\Psi_{(n+k)}=\{\Psi_j\}_{j=0}^{n+k}$, where

$$\Psi_j:=\sum_{l=\max\{0,j-n\}}^{\min\{j,k\}}\sum_{i=l}^{k}\binom{k}{i}\psi_{j-l}^{(k-i)}B_{i,l}^{\varphi},\quad j\in\{0,\dots,n+k\}.$$

Noticing that $B_{0,0}^{\varphi}=1$, $B_{1,0}^{\varphi}=0$ and $B_{1,1}^{\varphi}=\varphi'$, we have that

$$\psi_{(n)}^{[0]}=\psi_{(n)}\quad\text{and}\quad \psi_{(n)}^{[1]}=\Psi_{(n+1)}=\{\Psi_j\}_{j=0}^{n+1},$$

where

$$\text{(when } k=1\text{)}\quad \Psi_j(z)=\begin{cases}\psi_0'(z) & \text{if } j=0,\\ \psi_{j-1}(z)\varphi'(z)+\psi_j'(z) & \text{if } j\in\{1,\dots,n\},\\ \psi_n(z)\varphi'(z) & \text{if } j=n+1.\end{cases}\tag{2.4}$$

Due to Faà di Bruno's formula (1.1), we have

$$(T_{\psi_{(n)},\varphi}f)^{(k)}=T_{\psi_{(n)}^{[k]},\varphi}f,\quad f\in H(\mathbb{D}).\tag{2.5}$$

By (2.5) and the fact that $(f^{(k-i)})^{(i)} = f^{(k)}$, we have for $i \in \{0, \dots, k\}$,

$$(T_{\psi_{(n)}^{[k-i]},\varphi} f)^{(i)} = T_{\psi_{(n)}^{[k]},\varphi} f, \quad f \in H(\mathbb{D}).$$

Then

$$(\psi_{(n)}^{[k-i]})^{[i]} = \psi_{(n)}^{[k]}. \tag{2.6}$$

Note that if X is admissible and $\Psi_{(n+k)} = \psi_{(n)}^{[k]}$, then by (2.5) and (IV), for $z \in \mathbb{D}$ and $f \in X$,

$$|(T_{\psi_{(n)},\varphi} f)^{(k)}(z)| \le \sum_{j=0}^{n+k} |\Psi_j(z)||f^{(j)}(\varphi(z))| \le C \sum_{j=0}^{n+k} \frac{|\Psi_j(z)|K(\varphi(z))}{(1-|\varphi(z)|^2)^j} \|f\|_X. \tag{2.7}$$

We make the following assumption on the mapping $T_{\psi_{(n)},\varphi} \colon X \to \mathcal{W}_\mu^{(k)}$ for the theorems and corollaries in Sections 3, 4 and 5.

Assumption. Unless otherwise specified, we will always assume that $n, k \in \mathbb{N}_0$, X is admissible, μ is a weight, $\psi_j \in H(\mathbb{D})$ $(j \in \{0, 1, \dots, n\})$, $\varphi \in S(\mathbb{D})$ and $\Psi_{(n+k)} = \psi_{(n)}^{[k]}$.

3. Boundedness and norm estimate

In this section, we characterize the boundedness of $T_{\psi_{(n)},\varphi} \colon X \to \mathcal{W}_\mu^{(k)}$ (or $\mathcal{W}_{\mu,0}^{(k)}$) for $k \in \mathbb{N}_0$ and give its norm estimate. The following is our first result.

Theorem 3.1. *The following are equivalent:*

(i) $T_{\psi_{(n)},\varphi} \colon X \to \mathcal{W}_\mu^{(k)}$ *is bounded;*

(ii) $M_j := \sup_{z \in \mathbb{D}} \dfrac{\mu(z)|\Psi_j(z)|K(\varphi(z))}{(1-|\varphi(z)|^2)^j} < \infty$ *for* $j \in \{0, \dots, n+k\}$.

Moreover, if $T_{\psi_{(n)},\varphi} \colon X \to \mathcal{W}_\mu^{(k)}/\mathcal{P}_{k-1}$ *is bounded, then*

$$\|T_{\psi_{(n)},\varphi}\|_{X \to \mathcal{W}_\mu^{(k)}/\mathcal{P}_{k-1}} \asymp \sum_{j=0}^{n+k} M_j. \tag{3.1}$$

Proof. (i) $\Rightarrow$ (ii). Suppose that $T_{\psi_{(n)},\varphi} \colon X \to \mathcal{W}_\mu^{(k)}$ is bounded. We first prove $\Psi_j \in H_\mu^\infty$ by induction. It follows from the condition (I) that $z^j \in X$ and

$$\begin{aligned} \sup_{z \in \mathbb{D}} \mu(z) \left| \sum_{i=0}^{j} \frac{j!}{(j-i)!} \Psi_i(z) \varphi^{j-i}(z) \right| &\le \|T_{\psi_{(n)},\varphi} z^j\|_{\mathcal{W}_\mu^{(k)}} \\ &\le C \|T_{\psi_{(n)},\varphi}\|_{X \to \mathcal{W}_\mu^{(k)}} \end{aligned} \tag{3.2}$$

for $j \in \{0, 1, \ldots, n+k\}$. When $j = 0$, (3.2) means $\sup_{z\in\mathbb{D}} \mu(z)|\Psi_0(z)| \le C\|T_{\psi_{(n)},\varphi}\|_{X\to\mathcal{W}_\mu^{(k)}}$. Fixing $j \in \{1, \ldots, n+k\}$, assume that

$$\sup_{z\in\mathbb{D}} \mu(z)|\Psi_i(z)| \le C\|T_{\psi_{(n)},\varphi}\|_{X\to\mathcal{W}_\mu^{(k)}} \tag{3.3}$$

holds for $i \in \{0, 1, \ldots, j-1\}$. Notice that $\|\varphi\|_\infty \le 1$, it follows from (3.2), (3.3) that

$$\begin{aligned}
\sup_{z\in\mathbb{D}} \mu(z)|\Psi_j(z)| &\le \sup_{z\in\mathbb{D}} \mu(z)\left|\sum_{i=0}^{j} \frac{1}{(j-i)!}\Psi_i(z)\varphi^{j-i}(z)\right| + \sup_{z\in\mathbb{D}} \mu(z)\left|\sum_{i=0}^{j-1} \frac{1}{(j-i)!}\Psi_i(z)\varphi^{j-i}(z)\right| \\
&\le \sup_{z\in\mathbb{D}} \mu(z)\left|\sum_{i=0}^{j} \frac{1}{(j-i)!}\Psi_i(z)\varphi^{j-i}(z)\right| + \sum_{i=0}^{j-1} \frac{1}{(j-i)!}\sup_{z\in\mathbb{D}} \mu(z)|\Psi_i(z)| \\
&\le C\|T_{\psi_{(n)},\varphi}\|_{X\to\mathcal{W}_\mu^{(k)}}.
\end{aligned}$$

That is, (3.3) holds for $i = j$. Thus, for $j \in \{0, 1, \ldots, n+k\}$, $\Psi_j \in H_\mu^\infty$ and

$$\sup_{z\in\mathbb{D}} \mu(z)|\Psi_j(z)| \le C\|T_{\psi_{(n)},\varphi}\|_{X\to\mathcal{W}_\mu^{(k)}}. \tag{3.4}$$

Now we prove that $M_j \le C\|T_{\psi_{(n)},\varphi}\|_{X\to\mathcal{W}_\mu^{(k)}}$ by reverse induction. We first deal with the case $j = n+k$. By (2.1), we have that

$$\begin{aligned}
&\frac{\mu(w)|\Psi_{n+k}(w)|K(\varphi(w))|\varphi(w)|^{n+k}}{(1-|\varphi(w)|^2)^{n+k}} \\
&\le \frac{\mu(w)|\Psi_{n+k}(w)||f_{\varphi(w)}(\varphi(w))||\varphi(w)|^{n+k}}{(1-|\varphi(w)|^2)^{n+k}} + \varepsilon\frac{\mu(w)|\Psi_{n+k}(w)||\varphi(w)|^{n+k}}{(1-|\varphi(w)|^2)^{n+k}} \\
&\le \|T_{\psi_{(n)},\varphi}g_{\varphi(w),n+k}\|_{\mathcal{W}_\mu^{(k)}} + \varepsilon\|T_{\psi_{(n)},\varphi}f_{\varphi(w),n+k}\|_{\mathcal{W}_\mu^{(k)}} \\
&\le C\|T_{\psi_{(n)},\varphi}\|_{X\to\mathcal{W}_\mu^{(k)}},
\end{aligned}$$

where $f_{\varphi(w),n+k}$ and $g_{\varphi(w),n+k}$ are defined by (2.2) and (2.3), respectively. Thus,

$$\sup_{|\varphi(w)|>1/2} \frac{\mu(w)|\Psi_{n+k}(w)|K(\varphi(w))}{(1-|\varphi(w)|^2)^{n+k}} \le C\|T_{\psi_{(n)},\varphi}\|_{X\to\mathcal{W}_\mu^{(k)}}. \tag{3.5}$$

It follows from the boundedness of K on compact subsets of $\mathbb{D}$ and (3.4) that

$$\sup_{|\varphi(w)|\le 1/2} \frac{\mu(w)|\Psi_{n+k}(w)|K(\varphi(w))}{(1-|\varphi(w)|^2)^{n+k}} \le C\|T_{\psi_{(n)},\varphi}\|_{X\to\mathcal{W}_\mu^{(k)}}. \tag{3.6}$$

By (3.5) and (3.6), we have that $M_{n+k} \le C\|T_{\psi_{(n)},\varphi}\|_{X\to\mathcal{W}_\mu^{(k)}}$.

Fix $j \in \{0, \dots, n+k-1\}$, assume that $M_l \le C\|T_{\psi_{(n)},\varphi}\|_{X\to\mathcal{W}_\mu^{(k)}}$ for $l \in \{j+1, \dots, n+k\}$. Then by (IV),

$$\begin{aligned}
&\frac{\mu(w)|\Psi_j(w)|K(\varphi(w))|\varphi(w)|^j}{(1-|\varphi(w)|^2)^j} \\
&\le \frac{\mu(w)|\Psi_j(w)||f_{\varphi(w)}(\varphi(w))||\varphi(w)|^j}{(1-|\varphi(w)|^2)^j} + \varepsilon\frac{\mu(w)|\Psi_j(w)||\varphi(w)|^j}{(1-|\varphi(w)|^2)^j} \\
&\le \mu(w)\left|\sum_{l=j}^{n+k}\Psi_l(w)\binom{l}{j}\frac{\overline{\varphi(w)}^j}{(1-|\varphi(w)|^2)^j}f_{\varphi(w)}^{(l-j)}(\varphi(w))\right| \\
&\quad + \mu(w)\left|\sum_{l=j+1}^{n+k}\Psi_l(w)\binom{l}{j}\frac{\overline{\varphi(w)}^j}{(1-|\varphi(w)|^2)^j}f_{\varphi(w)}^{(l-j)}(\varphi(w))\right| + \varepsilon\frac{\mu(w)|\Psi_j(w)||\varphi(w)|^j}{(1-|\varphi(w)|^2)^j} \\
&\le \|T_{\psi_{(n)},\varphi}g_{\varphi(w),j}\|_{\mathcal{W}_\mu^{(k)}} + C\sum_{l=j+1}^{n+k}\frac{\mu(w)|\Psi_l(w)|}{(1-|\varphi(w)|^2)^j}\frac{K(\varphi(w))}{(1-|\varphi(w)|^2)^{l-j}} + \varepsilon\|T_{\psi_{(n)},\varphi}f_{\varphi(w),j}\|_{\mathcal{W}_\mu^{(k)}} \\
&\le C\|T_{\psi_{(n)},\varphi}\|_{X\to\mathcal{W}_\mu^{(k)}} + C\sum_{l=j+1}^{n+k}M_l \le C\|T_{\psi_{(n)},\varphi}\|_{X\to\mathcal{W}_\mu^{(k)}},
\end{aligned}$$

where $f_{\varphi(w),j}$ and $g_{\varphi(w),j}$ are defined by (2.2) and (2.3), respectively. Similar to the case $j = n+k$, replacing $n+k$ by j in (3.5) and (3.6), we can also prove that $M_j \le C\|T_{\psi_{(n)},\varphi}\|_{X\to\mathcal{W}_\mu^{(k)}}$ for $j \in \{0, \dots, n+k-1\}$. Hence,

$$\sum_{j=0}^{n+k} M_j \le C\|T_{\psi_{(n)},\varphi}\|_{X\to\mathcal{W}_\mu^{(k)}}. \tag{3.7}$$

(ii) $\Rightarrow$ (i). Suppose that (ii) holds. Then (2.7) implies that

$$\sum_{i=0}^{k-1}|(T_{\psi_{(n)},\varphi}f)^{(i)}(0)| \le C$$

and

$$b_{\mathcal{W}_\mu^{(k)}}(T_{\psi_{(n)},\varphi}f) \le C\sum_{j=0}^{n+k}M_j \tag{3.8}$$

for $f \in B_X$. Hence, $T_{\psi_{(n)},\varphi}\colon X \to \mathcal{W}_\mu^{(k)}$ is bounded.

Moreover, (3.7) and (3.8) imply the desired estimate (3.1). □

For the case $k = 0$, we have the following corollary.

Corollary 3.2. *$T_{\psi_{(n)},\varphi}\colon X \to H_\mu^\infty$ is bounded if and only if each $D_{\psi_j,\varphi}^j\colon X \to H_\mu^\infty$ is bounded, $j = 0, 1, \dots, n$.*

Proof. Since $\psi_{(n)}^{[0]} = \psi_{(n)}$, we have $\Psi_j = \psi_j$ in the definition of M_j for $j \in \{0, 1, \ldots, n\}$. Fixing any $j \in \{0, 1, \ldots, n\}$, let $\psi_i \equiv 0$ for $i \in \{0, 1, \ldots, n\} \setminus \{j\}$. Then Theorem 3.1 implies that $D^j_{\psi_j,\varphi}\colon X \to H^\infty_\mu$ is bounded if and only if $M_j < \infty$. This completes the proof. □

Note that Corollary 3.2 does not hold for the case $k \in \mathbb{N}$, see Example 4.5. For Corollaries 3.4, 4.2 and 4.4, we have similar proofs and notes.

The following is the corresponding little version of Theorem 3.1.

Theorem 3.3. *Suppose that X is polynomial dense. If $T_{\psi_{(n)},\varphi}\colon X \to \mathcal{W}^{(k)}_\mu$ is bounded, then the following are equivalent:*

(i) $T_{\psi_{(n)},\varphi}\colon X \to \mathcal{W}^{(k)}_{\mu,0}$ *is bounded;*

(ii) $\Psi_j \in H^\infty_{\mu,0}$ *for* $j \in \{0, 1, \ldots, n+k\}$.

Moreover, if $T_{\psi_{(n)},\varphi}\colon X \to \mathcal{W}^{(k)}_{\mu,0}/\mathcal{P}_{k-1}$ is bounded, then

$$\|T_{\psi_{(n)},\varphi}\|_{X\to\mathcal{W}^{(k)}_{\mu,0}/\mathcal{P}_{k-1}} \asymp \sum_{j=0}^{n+k} M_j. \tag{3.9}$$

Proof. (i) ⇒ (ii). Suppose that $T_{\psi_{(n)},\varphi}\colon X \to \mathcal{W}^{(k)}_{\mu,0}$ is bounded. Then

$$\lim_{|z|\to 1} \mu(z) \left| \sum_{i=0}^{j} \frac{j!}{(j-i)!} \Psi_i(z)\varphi^{j-i}(z) \right| \le \lim_{|z|\to 1} \mu(z) \left| \left(T_{\psi_{(n)},\varphi} z^j\right)^{(k)} \right| = 0 \tag{3.10}$$

for $j \in \{0, 1, \ldots, n+k\}$. When $j = 0$, (3.10) means that $\Psi_0 \in H^\infty_{\mu,0}$. Fixing $j \in \{1, \ldots, n+k\}$, assume that $\Psi_i \in H^\infty_{\mu,0}$ holds for $i \in \{0, 1, \ldots, j-1\}$. It follows from (3.10) that

$$\begin{aligned}
&\lim_{|z|\to 1} \mu(z)|\Psi_j(z)| \\
&\le \lim_{|z|\to 1} \mu(z) \left| \sum_{i=0}^{j} \frac{1}{(j-i)!} \Psi_i(z)\varphi^{j-i}(z) \right| + \lim_{|z|\to 1} \mu(z) \left| \sum_{i=0}^{j-1} \frac{1}{(j-i)!} \Psi_i(z)\varphi^{j-i}(z) \right| \\
&\le \lim_{|z|\to 1} \mu(z) \left| \sum_{i=0}^{j} \frac{1}{(j-i)!} \Psi_i(z)\varphi^{j-i}(z) \right| + \sum_{i=0}^{j-1} \frac{1}{(j-i)!} \lim_{|z|\to 1} \mu(z)|\Psi_i(z)| = 0.
\end{aligned}$$

Thus, $\Psi_j \in H^\infty_{\mu,0}$ for $j \in \{0, 1, \ldots, n+k\}$.

(ii) ⇒ (i). Assume $\Psi_j \in H^\infty_{\mu,0}$ for $j \in \{0, 1, \ldots, n+k\}$. For each polynomial p,

$$\mu(z)|(T_{\psi_{(n)},\varphi} p)^{(k)}(z)| \le \sum_{j=0}^{n+k} \mu(z)|\Psi_j(z)| \|p^{(j)}\|_\infty,$$

which implies that $T_{\psi_{(n)},\varphi}p \in \mathcal{W}^{(k)}_{\mu,0}$. Since X is polynomial dense, for every $f \in X$ there is a sequence $\{p_i\}_{i\in\mathbb{N}}$ of polynomials such that $p_i \to f$ as $i \to \infty$. Hence $T_{\psi_{(n)},\varphi}p_i \to T_{\psi_{(n)},\varphi}f$ as $i \to \infty$ and then $T_{\psi_{(n)},\varphi}(X) \subset \mathcal{W}^{(k)}_{\mu,0}$.

The estimate (3.9) follows from (3.1). $\square$

Corollary 3.4. *Suppose that X is polynomial dense. Then $T_{\psi_{(n)},\varphi}\colon X \to H^\infty_{\mu,0}$ is bounded if and only if each $D^j_{\psi_j,\varphi}\colon X \to H^\infty_{\mu,0}$ is bounded, $j = 0, 1, \dots, n$.*

Remark 3.5. For simplicity, we suppose that X is admissible in this paper. However, we do not need the condition (II) in this section.

4. Compactness and essential norm estimate

In this paper, we will follow the convention that $\sup_{z\in\emptyset} f(z) = 0$ for any non-negative function $f \geq 0$. Thus, if $\|\varphi\|_\infty < 1$ and $f \geq 0$, then

$$\limsup_{|\varphi(z)|\to 1} f(z) = \lim_{\delta\to 1^-} \sup_{|\varphi(z)|>\delta} f(z) = 0. \tag{4.1}$$

This implies that for all $\varphi \in S(\mathbb{D})$ and $f \geq 0$, we have

$$\limsup_{|\varphi(z)|\to 1} f(z) \leq \limsup_{|z|\to 1} f(z). \tag{4.2}$$

In this section, we characterize the compactness of $T_{\psi_{(n)},\varphi}\colon X \to \mathcal{W}^{(k)}_\mu$ (or $\mathcal{W}^{(k)}_{\mu,0}$) for $k \in \mathbb{N}_0$ and give its essential norm estimate. The following is the big version characterization.

Theorem 4.1. *If $T_{\psi_{(n)},\varphi}\colon X \to \mathcal{W}^{(k)}_\mu$ is bounded, then the following are equivalent:*

(i) *$T_{\psi_{(n)},\varphi}\colon X \to \mathcal{W}^{(k)}_\mu$ is compact;*

(ii) $\displaystyle\lim_{|\varphi(z)|\to 1} \frac{\mu(z)|\Psi_j(z)|K(\varphi(z))}{(1-|\varphi(z)|^2)^j} = 0$ *for $j \in \{0, 1, \dots, n+k\}$.*

Proof. (i) $\Rightarrow$ (ii). Suppose that $T_{\psi_{(n)},\varphi}\colon X \to \mathcal{W}^{(k)}_\mu$ is compact. Consider a sequence $\{z_i\}_{i\in\mathbb{N}}$ satisfying the condition $|\varphi(z_i)| \to 1$ as $i \to \infty$. If such sequence does not exist, then (ii) obviously holds according to (4.1). Define $f_{\varphi(z_i),j}$ and $g_{\varphi(z_i),j}$ by (2.2) and (2.3), respectively. Then both $\{f_{\varphi(z_i),j}\}_{i\in\mathbb{N}}$ and $\{g_{\varphi(z_i),j}\}_{i\in\mathbb{N}}$ converge to zero uniformly on compacts of $\mathbb{D}$ as $i \to \infty$ for $j \in \{0, 1, \dots, n+k\}$. From Lemma 2.1 it follows that $\|T_{\psi_{(n)},\varphi}f_{\varphi(z_i),j}\|_{\mathcal{W}^{(k)}_\mu} \to 0$ and $\|T_{\psi_{(n)},\varphi}g_{\varphi(z_i),j}\|_{\mathcal{W}^{(k)}_\mu} \to 0$ as $i \to \infty$ for $j \in \{0, 1, \dots, n+k\}$.

Thus,

$$\begin{aligned}
&\frac{\mu(z_i)|\Psi_{n+k}(z_i)|K(\varphi(z_i))|\varphi(z_i)|^{n+k}}{(1-|\varphi(z_i)|^2)^{n+k}}\\
&\le \frac{\mu(z_i)|\Psi_{n+k}(z_i)||f_{\varphi(z_i)}(\varphi(z_i))||\varphi(z_i)|^{n+k}}{(1-|\varphi(z_i)|^2)^{n+k}}+\varepsilon\frac{\mu(z_i)|\Psi_{n+k}(z_i)||\varphi(z_i)|^{n+k}}{(1-|\varphi(z_i)|^2)^{n+k}}\\
&\le \|T_{\psi_{(n)},\varphi}g_{\varphi(z_i),n+k}\|_{\mathcal{W}_\mu^{(k)}}+\varepsilon\|T_{\psi_{(n)},\varphi}f_{\varphi(z_i),n+k}\|_{\mathcal{W}_\mu^{(k)}}\to 0
\end{aligned}$$

as $i\to\infty$, from which it follows that

$$\lim_{i\to\infty}\frac{\mu(z_i)|\Psi_j(z_i)|K(\varphi(z_i))}{(1-|\varphi(z_i)|^2)^j}=0 \tag{4.3}$$

for $j=n+k$. Similar to the proof of Theorem 3.1, we can prove by reverse induction that (4.3) holds for $j\in\{0,1,\dots,n+k\}$. From this it follows that (ii) holds.

(ii) $\Rightarrow$ (i). Now assume that (ii) holds. Then for every $\varepsilon>0$ there is an $r\in(0,1)$ such that when $r<|\varphi(z)|<1$,

$$\frac{\mu(z)|\Psi_j(z)|K(\varphi(z))}{(1-|\varphi(z)|^2)^j}<\frac{\varepsilon}{n+k+1} \tag{4.4}$$

for $j\in\{0,1,\dots,n+k\}$. Assume that $\{h_i\}_{i\in\mathbb{N}}$ converges to zero uniformly on compacts of $\mathbb{D}$ as $i\to\infty$ and $\sup_{i\in\mathbb{N}}\|h_i\|_X\le L$. It follows from the Weierstrass Theorem that $\{h_i^{(j)}\}_{i\in\mathbb{N}}$ also converges to zero uniformly on compacts of $\mathbb{D}$ as $i\to\infty$ for $j\in\{0,1,\dots,n+k\}$. Then by (2.7) and (4.4), for $r<|\varphi(z)|<1$,

$$\mu(z)|(T_{\psi_{(n)},\varphi}h_i)^{(k)}(z)|<CL\varepsilon. \tag{4.5}$$

If $|\varphi(z)|\le r$, by (2.7) and (3.4), we have

$$\mu(z)|(T_{\psi_{(n)},\varphi}h_i)^{(k)}(z)|\le C\sum_{j=0}^{n+k}\sup_{|w|\le r}|h_i^{(j)}(w)|\to 0 \tag{4.6}$$

as $i\to\infty$. From (4.5) and (4.6) it follows that

$$b_{\mathcal{W}_\mu^{(k)}}(T_{\psi_{(n)},\varphi}h_i)\to 0 \tag{4.7}$$

as $i\to\infty$. Since $\{\varphi(0)\}$ is a compact subset of $\mathbb{D}$, we have

$$\sum_{l=0}^{k-1}|(T_{\psi_{(n)},\varphi}h_i)^{(l)}(0)|\le C\sum_{l=0}^{n+k-1}|h_i^{(l)}(\varphi(0))|\to 0 \tag{4.8}$$

as $i\to\infty$. From (4.7) and (4.8) it follows that $\|T_{\psi_{(n)},\varphi}h_i\|_{\mathcal{W}_\mu^{(k)}}\to 0$ as $i\to\infty$, which completes the proof by Lemma 2.1. □

Corollary 4.2. $T_{\psi_{(n)},\varphi}\colon X \to H_\mu^\infty$ *is compact if and only if each* $D^j_{\psi_j,\varphi}\colon X \to H_\mu^\infty$ *is compact,* $j = 0, 1, \dots, n$.

Here is the corresponding little version of Theorem 4.1.

Theorem 4.3. *The following are equivalent:*

(i) $T_{\psi_{(n)},\varphi}\colon X \to \mathcal{W}^{(k)}_{\mu,0}$ *is compact;*

(ii) $\displaystyle\lim_{|z|\to 1} \frac{\mu(z)|\Psi_j(z)|K(\varphi(z))}{(1-|\varphi(z)|^2)^j} = 0$ *for* $j \in \{0, 1, \dots, n+k\}$.

Proof. (i) $\Rightarrow$ (ii). Suppose that $T_{\psi_{(n)},\varphi}\colon X \to \mathcal{W}^{(k)}_{\mu,0}$ is compact. By Theorem 4.1, for every $\varepsilon > 0$ there exists an $r \in (0,1)$ such that for $j \in \{0, 1, \dots, n+k\}$, whenever $r < |\varphi(z)| < 1$,

$$\frac{\mu(z)|\Psi_j(z)|K(\varphi(z))}{(1-|\varphi(z)|^2)^j} < \varepsilon. \tag{4.9}$$

By (i) $\Rightarrow$ (ii) of Theorem 3.3 (note that this implication does not need the polynomial density of X), we have $\Psi_j \in H^\infty_{\mu,0}$ for $j \in \{0, 1, \dots, n+k\}$. Then there exists a $\rho \in (0,1)$ such that for $j \in \{0, 1, \dots, n+k\}$, whenever $\rho < |z| < 1$ and $|\varphi(z)| \le r$, we have

$$\frac{\mu(z)|\Psi_j(z)|K(\varphi(z))}{(1-|\varphi(z)|^2)^j} \le C\mu(z)|\Psi_j(z)| < \varepsilon. \tag{4.10}$$

Inequalities (4.9) and (4.10) imply (ii) holds.

(ii) $\Rightarrow$ (i). Assume that (ii) holds. Then Theorem 3.1 implies that $T_{\psi_{(n)},\varphi}\colon X \to \mathcal{W}^{(k)}_{\mu}$ is bounded. It follows from (2.7) that $T_{\psi_{(n)},\varphi}(X) \subset \mathcal{W}^{(k)}_{\mu,0}$ and

$$\lim_{|z|\to 1} \sup_{f\in B_X} \mu(z)|(T_{\psi_{(n)},\varphi} f)^{(k)}(z)| \le \lim_{|z|\to 1} \sum_{j=0}^{n+k} \frac{\mu(z)|\Psi_j(z)|K(\varphi(z))}{(1-|\varphi(z)|^2)^j} = 0.$$

By Lemma 2.2, $T_{\psi_{(n)},\varphi}\colon X \to \mathcal{W}^{(k)}_{\mu,0}$ is compact. □

Corollary 4.4. $T_{\psi_{(n)},\varphi}\colon X \to H^\infty_{\mu,0}$ *is compact if and only if each* $D^j_{\psi_j,\varphi}\colon X \to H^\infty_{\mu,0}$ *is compact,* $j = 0, 1, \dots, n$.

The above corollaries in Sections 3 and 4 show that the boundedness (resp., compactness) of the sum operator is equivalent to the boundedness (resp., compactness) of all the summands for the case $k = 0$. However, this equivalence can not be expected for $k \in \mathbb{N}$. We construct an example for $k = 1$ as follows.

Example 4.5. $D^0_{\psi_0,\varphi}\colon H^1 \to \mathcal{B}_{1/2}$ and $D^1_{\psi_1,\varphi}\colon H^1 \to \mathcal{B}_{1/2}$ are unbounded but $D^0_{\psi_0,\varphi} + D^1_{\psi_1,\varphi}\colon H^1 \to \mathcal{B}_{1/2,0}$ is compact, where $\psi_0 \equiv -1$, $\psi_1(z) = \varphi(z) = \frac{1}{M}(1-z)^{1/2}\log(1-z)$ for $z \in \mathbb{D}$ and $M = 1 + \sup_{z\in\mathbb{D}}(1-z)^{1/2}\log(1-z) < \infty$.

Proof. Since $K_{H^1}(z) = (1-|z|^2)^{-1}$, by Theorems 3.1, 4.3 and (2.4), we have that

(i) $D^0_{\psi_0,\varphi}\colon H^1 \to \mathcal{B}_{1/2}$ is unbounded, since $\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^{1/2}|\psi_0(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^2} = \infty$.

(ii) $D^1_{\psi_1,\varphi}\colon H^1 \to \mathcal{B}_{1/2}$ is unbounded, since $\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^{1/2}|\psi_1'(z)|}{(1-|\varphi(z)|^2)^2} = \infty$.

(iii) $D^0_{\psi_0,\varphi} + D^1_{\psi_1,\varphi}\colon H^1 \to \mathcal{B}_{1/2,0}$ is compact, i.e., $\lim_{|z|\to 1} \frac{(1-|z|^2)^{1/2}|\psi_0'(z)|}{1-|\varphi(z)|^2} = 0$, $\lim_{|z|\to 1} \frac{(1-|z|^2)^{1/2}|\psi_0(z)\varphi'(z)+\psi_1'(z)|}{(1-|\varphi(z)|^2)^2} = 0$, $\lim_{|z|\to 1} \frac{(1-|z|^2)^{1/2}|\psi_1(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^3} = 0$.

The verification is easy and is left to the reader. □

Our characterizations of boundedness and compactness of $T_{\psi_{(n)},\varphi}\colon X \to \mathcal{W}^{(k)}_\mu$ (or $\mathcal{W}^{(k)}_{\mu,0}$) are in terms of $\psi^{[k]}_{(n)}$. By (2.6), $(\psi^{[k-i]}_{(n)})^{[i]} = \psi^{[k]}_{(n)}$. This shows that the boundedness and compactness of $T_{\psi^{[k-i]}_{(n)},\varphi}\colon X \to \mathcal{W}^{(i)}_{\mu,0}$ (or $\mathcal{W}^{(i)}_{\mu,0}$) have the same characterizations for any $i \in \{0,\dots,k-1\}$. Thus, we have the following corollary.

Corollary 4.6. (i) *$T_{\psi_{(n)},\varphi}\colon X \to \mathcal{W}^{(k)}_\mu$ is bounded (resp., compact) if and only if $T_{\psi^{[k-i]}_{(n)},\varphi}\colon X \to \mathcal{W}^{(i)}_\mu$ is bounded (resp., compact) for any $i \in \{0,\dots,k-1\}$.*

(ii) *$T_{\psi_{(n)},\varphi}\colon X \to \mathcal{W}^{(k)}_{\mu,0}$ is bounded if and only if $T_{\psi^{[k-i]}_{(n)},\varphi}\colon X \to \mathcal{W}^{(i)}_{\mu,0}$ is bounded for any $i \in \{0,\dots,k-1\}$ whenever X is polynomial dense.*

(iii) *$T_{\psi_{(n)},\varphi}\colon X \to \mathcal{W}^{(k)}_{\mu,0}$ is compact if and only if $T_{\psi^{[k-i]}_{(n)},\varphi}\colon X \to \mathcal{W}^{(i)}_{\mu,0}$ is compact for any $i \in \{0,\dots,k-1\}$.*

In order to study the essential norm of $T_{\psi_{(n)},\varphi}\colon X \to \mathcal{W}^{(k)}_\mu$ (or $\mathcal{W}^{(k)}_{\mu,0}$), we denote by C_r the composition operator $C_{r(id)}$ (i.e., $(C_r(f))(z) = f(rz)$, $z \in \mathbb{D}$) and introduce two additional conditions on X:

(V) C_r is compact on X for all $0 < r < 1$;

(VI) There exists $C > 0$ such that $K(rz) \le CK(z)$ for all $z \in \mathbb{D}$ and $0 < r < 1$.

Obviously, (VI) holds for the spaces H^∞, $\mathcal{A}^{-\alpha}$ ($\alpha > 0$), H^p and A^p_α ($1 \le p < \infty$, $-1 < \alpha < \infty$). It follows from Lemma 2.1 that (V) also holds for these spaces.

Now we give an essential norm estimate of $T_{\psi_{(n)},\varphi}\colon X \to \mathcal{W}^{(k)}_\mu$ (or $\mathcal{W}^{(k)}_{\mu,0}$).

Theorem 4.7. *Suppose that X satisfies* (V) *and* (VI)*.*

(i) *If $T_{\psi_{(n)},\varphi}\colon X \to \mathcal{W}^{(k)}_\mu$ is bounded, then*

$$\|T_{\psi_{(n)},\varphi}\|_{e,X\to\mathcal{W}^{(k)}_\mu} \asymp \sum_{j=0}^{n+k} \limsup_{|\varphi(z)|\to 1} \frac{\mu(z)|\Psi_j(z)|K(\varphi(z))}{(1-|\varphi(z)|^2)^j}.$$

(ii) *If* $T_{\psi_{(n)},\varphi}\colon X \to \mathcal{W}_{\mu,0}^{(k)}$ *is bounded, then*

$$\|T_{\psi_{(n)},\varphi}\|_{e,X\to\mathcal{W}_{\mu,0}^{(k)}} \asymp \sum_{j=0}^{n+k} \limsup_{|z|\to 1} \frac{\mu(z)|\Psi_j(z)|K(\varphi(z))}{(1-|\varphi(z)|^2)^j}.$$

Proof. (i). We begin by showing the upper estimate. Since X contains the constant functions, we have $K(z) \geq 1/\|1\| > 0$ for $z \in \mathbb{D}$. For any $|\varphi(0)| < \delta < 1$, let

$$r = 1 - \frac{K(\varphi(0))(1-\delta)^{n+k+2}}{\sup_{|w|\leq\delta} K(w)} \in (0,1).$$

By the mean value theorem and (IV), if $f \in B_X$ and $|\varphi(z)| \leq \delta$, then

$$\begin{aligned}
|((I-C_r)f)^{(j)}(\varphi(z))| &= |f^{(j)}(\varphi(z)) - r^j f^{(j)}(r\varphi(z))| \\
&\leq (1-r^j)|f^{(j)}(\varphi(z))| + |f^{(j)}(\varphi(z)) - f^{(j)}(r\varphi(z))| \\
&\leq (1-r^j)|f^{(j)}(\varphi(z))| + (1-r) \sup_{|w|\leq|\varphi(z)|} |f^{(j+1)}(w)| \\
&\leq C(1-r)\left[\frac{K(\varphi(z))}{(1-|\varphi(z)|^2)^j} + \sup_{|w|\leq|\varphi(z)|} \frac{K(w)}{(1-|w|^2)^{j+1}}\right] \\
&\leq C(1-r) \sup_{|w|\leq\delta} \frac{K(w)}{(1-\delta)^{j+1}} \leq C(1-\delta)
\end{aligned}$$

for $j \in \{0,1,\ldots,n+k\}$. By (V), $T_{\psi_{(n)},\varphi}C_r\colon X \to \mathcal{W}_\mu^{(k)}$ is compact. Thus, from (IV), (VI) and the fact that $\Psi_j \in H_\mu^\infty$ for $j \in \{0,1,\ldots,n+k\}$ implied by Theorem 3.1, it follows that

$$\begin{aligned}
\|T_{\psi_{(n)},\varphi}\|_{e,X\to\mathcal{W}_\mu^{(k)}} &\leq \|T_{\psi_{(n)},\varphi} - T_{\psi_{(n)},\varphi}C_r\|_{X\to\mathcal{W}_\mu^{(k)}} = \sup_{f\in B_X} \|T_{\psi_{(n)},\varphi}(I-C_r)f\|_{\mathcal{W}_\mu^{(k)}} \\
&\leq C \sum_{i=0}^{n+k-1} \sup_{f\in B_X} |((I-C_r)f)^{(i)}(\varphi(0))| \\
&\quad + \sum_{j=0}^{n+k} \sup_{f\in B_X} \sup_{|\varphi(z)|\leq\delta} \mu(z)|\Psi_j(z)||((I-C_r)f)^{(j)}(\varphi(z))| \\
&\quad + \sum_{j=0}^{n+k} \sup_{f\in B_X} \sup_{|\varphi(z)|>\delta} \mu(z)|\Psi_j(z)||((I-C_r)f)^{(j)}(\varphi(z))| \\
&\leq C(1-\delta) + \sum_{j=0}^{n+k} \sup_{f\in B_X} \sup_{|\varphi(z)|>\delta} \mu(z)|\Psi_j(z)|\big(|f^{(j)}(\varphi(z))| + |f^{(j)}(r\varphi(z))|\big) \\
&\leq C(1-\delta) + C\sum_{j=0}^{n+k} \sup_{|\varphi(z)|>\delta} \frac{\mu(z)|\Psi_j(z)|K(\varphi(z))}{(1-|\varphi(z)|^2)^j}.
\end{aligned}$$

Letting $\delta \to 1$, we get the desired upper estimate.

We next prove the lower estimate. Let $T\colon X \to \mathcal{W}_\mu^{(k)}$ be any compact operator. It follows from Lemma 2.1 that $\|Tf_{\varphi(z),j}\|_{\mathcal{W}_\mu^{(k)}} \to 0$ and $\|Tg_{\varphi(z),j}\|_{\mathcal{W}_\mu^{(k)}} \to 0$ as $|\varphi(z)| \to 1$ for $j \in \{0, 1, \ldots, n+k\}$, where $f_{\varphi(z),j}$ and $g_{\varphi(z),j}$ are defined by (2.2) and (2.3), respectively. Thus,

$$\begin{aligned}
&\|T_{\psi_{(n)},\varphi} - T\|_{X\to\mathcal{W}_\mu^{(k)}} \\
&\geq C \limsup_{|\varphi(z)|\to 1} \Big(\|(T_{\psi_{(n)},\varphi} - T) g_{\varphi(z),n+k}\|_{\mathcal{W}_\mu^{(k)}} + \varepsilon \|(T_{\psi_{(n)},\varphi} - T) f_{\varphi(z),n+k}\|_{\mathcal{W}_\mu^{(k)}} \Big) \\
&\geq C \limsup_{|\varphi(z)|\to 1} \Big(\|T_{\psi_{(n)},\varphi} g_{\varphi(z),n+k}\|_{\mathcal{W}_\mu^{(k)}} + \varepsilon \|T_{\psi_{(n)},\varphi} f_{\varphi(z),n+k}\|_{\mathcal{W}_\mu^{(k)}} \Big) \\
&\geq C \limsup_{|\varphi(z)|\to 1} \frac{\mu(z)|\Psi_{n+k}(z)|\big(|f_{\varphi(z)}(\varphi(z))| + \varepsilon\big)|\varphi(z)|^{n+k}}{(1-|\varphi(z)|^2)^{n+k}} \\
&\geq C \limsup_{|\varphi(z)|\to 1} \frac{\mu(z)|\Psi_{n+k}(z)|K(\varphi(z))}{(1-|\varphi(z)|^2)^{n+k}}.
\end{aligned}$$

Similar to the proof of Theorem 3.1, we can prove by reverse induction that

$$\|T_{\psi_{(n)},\varphi} - T\|_{X\to\mathcal{W}_\mu^{(k)}} \geq C \limsup_{|\varphi(z)|\to 1} \frac{\mu(z)|\Psi_j(z)|K(\varphi(z))}{(1-|\varphi(z)|^2)^j}$$

for $j \in \{0, 1, \ldots, n+k\}$. Adding them and taking the infimum over the set of all compact operators $T\colon X \to \mathcal{W}_\mu^{(k)}$, we obtain the desired lower estimate.

(ii). Assume that $\{z_i^{(j)}\}_{i\in\mathbb{N}}$, $j \in \{0, 1, \ldots, n+k\}$ are $n+k+1$ sequences in $\mathbb{D}$ such that

$$\sum_{j=0}^{n+k} \lim_{i\to\infty} \frac{\mu(z_i^{(j)})|\Psi_j(z_i^{(j)})|K(\varphi(z_i^{(j)}))}{\big(1-|\varphi(z_i^{(j)})|^2\big)^j} = \sum_{j=0}^{n+k} \limsup_{|z|\to 1} \frac{\mu(z)|\Psi_j(z)|K(\varphi(z))}{(1-|\varphi(z)|^2)^j}. \tag{4.11}$$

Let $\Gamma = \big\{j \in \{0, 1, \ldots, n+k\} : \sup_{i\in\mathbb{N}} |\varphi(z_i^{(j)})| = 1\big\}$. If $\Gamma = \emptyset$, in view of the fact that $\Psi_j \in H_{\mu,0}^\infty$ for $j \in \{0, 1, \ldots, n+k\}$, the above two quantities are zero, which implies by Theorem 4.3 that $T_{\psi_{(n)},\varphi}\colon X \to \mathcal{W}_{\mu,0}^{(k)}$ is compact and (ii) holds. If $\Gamma \neq \emptyset$, there are subsequences $\{z_{i_l^{(j)}}^{(j)}\}_{l\in\mathbb{N}}$ of $\{z_i^{(j)}\}_{i\in\mathbb{N}}$ such that $|\varphi(z_{i_l^{(j)}}^{(j)})| \to 1$ as $l \to \infty$ for $j \in \Gamma$. Then by (4.11), we have that

$$\begin{aligned}
\sum_{j=0}^{n+k} \limsup_{|z|\to 1} \frac{\mu(z)|\Psi_j(z)|K(\varphi(z))}{(1-|\varphi(z)|^2)^j} &= \sum_{j\in\Gamma} \lim_{l\to\infty} \frac{\mu(z_{i_l^{(j)}}^{(j)})|\Psi_j(z_{i_l^{(j)}}^{(j)})|K(\varphi(z_{i_l^{(j)}}^{(j)}))}{\big(1-|\varphi(z_{i_l^{(j)}}^{(j)})|^2\big)^j} \\
&\leq \sum_{j\in\Gamma} \limsup_{|\varphi(z)|\to 1} \frac{\mu(z)|\Psi_j(z)|K(\varphi(z))}{(1-|\varphi(z)|^2)^j} \\
&= \sum_{j=0}^{n+k} \limsup_{|\varphi(z)|\to 1} \frac{\mu(z)|\Psi_j(z)|K(\varphi(z))}{(1-|\varphi(z)|^2)^j}.
\end{aligned}$$

It follows from (4.2) that

$$\sum_{j=0}^{n+k}\limsup_{|z|\to 1}\frac{\mu(z)|\Psi_j(z)|K(\varphi(z))}{(1-|\varphi(z)|^2)^j}\geq \sum_{j=0}^{n+k}\limsup_{|\varphi(z)|\to 1}\frac{\mu(z)|\Psi_j(z)|K(\varphi(z))}{(1-|\varphi(z)|^2)^j}.$$

Hence,

$$\sum_{j=0}^{n+k}\limsup_{|z|\to 1}\frac{\mu(z)|\Psi_j(z)|K(\varphi(z))}{(1-|\varphi(z)|^2)^j}= \sum_{j=0}^{n+k}\limsup_{|\varphi(z)|\to 1}\frac{\mu(z)|\Psi_j(z)|K(\varphi(z))}{(1-|\varphi(z)|^2)^j}.$$

Similar to (i), we can prove that

$$\|T_{\psi_{(n)},\varphi}\|_{e,X\to\mathcal{W}^{(k)}_{\mu,0}}\asymp \sum_{j=0}^{n+k}\limsup_{|\varphi(z)|\to 1}\frac{\mu(z)|\Psi_j(z)|K(\varphi(z))}{(1-|\varphi(z)|^2)^j}.$$

Thus, (ii) follows. The proof is completed. □

Remark 4.8. For the case $k=0$ of Theorem 4.7, we do not need the condition (V). Indeed, by (VI), Theorems 3.1 and 4.1, we have that $T_{\psi_{(n)},r\varphi}\colon X\to H^\infty_\mu$ is compact. Thus,

$$\|T_{\psi_{(n)},\varphi}\|_{e,X\to H^\infty_\mu}\leq \|T_{\psi_{(n)},\varphi}-T_{\psi_{(n)},r\varphi}\|_{X\to H^\infty_\mu}.$$

The rest of proof is similar and is left to the reader.

5. Order boundedness

In this section, we characterize the order boundedness of $T_{\psi_{(n)},\varphi}\colon X\to\mathcal{W}^{(k)}_\mu$ (or $\mathcal{W}^{(k)}_{\mu,0}$). It is obvious that K is continuous in $\mathbb{D}$ for the spaces H^∞, $\mathcal{A}^{-\alpha}$ ($\alpha>0$), H^p and A^p_α ($1\leq p<\infty$, $-1<\alpha<\infty$).

Theorem 5.1. *Suppose that K is continuous in $\mathbb{D}$. Then*

(i) $T_{\psi_{(n)},\varphi}\colon X\to\mathcal{W}^{(k)}_\mu$ *is order bounded if and only if* $T_{\psi_{(n)},\varphi}\colon X\to\mathcal{W}^{(k)}_\mu$ *is bounded.*

(ii) $T_{\psi_{(n)},\varphi}\colon X\to\mathcal{W}^{(k)}_{\mu,0}$ *is (big) order bounded if and only if* $T_{\psi_{(n)},\varphi}\colon X\to\mathcal{W}^{(k)}_{\mu,0}$ *is bounded.*

(iii) $T_{\psi_{(n)},\varphi}\colon X\to\mathcal{W}^{(k)}_{\mu,0}$ *is (little) order bounded if and only if* $T_{\psi_{(n)},\varphi}\colon X\to\mathcal{W}^{(k)}_{\mu,0}$ *is compact.*

Proof. (i). Suppose that $T_{\psi_{(n)},\varphi}\colon X\to\mathcal{W}^{(k)}_\mu$ is order bounded. Then there exists $h\in \mathrm{BC}_\mu$ such that

$$|(T_{\psi_{(n)},\varphi}f)^{(k)}|\leq h \tag{5.1}$$

for $f \in B_X$. Multiplying two sides by $\mu(z)$ and taking the supremum over $\mathbb{D}$, we have that $b_{\mathcal{W}_\mu^{(k)}}(T_{\psi_{(n)},\varphi} f) \leq \|h\|_\mu$ for $f \in B_X$. It follows from (2.7) that $\sum_{i=0}^{k-1} |(T_{\psi_{(n)},\varphi} f)^{(i)}(0)| \leq C\|f\|_X$. Thus, $T_{\psi_{(n)},\varphi}\colon X \to \mathcal{W}_\mu^{(k)}$ is bounded with $\|T_{\psi_{(n)},\varphi}\|_{X \to \mathcal{W}_\mu^{(k)}} \leq C + \|h\|_\mu$.

Conversely, suppose that $T_{\psi_{(n)},\varphi}\colon X \to \mathcal{W}_\mu^{(k)}$ is bounded. Set

$$h(z) = C \sum_{j=0}^{n+k} \frac{|\Psi_j(z)| K(\varphi(z))}{(1-|\varphi(z)|^2)^j}, \tag{5.2}$$

where the constant C is taken from (2.7). Theorem 3.1 implies that $h \in \mathrm{BC}_\mu$. It follows from (2.7) that (5.1) holds for $f \in B_X$, i.e., $T_{\psi_{(n)},\varphi}\colon X \to \mathcal{W}_\mu^{(k)}$ is order bounded.

(ii). It is obvious that $T_{\psi_{(n)},\varphi}\colon X \to \mathcal{W}_{\mu,0}^{(k)}$ is bounded (resp., (big) order bounded) if and only if $T_{\psi_{(n)},\varphi}(X) \subset \mathcal{W}_{\mu,0}^{(k)}$ and $T_{\psi_{(n)},\varphi}\colon X \to \mathcal{W}_\mu^{(k)}$ is bounded (resp., order bounded). Thus, (ii) follows from (i).

(iii). Assume that $T_{\psi_{(n)},\varphi}\colon X \to \mathcal{W}_{\mu,0}^{(k)}$ is (little) order bounded. Then there exists $h \in \mathrm{BC}_{\mu,0}$ such that (5.1) holds for $f \in B_X$ and

$$\lim_{|z| \to 1} \sup_{f \in B_X} \mu(z) |(T_{\psi_{(n)},\varphi} f)^{(k)}(z)| \leq \lim_{|z| \to 1} \mu(z) h(z) = 0.$$

Since $T_{\psi_{(n)},\varphi}\colon X \to \mathcal{W}_\mu^{(k)}$ is order bounded, (i) implies that $T_{\psi_{(n)},\varphi}\colon X \to \mathcal{W}_\mu^{(k)}$ is bounded. By Lemma 2.2, $T_{\psi_{(n)},\varphi}\colon X \to \mathcal{W}_{\mu,0}^{(k)}$ is compact.

Conversely, suppose that $T_{\psi_{(n)},\varphi}\colon X \to \mathcal{W}_{\mu,0}^{(k)}$ is compact. Define h by (5.2). Theorem 4.3 implies that $h \in \mathrm{BC}_{\mu,0}$. It follows from (2.7) that (5.1) holds for $f \in B_X$, i.e., $T_{\psi_{(n)},\varphi}\colon X \to \mathcal{W}_{\mu,0}^{(k)}$ is (little) order bounded. □

Theorem 5.1 shows that for the operator $T_{\psi_{(n)},\varphi}\colon X \to \mathcal{W}_\mu^{(k)}$ (or $\mathcal{W}_{\mu,0}^{(k)}$), we can turn the study of the order boundedness into that of the boundedness and compactness.

References

[1] F. Colonna and M. Tjani, *Operator norms and essential norms of weighted composition operators between Banach spaces of analytic functions*, J. Math. Anal. Appl. **434** (2016), no. 1, 93–124.

[2] M. D. Contreras, J. A. Peláez, C. Pommerenke and J. Rättyä, *Integral operators mapping into the space of bounded analytic functions*, J. Funct. Anal. **271** (2016), no. 10, 2899–2943.

[3] C. C. Cowen and B. D. MacCluer, *Composition Operators on Spaces of Analytic Functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1995.

[4] T. Eklund, P. Galindo, M. Lindström and I. Nieminen, *Norm, essential norm and weak compactness of weighted composition operators between dual Banach spaces of analytic functions*, J. Math. Anal. Appl. **451** (2017), no. 1, 1–13.

[5] H. Hedenmalm, B. Korenblum and K. Zhu, *Theory of Bergman Spaces*, Graduate Texts in Mathematics **199**, Springer-Verlag, New York, 2000.

[6] H. Jarchow and J. Xiao, *Composition operators between Nevanlinna classes and Bergman spaces with weights*, J. Operator Theory **46** (2001), no. 3, suppl., 605–618.

[7] Z.-J. Jiang, *Product-type operators from logarithmic Bergman-type spaces to Zygmund-Orlicz spaces*, Mediterr. J. Math. **13** (2016), no. 6, 4639–4659.

[8] ______, *Product-type operators from Zygmund spaces to Bloch-Orlicz spaces*, Complex Var. Elliptic Equ. **62** (2017), no. 11, 1645–1664.

[9] Y.-X. Liang and Z.-H. Zhou, *New estimate of essential norm of composition followed by differentiation between Bloch-type spaces*, Banach J. Math. Anal. **8** (2014), no. 1, 118–137.

[10] Y. Liu, X. Liu and Y. Yu, *On an extension of Stević-Sharma operator from the mixed-norm space to weighted-type spaces*, Complex Var. Elliptic Equ. **62** (2017), no. 5, 670–694.

[11] Y. Liu and Y. Yu, *Products of composition, multiplication and radial derivative operators from logarithmic Bloch spaces to weighted-type spaces on the unit ball*, J. Math. Anal. Appl. **423** (2015), no. 1, 76–93.

[12] ______, *On an extension of Stević-Sharma operator from the general space to weighted-type spaces on the unit ball*, Complex Anal. Oper. Theory **11** (2017), no. 2, 261–288.

[13] K. Madigan and A. Matheson, *Compact composition operators on the Bloch space*, Trans. Amer. Math. Soc. **347** (1995), no. 7, 2679–2687.

[14] J. H. Shapiro, *The essential norm of a composition operator*, Ann. of Math. (2) **125** (1987), no. 2, 375–404.

[15] A. K. Sharma, *On order bounded weighted composition operators between Dirichlet spaces*, Positivity **21** (2017), no. 3, 1213–1221.

[16] S. Stević, *Weighted differentiation composition operators from H^∞ and Bloch spaces to nth weighted-type spaces on the unit disk*, Appl. Math. Comput. **216** (2010), no. 12, 3634–3641.

[17] S. Stević, A. K. Sharma and A. Bhat, *Products of multiplication composition and differentiation operators on weighted Bergman spaces*, Appl. Math. Comput. **217** (2011), no. 20, 8115–8125.

[18] Y. Yu and Y. Liu, *On Stević type operator from H^∞ space to the logarithmic Bloch spaces*, Complex Anal. Oper. Theory **9** (2015), no. 8, 1759–1780.

[19] F. Zhang and Y. Liu, *Products of multiplication, composition and differentiation operators from mixed-norm spaces to weighted-type spaces*, Taiwanese J. Math. **18** (2014), no. 6, 1927–1940.

[20] ______, *On a Stević-Sharma operator from Hardy spaces to Zygmund-type spaces on the unit disk*, Complex Anal. Oper. Theory **12** (2018), no. 1, 81–100.

[21] K. H. Zhu, *Bloch type spaces of analytic functions*, Rocky Mountain J. Math. **23** (1993), no. 3, 1143–1177.

[22] X. Zhu, *Products of differentiation, composition and multiplication from Bergman type spaces to Bers type spaces*, Integral Transforms Spec. Funct. **18** (2007), no. 3-4, 223–231.

[23] N. Zorboska, *Intrinsic operators from holomorphic function spaces to growth spaces*, Integral Equations Operator Theory **87** (2017), no. 4, 581–600.

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