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Almost Periodicity of All L^2-bounded Solutions of a Functional Heat Equation

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Abstract. In this paper, we continue the investigations done in the literature about the so called Bohr-Neugebauer property for almost periodic differential equations. More specifically, for a class of functional heat equations, we prove that each L^2-bounded solution is almost periodic. This extends a result in [5] to the delay case.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set with smooth boundary, τ be a positive constant and $\mathcal{C} = C([-\tau,0], L^2(\Omega,\mathbb{R}))$ denote the space of continuous functions $\varphi\colon [-\tau,0] \to L^2(\Omega,\mathbb{R})$ with the norm defined by $\|\varphi\| = \sup_{-\tau \leq \theta \leq 0} \|\varphi(\theta)\|_{L^2}$, here $\|\varphi(\theta)\|_{L^2} = \left(\int_\Omega \varphi^2(\theta,x)\,\mathrm{d}x\right)^{1/2}$ for $\theta \in [-\tau,0]$.

In this paper, we consider the boundary problem of partial functional differential equation

$$\begin{cases} \frac{\partial}{\partial t} u(t,x) = \Delta u + f(t,x,u_t) & \text{if } (t,x) \in \mathbb{R} \times \Omega, \\ u(t,x) = 0 & \text{if } (t,x) \in \mathbb{R} \times \partial\Omega, \end{cases} \tag{1.1}$$

where Δ is the Laplace operator acting on the variable $x \in \Omega$, $f\colon \mathbb{R} \times \overline{\Omega} \times \mathcal{C} \to \mathbb{R}$ is continuous, and the time delay function $u_t \in \mathcal{C}$ defined by $u_t(\theta)(\cdot) = u(t+\theta,\cdot) \in L^2(\Omega,\mathbb{R})$ for $\theta \in [-\tau,0]$.

There have been much research activity for the qualitative behavior of partial differential equations with or without delays, see, e.g., the references [1–3, 6, 8, 9, 13, 14]. It is worth mentioning that the authors in [4, 7, 11, 12, 15] studied the Bohr-Neugebauer property for some special abstract differential equations. A differential equation is said to has Bohr-Neugebauer property if its any bounded solution is almost periodic. This issue also

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occurred in Corduneanu's monograph [5, Chapter 7], where the author considered the following heat equation

$$(1.2)\qquad \begin{cases} \frac{\partial}{\partial t}u(t,x) = \Delta u + \widetilde{f}(t,x,u) & \text{if } (t,x) \in \mathbb{R}\times\Omega, \\ u(t,x) = 0 & \text{if } (t,x) \in \mathbb{R}\times\partial\Omega \end{cases}$$

and, under the assumption that $u(t,x)$ was a solution of (1.2) with the property

$$\sup_{t\in\mathbb{R}} \int_\Omega u^2(t,x)\,\mathrm{d}x < \infty,$$

obtained a conclusion that this L^2-bounded solution $u(t,x)$ was almost periodic.

The main objective of this paper is to extend the conclusion of (1.2) to (1.1). For this purpose, we assume that

(H1) $b \in C(\mathbb{R}\times\overline{\Omega},\mathbb{R})$ with $b(t,x) \ge 0$ for $(t,x) \in \mathbb{R}\times\overline{\Omega}$;

(H2) for any $\varphi_1, \varphi_2 \in \mathcal{C}$ and $(t,x) \in \mathbb{R}\times\overline{\Omega}$,

$$|f(t,x,\varphi_1) - f(t,x,\varphi_2)| \le b(t,x)\|\varphi_1 - \varphi_2\|;$$

(H3) $\lambda > 0$ is the smallest eigenvalue of the boundary-value problem [9, 10]

$$(1.3)\qquad \begin{cases} \Delta w + \lambda w = 0 & \text{if } x \in \Omega, \\ w = 0 & \text{if } x \in \partial\Omega; \end{cases}$$

(H4) $\left(\int_\Omega b^2(t,x)\,\mathrm{d}x\right)^{1/2} \le b_0 < \lambda$ for all $t \in \mathbb{R}$.

We remark that the existence of L^2-solutions of functional heat equations had been studied in monograph [13]. Next we consider only the almost periodicity of L^2-solutions of (1.1).

2. Main results

As usual, by $C([-\tau,0],\mathbb{R})$ we denote the Banach space of real-valued functions on $[-\tau,0]$ with supremum norm. In what follows, we will require an important conclusion, which extends the result in [5, Proposition 6.5].

Lemma 2.1. *Let $\psi\colon \mathbb{R}\to\mathbb{R}_+$ be bounded, differential and satisfy*

$$\psi'(t) \le \omega(\psi_t), \quad t \in \mathbb{R},$$

where $\psi_t \in C([-\tau,0],\mathbb{R})$ is defined by $\psi_t(\theta) = \psi(t+\theta)$ for $\theta \in [-\tau,0]$, $\omega\colon C([-\tau,0],\mathbb{R}) \to \mathbb{R}$ is continuous, and $\omega(\psi) < 0$ for $\|\psi\| > \mu > 0$. Then

$$\psi(t) \le \mu, \quad t \in \mathbb{R}.$$

Proof. We first consider the case that

$$\psi(t) \le \psi(t_M), \quad t \in \mathbb{R},$$

where t_M is some point in $\mathbb{R}$. That is, ψ obtains its maximum value at the point t_M. Then, we have

$$0 = \psi'(t_M) \le \omega(\psi_{t_M}),$$

which, together with the assumption $\omega(\psi) < 0$ for $\|\psi\| > \mu > 0$, results in

$$\mu \ge \|\psi_{t_M}\| = \psi(t_M) \ge \psi(t), \quad t \in \mathbb{R}.$$

In the case that $\limsup_{t\to\infty} \psi(t) = \sup\{\psi(t) : t \in \mathbb{R}\}$, we can choose a sequence $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$, such that $\lim_{n\to\infty} \psi(t_n) = \sup\{\psi(t) : t \in \mathbb{R}\}$, and

$$\psi'(t_n) \ge 0 \quad \text{for sufficiently large } n.$$

Similarly, by

$$0 \le \psi'(t_n) \le \omega(\psi_{t_n}) \quad \text{for sufficiently large } n,$$

we obtain that

$$\mu \ge \|\psi_{t_n}\| \ge \psi(t_n) \quad \text{for sufficiently large } n,$$

which means

$$\mu \ge \sup\{\psi(t) : t \in \mathbb{R}\}.$$

In case $\limsup_{t\to-\infty} \psi(t) = \sup\{\psi(t) : t \in \mathbb{R}\}$, we assert that

$$\sup\{\psi(t) : t \in \mathbb{R}\} \le \mu \quad \text{for } t \in \mathbb{R}. \tag{2.1}$$

Otherwise, there exists a $t_N < 0$ such that $\psi(t_N) > \mu$, which yields

$$\psi'(t_N) \le \omega(\psi_{t_N}) < 0$$

and leads to

$$\psi(t_N) \le \psi(t) \le \sup\{\psi(t) : t \in \mathbb{R}\} \quad \text{for } t \le t_N.$$

Now by the assumption on ω, we have $\omega(\psi_t) \le -m < 0$ for $t \le t_N$, and this, in combination with the assumption $\psi'(t) \le \omega(\psi_t)$, induces $\sup\{\psi(t) : t \in \mathbb{R}\} = \infty$, which conflicts with our assumption on ψ. In other words, the assertion (2.1) is true. The proof is complete. □

Referring to [5, Chapter 7], by an $L^2(\Omega)$-almost periodic function $f(t,x,\varphi)$ in t uniformly with respect to $\varphi \in \mathcal{C}$ we mean that, for each $\varepsilon > 0$, there exists a number $l = l(\varepsilon) > 0$ such that any interval $[\mu, \mu + l] \subset \mathbb{R}$ contains a point σ with the property

$$\int_\Omega |f(t+\sigma, x, \varphi) - f(t, x, \varphi)|^2 \,\mathrm{d}x < \varepsilon^2 \quad \text{for all } (t,\varphi) \in \mathbb{R} \times \mathcal{C}. \tag{2.2}$$

Theorem 2.2. *Suppose that $f(t,x,\varphi)$ is $L^2(\Omega)$-almost periodic in t uniformly with respect to $\varphi \in \mathcal{C}$. Then, under the assumptions* (H1)–(H4)*, each L^2-bounded solution $u(t,x)$ of* (1.1) *is almost periodic in the sense of mapping $t \in \mathbb{R} \to u(t,\cdot\,) \in L^2(\Omega,\mathbb{R})$.*

Proof. The proof is similar to that in [5, Theorem 7.5]. By the assumption on $f(t,x,\varphi)$, for each $\varepsilon > 0$, there exists an $l = l(\varepsilon) > 0$ such that any interval $[\mu, \mu + l] \subset \mathbb{R}$ contains a point σ with the property (2.2). For the fixed $\sigma \in \mathbb{R}$ we define

$$v(t,x) = u(t+\sigma,x) - u(t,x).$$

Then we have

$$\begin{cases} \frac{\partial}{\partial t} v(t,x) = \Delta v + f(t+\sigma,x,u_{t+\sigma}) - f(t,x,u_t) & \text{if } (t,x) \in \mathbb{R}\times\Omega, \\ v(t,x) = 0 & \text{if } (t,x) \in \mathbb{R}\times\partial\Omega. \end{cases} \tag{2.3}$$

Let

$$V(t) = \int_\Omega v^2(t,x)\,\mathrm{d}x, \quad t \in \mathbb{R}$$

and

$$\|V_t\| = \sup_{-\tau\le\theta\le 0} \int_\Omega |u(t+\sigma+\theta,x) - u(t+\theta,x)|^2\,\mathrm{d}x, \quad t \in \mathbb{R}.$$

Then

$$\sqrt{\|V_t\|} = \|u_{t+\sigma} - u_t\|,$$

and $V(t)$ is bounded on $\mathbb{R}$.

Now invoking (2.3), we get

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}V}{\mathrm{d}t} &= \int_\Omega v\frac{\partial v}{\partial t}\,\mathrm{d}x \\ &= \int_\Omega v\Delta v\,\mathrm{d}x + \int_\Omega v(f(t+\sigma,x,u_{t+\sigma}) - f(t,x,u_t))\,\mathrm{d}x. \end{aligned} \tag{2.4}$$

Note that, from Green's formula and Poincaré's inequality, it follows that

$$\lambda \int_\Omega v^2(t,x)\,\mathrm{d}x \le \int_\Omega |\operatorname{grad} v(t,x)|^2\,\mathrm{d}x = -\int_\Omega v\Delta v\,\mathrm{d}x,$$

where λ is the smallest eigenvalue of (1.3). Consequently, from (2.4) we derive

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}V}{\mathrm{d}t} \le -\lambda &\int_\Omega v^2(t,x)\,\mathrm{d}x \\ &+ \int_\Omega v(f(t+\sigma,x,u_{t+\sigma}) - f(t+\sigma,x,u_t))\,\mathrm{d}x \\ &+ \int_\Omega v(f(t+\sigma,x,u_t) - f(t,x,u_t))\,\mathrm{d}x. \end{aligned} \tag{2.5}$$

In addition, the Hölder inequality leads us to

$$\begin{aligned}
&\int_\Omega v(f(t+\sigma,x,u_{t+\sigma}) - f(t+\sigma,x,u_t))\,\mathrm{d}x \\
&\le \left(\int_\Omega v^2\,\mathrm{d}x\right)^{1/2}\left(\int_\Omega b^2(t+\sigma,x)\,\mathrm{d}x\right)^{1/2}\|u_{t+\sigma}-u_t\| \\
&\le b_0\left(\int_\Omega v^2\,\mathrm{d}x\right)^{1/2}\|u_{t+\sigma}-u_t\|
\end{aligned}$$

and

$$\begin{aligned}
&\int_\Omega v(f(t+\sigma,x,u_t) - f(t,x,u_t))\,\mathrm{d}x \\
&\le \left(\int_\Omega v^2\,\mathrm{d}x\right)^{1/2}\left(\int_\Omega |f(t+\sigma,x,u_t)-f(t,x,u_t)|^2\,\mathrm{d}x\right)^{1/2},
\end{aligned}$$

where for the first two inequalities we have imposed the assumptions (H2) and (H4), respectively. Hence, from (2.5) we obtain

$$\frac{1}{2}\frac{\mathrm{d}V}{\mathrm{d}t} \le -\lambda V + b_0\sqrt{V}\|u_{t+\sigma}-u_t\| + \varepsilon\sqrt{V}$$

and then

$$\frac{1}{2}\frac{\mathrm{d}V}{\mathrm{d}t} \le -\lambda V + b_0\|V_t\| + \varepsilon\sqrt{\|V_t\|}, \tag{2.6}$$

where we have used (2.2) for the first inequality and $\sqrt{V(t)} \le \sqrt{\|V_t\|} = \|u_{t+\sigma}-u_t\|$ for the second one. Since

$$-\lambda V + b_0\|V_t\| + \varepsilon\sqrt{\|V_t\|} \ge -\lambda\|V_t\| + b_0\|V_t\| + \varepsilon\sqrt{\|V_t\|},$$

we first consider

$$-\lambda\|V_t\| + b_0\|V_t\| + \varepsilon\sqrt{\|V_t\|} \ge 0$$

and get

$$\sqrt{\|V_t\|} \le \frac{\varepsilon}{\lambda - b_0}. \tag{2.7}$$

On the other hand, by the boundedness of $V(t)$ we have

$$V(t_M) = \sup\{V(t) : t\in\mathbb{R}\} \quad \text{for some } t_M \in \mathbb{R},$$

or

$$\limsup_{t\to\infty} V(t) = \sup\{V(t) : t\in\mathbb{R}\}, \quad \text{or} \quad \limsup_{t\to-\infty} V(t) = \sup\{V(t) : t\in\mathbb{R}\},$$

which induce

$$\{V : -\lambda V + b_0\|V_t\| + \varepsilon\sqrt{\|V_t\|} \geq 0\} = \{V : -\lambda\|V_t\| + b_0\|V_t\| + \varepsilon\sqrt{\|V_t\|} \geq 0\}.$$

Hence, by Lemma 2.1, (2.6) and (2.7), we learn that

$$V(t) \leq \left(\frac{\varepsilon}{\lambda - b_0}\right)^2, \quad t \in \mathbb{R},$$

namely,

$$\int_\Omega |u(t+\sigma, x) - u(t,x)|^2 \,\mathrm{d}x \leq \left(\frac{\varepsilon}{\lambda - b_0}\right)^2, \quad t \in \mathbb{R},$$

which shows that the L^2-bounded solution $u(t,x)$ of (1.1) is almost periodic. The proof is complete. □

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