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Chromatic Number and Orientations of Graphs and Signed Graphs

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Abstract. Assume D is a digraph, and D' is a spanning sub-digraph of D. We say D' is a modulo-k Eulerian sub-digraph of D if for each vertex v of D', $d^+_{D'}(v) \equiv d^-_{D'}(v) \pmod{k}$. A modulo-$k$ Eulerian sub-digraph D' of D is special if for every vertex v, $d^+_D(v) = 0$ implies $d^-_{D'}(v) = 0$ and $d^+_{D'}(v) = d^+_D(v) > 0$ implies $d^-_{D'}(v) > 0$. We denote by $\mathrm{OE}_k(D)$ or $\mathrm{EE}_k(D)$ (respectively, $\mathrm{OE}^s_k(D)$ or $\mathrm{EE}^s_k(D)$) the sets of spanning modulo-k Eulerian sub-digraphs (respectively, the sets of spanning special modulo-k Eulerian sub-digraphs) of D with an odd number or even number of edges. Matiyasevich [A criterion for vertex colorability of a graph stated in terms of edge orientations, (in Russia), Diskretnyi Analiz, issue 26, 65–71 (1974)] proved that a graph G is k-colourable if and only if G has an orientation D such that $|\mathrm{OE}_k(D)| \neq |\mathrm{EE}_k(D)|$. In this paper, we give another characterization of k-colourable graphs: a graph G is k-colourable if and only if G has an orientation D such that $|\mathrm{OE}^s_{k-1}(D)| \neq |\mathrm{EE}^s_{k-1}(D)|$. We extend the characterizations of k-colourable graphs to k-colourable signed graphs: If k is an even integer, then a signed graph (G,σ) is k-colourable if and only if G has an orientation D such that $|\mathrm{OE}_k(D)| \neq |\mathrm{EE}_k(D)|$; if k is an odd integer, then (G,σ) is k-colourable if and only if G has an orientation D such that $|\mathrm{OE}^s_{k-1}(D)| \neq |\mathrm{EE}^s_{k-1}(D)|$, where a (special) modulo-k Eulerian sub-digraph is even or odd if it has an even or odd number of positive edges. The characterization of k-colourable signed graphs for even k (respectively, for odd k) fails for odd k (respectively, for even k).

1. Introduction

Colourings and orientations of graphs are closely related to each other. In 1962, Minty [8] and Vitaver [12] proved that a graph G is k-colourable if and only if G has an orientation such that every cycle C of G contains at least $|C|/k$ edges in each of the two directions around the cycle. This result is generalized by Goddyn and Zhang [4] to circular colouring of graphs, who proved that a graph G is circular r-colourable if and only if G has an

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orientation such that every cycle C of G contains at least $|C|/r$ edges in each of the two directions around the cycle. The result of Minty and Vitaver is strengthened in another direction by Tuza in [11], where it was proved that it suffices to put this constraint on cycles of lengths $1 \pmod{k}$, and in [14], the result of Goddyn and Zhang is strengthened, where it was proved that for $r = k/d$ (with $(k, d) = 1$), it suffices to put this constraint on cycles of lengths $q \pmod{k}$ for $q \in \{1, 2, \ldots, 2d-1\}$. In 1966–67, Gallai [3] and Roy [10] proved that a graph G is k-colourable if and only if it has an acyclic orientation which contains no directed path of length k. A digraph D is called *Eulerian* if for each vertex v of D, $d_D^+(v) = d_D^-(v)$. For a digraph D, we denote by $\mathrm{OE}(D)$ (respectively, $\mathrm{EE}(D)$) the set of spanning Eulerian sub-digraphs of D with an odd (respectively, an even) number of edges. In 1992, Alon-Tarsi [1] proved that if a graph G has an orientation D with maximum out-degree $\Delta^+(D) < k$ and with $|\mathrm{EE}(D)| \neq |\mathrm{OE}(D)|$, then G is k-choosable. We call a digraph D *modulo-k Eulerian* if for each vertex v of D, $d_D^+(v) \equiv d_D^-(v) \pmod{k}$. For a digraph D, we denote by $\mathrm{OE}_k(D)$ (respectively, $\mathrm{EE}_k(D)$) the set of spanning modulo-k Eulerian sub-digraphs of D with an odd number (respectively, an even number) of edges. The following result was proved by Matiyasevich [7] in 1974.

Theorem 1.1. [7] *A graph G is k-colourable if and only if G has an orientation D such that* $|\mathrm{OE}_k(D)| \neq |\mathrm{EE}_k(D)|$.

A *signed graph* is a pair (G, σ), where G is a graph and $\sigma\colon E(G) \to \{1, -1\}$ is a *signature* of G which assigns to each edge e of G a sign $\sigma_e \in \{1, -1\}$. Colouring of signed graphs was first studied by Zaslavsky in the 1980's [13], and has attracted some recent attention [2,5,6,9]. There are a few definitions of k-colourings of signed graphs that are not equivalent. We adopt the definition given in [6]. For a positive integer k, if $k = 2q$ is even, then let $M_k = \{\pm 1, \pm 2, \ldots, \pm q\}$; if $k = 2q+1$ is odd, then let $M_k = \{0, \pm 1, \pm 2, \ldots, \pm q\}$. A *$k$-colouring* ϕ of G is a mapping $\phi\colon V(G) \to M_k$ such that for each edge $e = uv$, $\phi(u) \neq \sigma_e \phi(v)$. The *chromatic number* $\chi(G, \sigma)$ of a signed graph (G, σ) is the minimum integer k such that (G, σ) has a k-colouring.

Assume (G, σ) is a signed graph and D is an orientation of G. A modulo-k Eulerian sub-digraph of D is called *even* or *odd* if it has an even or odd number of positive edges. We denote by $\mathrm{OE}_k(D)$ and $\mathrm{EE}_k(D)$ the set of odd and even spanning modulo-k Eulerian sub-digraphs of D, respectively.

We first show that when k is an even integer, the conclusion of Theorem 1.1 holds also for colourability of signed graphs.

Theorem 1.2. *Assume $k = 2q$ is an even integer. A signed graph (G, σ) is k-colourable if and only if G has an orientation D such that* $|\mathrm{OE}_k(D)| \neq |\mathrm{EE}_k(D)|$.

Our proof of Theorem 1.2 also gives an alternate proof of Theorem 1.1, as when (G, σ)

has no negative edges, then the condition that k be even is not needed.

When k is odd, one direction of Theorem 1.2 still holds, but the other direction fails.

Theorem 1.3. *Assume $k = 2q + 1$ is an odd integer. If (G, σ) is k-colourable then G has an orientation D such that $|\operatorname{OE}_k(D)| \neq |\operatorname{EE}_k(D)|$. However, for any odd integer k, there is a signed graph (G, σ) which has an orientation D with $|\operatorname{OE}_k(D)| \neq |\operatorname{EE}_k(D)|$ and yet (G, σ) is not k-colourable.*

Assume D is a digraph and D' is a modulo-k Eulerian sub-digraph of D. We say D' is a *special modulo-k Eulerian sub-digraph* of D if the following hold:

- If $d^+_D(v) = 0$, then $d^-_{D'}(v) = 0$.
- If $d^+_D(v) = d^+_{D'}(v) > 0$, then $d^-_{D'}(v) > 0$.

We denote by $\operatorname{OE}^s_k(D)$ and $\operatorname{EE}^s_k(D)$ the set of odd and even spanning special modulo-k Eulerian sub-digraphs of D, respectively. We present a new characterization of k-colourable graphs.

Theorem 1.4. *A graph G is k-colourable if and only if G has an orientation D such that $|\operatorname{OE}^s_{k-1}(D)| \neq |\operatorname{EE}^s_{k-1}(D)|$.*

If k is an odd integer, then Theorem 1.4 holds for signed graphs.

Theorem 1.5. *Assume $k = 2q+1$ is an odd integer. A signed graph (G, σ) is k-colourable if and only if G has an orientation D such that $|\operatorname{OE}^s_{k-1}(D)| \neq |\operatorname{EE}^s_{k-1}(D)|$.*

When k is even, both directions of Theorem 1.5 fail.

Theorem 1.6. *For any even integer k, there is a signed graph (G, σ) which has an orientation D with $|\operatorname{OE}^s_{k-1}(D)| \neq |\operatorname{EE}^s_{k-1}(D)|$ and yet (G, σ) is not k-colourable, and there is also a signed graph (G, σ) which is k-colourable, but G has no orientation D with $|\operatorname{OE}^s_{k-1}(D)| \neq |\operatorname{EE}^s_{k-1}(D)|$.*

2. Proofs

The result of Alon-Tarsi [1] and the result of Matiyasevich [7] are both proved by using the graph polynomial. Our proof of Theorem 1.2 also uses graph polynomial. As the class of signed graphs contains unsigned graphs as special cases, our proof of Theorem 1.2 also provides an alternate proof of Theorem 1.1 (when the signed graph (G, σ) has no negative edges, i.e., it is an unsigned graph, then the condition that k be even is not needed). Indeed, we did not know Matiyasevich's result (it was published in Russian) when we proved it. We thank Professor Reza Naserasr who informed us Matiyasevich's paper [7] and sent us a copy of the English translation of the paper.

Proof of Theorem 1.2. Assume the vertex set of G is $V = \{v_1, v_2, \ldots, v_n\}$. The *graph polynomial* of (G, σ) is defined as

$$P_{G,\sigma}(x_1, x_2, \ldots, x_n) = \prod_{\substack{e=v_iv_j \in E(G) \\ i<j}} (x_i - \sigma_e x_j).$$

Observe that $P_{G,\sigma}(x_1, x_2, \ldots, x_n)$ is a homogeneous polynomial of degree $|E|$. Let $I = \{(i_1, i_2, \ldots, i_n) : i_j \geq 0, \sum_{j=1}^n i_j = |E(G)|\}$. Assume

$$P_{G,\sigma}(x_1, x_2, \ldots, x_n) = \sum_{(i_1, i_2, \ldots, i_n) \in I} c_{i_1, i_2, \ldots, i_n} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}.$$

For a non-negative integer a, let m_a, r_a be the unique non-negative integers such that

$$a = m_a k + r_a, \quad 0 \leq r_a \leq k-1.$$

Let

$$\overline{P}_{G,\sigma}(x_1, x_2, \ldots, x_n) = \sum_{(i_1, i_2, \ldots, i_n) \in I} c_{i_1, i_2, \ldots, i_n} x_1^{r_{i_1}} x_2^{r_{i_2}} \cdots x_n^{r_{i_n}}$$

and

$$Q(x_1, x_2, \ldots, x_n) = \sum_{(i_1, i_2, \ldots, i_n) \in I} c_{i_1, i_2, \ldots, i_n} x_1^{r_{i_1}} x_2^{r_{i_2}} \cdots x_n^{r_{i_n}} \left(x_1^{m_{i_1} k} x_2^{m_{i_2} k} \cdots x_n^{m_{i_n} k} - 1 \right).$$

Then

$$P_{G,\sigma}(x_1, x_2, \ldots, x_n) = \overline{P}_{G,\sigma}(x_1, x_2, \ldots, x_n) + Q(x_1, x_2, \ldots, x_n).$$

Let $\omega_1, \omega_2, \ldots, \omega_k$ be the k roots of the polynomial $x^k - 1$ in the complex field $\mathbb{C}$. For a polynomial $P(x_1, x_2, \ldots, x_n) \in \mathbb{C}[x_1, x_2, \ldots, x_n]$ and for $\phi \colon V(G) \to \{\omega_1, \omega_2, \ldots, \omega_k\}$, let $P(\phi) = P(\phi(v_1), \phi(v_2), \ldots, \phi(v_n))$. When k is even, then (G, σ) is k-colourable if and only if there is a mapping $\phi \colon V(G) \to \{\omega_1, \omega_2, \ldots, \omega_k\}$ such that $P_{G,\sigma}(\phi) \neq 0$. (If G is an ordinary graph, i.e., there is no negative edges, then the condition k be even is not needed.)

Observe that for any $\phi \colon V(G) \to \{\omega_1, \omega_2, \ldots, \omega_k\}$, $Q(\phi) = 0$. So G is k-colourable if and only if there is a mapping $\phi \colon V(G) \to \{\omega_1, \omega_2, \ldots, \omega_k\}$ such that $\overline{P}_{G,\sigma}(\phi) \neq 0$. Since in the polynomial $\overline{P}_{G,\sigma}(x_1, x_2, \ldots, x_n)$, each variable x_i has degree at most $k-1$, we conclude that either $\overline{P}_{G,\sigma}(x_1, x_2, \ldots, x_n)$ is a zero polynomial or there exists $\phi \colon V(G) \to \{\omega_1, \omega_2, \ldots, \omega_k\}$ such that $\overline{P}_{G,\sigma}(\phi) \neq 0$. Therefore G is k-colourable if and only if $\overline{P}_{G,\sigma}(x_1, x_2, \ldots, x_n)$ is not a zero polynomial.

Observe that in the expression $\overline{P}_{G,\sigma}(x_1, x_2, \ldots, x_n) = \sum_{(i_1, i_2, \ldots, i_n) \in I} c_{i_1, i_2, \ldots, i_n} x_1^{r_{i_1}} x_2^{r_{i_2}} \cdots x_n^{r_{i_n}}$, there may be like-terms not combined yet. This is so because there may be distinct vectors $(i_1, i_2, \ldots, i_n), (j_1, j_2, \ldots, j_n) \in I$ such that $(r_{i_1}, r_{i_2}, \ldots, r_{i_n}) = (r_{j_1}, r_{j_2}, \ldots, r_{j_n})$.

Let $I' = \{(s_1, s_2, \ldots, s_n) : 0 \leq s_j \leq k-1\}$. Assume (after combining like-terms)

$$\overline{P}_{G,\sigma}(x_1, x_2, \ldots, x_n) = \sum_{(s_1,s_2,\ldots,s_n)\in I'} c'_{s_1,s_2,\ldots,s_n} x_1^{s_1} x_2^{s_2} \cdots x_n^{s_n}.$$

It follows from the definition that

$$c'_{s_1,s_2,\ldots,s_n} = \sum_{\substack{(i_1,i_2,\ldots,i_n)\in I \\ (r_{i_1},r_{i_2},\ldots,r_{i_n})=(s_1,s_2,\ldots,s_n)}} c_{i_1,i_2,\ldots,i_n}.$$

The polynomial $\overline{P}_{G,\sigma}(x_1, x_2, \ldots, x_n)$ is not a zero polynomial if and only if for some $(s_1, s_2, \ldots, s_n) \in I'$, $c'_{s_1,s_2,\ldots,s_n} = \sum_{(i_1,i_2,\ldots,i_n)\in I,(r_{i_1},r_{i_2},\ldots,r_{i_n})=(s_1,s_2,\ldots,s_n)} c_{i_1,i_2,\ldots,i_n} \neq 0$.

The coefficients of monomials in the expansion of $P_{G,\sigma}(x_1, x_2, \ldots, x_n)$ can be expressed in terms of orientations of G. For an oriented edge $e = (v_i, v_j)$ of G, if $i < j$, then let $w(e) = x_i$; if $i > j$, then let $w(e) = -\sigma_e x_i$. For an orientation D of G, let $w(D) = \prod_{e\in E(D)} w(e)$. Then $P_{G,\sigma}(x_1, x_2, \ldots, x_n) = \sum w(D)$, where the summation is taken over all orientations D of G. We call an oriented edge (v_i, v_j) of G *decreasing* if $i > j$. An orientation D of G is called even if it has an even number of positive decreasing edges; otherwise, it is called odd.

An orientation D contributes 1 or -1 to the coefficient $c_{i_1,i_2,\ldots,i_n}$ if and only if D is an even or an odd orientation of G with $d^+_D(v_j) = i_j$ for $j = 1, 2, \ldots, n$. Thus D contributes 1 or -1 to $c'_{s_1,s_2,\ldots,s_n}$ if and only if D is an even or an odd orientation of G with $r_{d^+_D(v_j)} = s_j$ for $j = 1, 2, \ldots, n$.

Assume D is an orientation with $r_{d^+_D(v_j)} = s_j$ for $j = 1, 2, \ldots, n$. Hence D makes a contribution to the coefficient $c'_{s_1,s_2,\ldots,s_n}$. By symmetry, we may assume the contribution is 1, i.e., D is an even orientation. If D' is another orientation which also makes a contribution to the coefficient $c'_{s_1,s_2,\ldots,s_n}$, then for any vertex v_j of G, $d^+_{D'}(v_j) \equiv d^+_D(v_j) \pmod{k}$. Hence the edge set

$$E' = \{e : D \text{ and } D' \text{ orient } e \text{ in opposite directions}\}$$

induces a modulo-k Eulerian sub-digraph of D.

Conversely, if E' is a subset of $E(D)$ which induces a modulo-k Eulerian sub-digraph of D, and D' is obtained from D by reversing the directions of edges in E', then D' makes a contributes to $c'_{s_1,s_2,\ldots,s_n}$. Moreover, if $|E'|$ contains an even number of positive edges, then the contribution of D' to $c'_{s_1,s_2,\ldots,s_n}$ is 1, and if $|E'|$ is contains an odd number of positive edges, then the contribution of D' to $c'_{s_1,s_2,\ldots,s_n}$ is -1. Therefore

$$c'_{s_1,s_2,\ldots,s_n} = |\operatorname{EE}_k(D)| - |\operatorname{OE}_k(D)|.$$

So $\overline{P}_{G,\sigma}$ is not a zero polynomial if and only if there is an orientation D of G such that $|\operatorname{EE}_k(D)| - |\operatorname{OE}_k(D)| \neq 0$. □

Assume D is a digraph, k is a positive integer and $f\colon V(G) \to \mathbb{Z}_k$ is a mapping. We say D is a *modulo-k f-sub-digraph* if for every vertex v, $d^+_D(v) - d^-_D(v) \equiv f(v) \pmod{k}$. For an orientation D of a signed graph (G,σ), a modulo-k f-sub-digraphs D' of D is odd or even if D' has an odd number or an even number of positive edges.

Corollary 2.1. *Assume (G,σ) is a signed graph, and k is an even positive integer, and D is an orientation of G. Then (G,σ) is k-colourable if and only if there is a mapping $f\colon V(G) \to \mathbb{Z}_k$ for which $|\operatorname{EE}^f_k(D)| \neq |\operatorname{OE}^f_k(D)|$.*

Proof. Assume E' is the edge set of a modulo-k f-sub-digraph of D, and let D' be the obtained from D by reserving the orientations of the edges in E'. It is easy to verify that if $|E'|$ is even, then $\operatorname{EE}_k(D') = \operatorname{EE}^f_k(D)$ and $\operatorname{OE}_k(D') = \operatorname{OE}^f_k(D)$; if $|E'|$ is odd, then $\operatorname{EE}_k(D') = \operatorname{OE}^f_k(D)$ and $\operatorname{OE}_k(D') = \operatorname{EE}^f_k(D)$. So the conclusion follows from Theorem 1.2. $\square$

Proof of Theorem 1.3. Assume $k = 2q+1$ is a positive odd integer. Let $\omega_1, \omega_2, \ldots, \omega_k$ be the k roots of $x^k - 1$. We may assume that $\omega_1 = 1$ and for $i = 1,2,\ldots,q$, $\omega_{2i} = -\omega_{2i+1}$.

By the proof of Theorem 1.2, there is an orientation D of G with $|\operatorname{OE}_k(D)| \neq |\operatorname{EE}_k(D)|$ if and only if there is a mapping $\phi\colon V(G) \to \{\omega_1, \omega_2, \ldots, \omega_k\}$ such that $P_{G,\sigma}(\phi) \neq 0$. Assume $\psi\colon V(G) \to M_k$ is a k-colouring of (G,σ). Let $\phi\colon V(G) \to \{\omega_1,\omega_2,\ldots,\omega_k\}$ be defined as

$$\phi(v) = \begin{cases} 1 & \text{if } \psi(v) = 0, \\ \omega_{2i} & \text{if } \psi(v) = i, \\ \omega_{2i+1} & \text{if } \psi(v) = -i. \end{cases}$$

Then it follows from the definition of $P_{G,\sigma}$ that $P_{G,\sigma}(\phi) \neq 0$. Hence there is an orientation D of G with $|\operatorname{OE}_k(D)| \neq |\operatorname{EE}_k(D)|$.

Let (G,σ) be the signed graph, where $G = K_{2k}$ is a complete graph with vertex set $\{v_1, v_2, \ldots, v_k, u_1, u_2, \ldots, u_k\}$ and σ is the signature of G defined as $\sigma_{u_iv_i} = -1$ for $i = 1,2,\ldots,k$ and $\sigma_e = 1$ otherwise.

Let ϕ be the mapping defined as $\phi(v_i) = \phi(u_i) = \omega_i$ for $i = 1,2,\ldots,k$. It is straightforward to verify (by using the definition of $P_{G,\sigma}$) that $P_{G,\sigma}(\phi) \neq 0$. So there is an orientation D of G with $|\operatorname{OE}_k(D)| \neq |\operatorname{EE}_k(D)|$. However, (G,σ) is not k-colourable, because colour 0 can be used on only one vertex, and each other colour class cannot contain a positive edge, and hence contains at most two vertices. $\square$

Proof of Theorem 1.4. Assume the vertex set of G is $V = \{v_1, v_2, \ldots, v_n\}$ and $P_G(x_1, x_2, \ldots, x_n)$ be the graph polynomial defined as $P_G(x_1,x_2,\ldots,x_n) = \prod_{e=v_iv_j \in E(G), i<j}(x_i - x_j)$. Let $I = \{(i_1,i_2,\ldots,i_n) : i_j \geq 0, \sum_{j=1}^n i_j = |E(G)|\}$. Assume

$$P_G(x_1,x_2,\ldots,x_n) = \sum_{(i_1,i_2,\ldots,i_n)\in I} c_{i_1,i_2,\ldots,i_n} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}.$$

For a non-negative integer a, let m'_a, r'_a be the unique non-negative integers as follows:

- If $a = 0$, then $m'_a = r'_a = 0$.
- If $a > 0$, then $a = m'_a(k-1) + r'_a$, $1 \le r'_a \le k-1$.

Let

$$\overline{P}_G(x_1, x_2, \ldots, x_n) = \sum_{(i_1, i_2, \ldots, i_n) \in I} c_{i_1, i_2, \ldots, i_n} x_1^{r'_{i_1}} x_2^{r'_{i_2}} \cdots x_n^{r'_{i_n}}$$

and

$$\begin{aligned} &Q(x_1, x_2, \ldots, x_n) \\ &= \sum_{(i_1, i_2, \ldots, i_n) \in I} c_{i_1, i_2, \ldots, i_n} x_1^{r'_{i_1}} x_2^{r'_{i_2}} \cdots x_n^{r'_{i_n}} \left(x_1^{m'_{i_1}(k-1)} x_2^{m'_{i_2}(k-1)} \cdots x_n^{m'_{i_n}(k-1)} - 1 \right). \end{aligned}$$

Then

$$P_G(x_1, x_2, \ldots, x_n) = \overline{P}_G(x_1, x_2, \ldots, x_n) + Q(x_1, x_2, \ldots, x_n).$$

Let $\omega_1, \omega_2, \ldots, \omega_k$ be the k roots of the polynomial $x^k - x$ in the complex field $\mathbb{C}$. Then G is k-colourable if and only if there is a mapping $\phi\colon V(G) \to \{\omega_1, \omega_2, \ldots, \omega_k\}$ such that $P_G(\phi) \neq 0$. Now we show that for any $\phi\colon V(G) \to \{\omega_1, \omega_2, \ldots, \omega_k\}$, $Q(\phi) = 0$. For this purpose, we prove that for any $(i_1, i_2, \ldots, i_n) \in I$, $\phi(v_1)^{r'_{i_1}} \phi(v_2)^{r'_{i_2}} \cdots \phi(v_n)^{r'_{i_n}} \big(\phi(v_1)^{m'_{i_1}(k-1)}$ $\phi(v_2)^{m'_{i_2}(k-1)} \cdots \phi(v_n)^{m'_{i_n}(k-1)} - 1\big) = 0$. If there is an index $1 \le j \le n$ such that $\phi(v_j) = 0$ and $r'_{i_j} \ge 1$, then $\phi(v_1)^{r'_{i_1}} \phi(v_2)^{r'_{i_2}} \cdots \phi(v_n)^{r'_{i_n}} = 0$. Assume for any index j, either $\phi(v_j) \neq 0$ or $r'_{i_j} = 0$ (and hence $m'_{i_j} = 0$). Then for each $1 \le j \le n$, $\phi(v_j)^{m'_{i_j}(k-1)} = 1$. Hence

$$\phi(v_1)^{m'_{i_1}(k-1)} \phi(v_2)^{m'_{i_2}(k-1)} \cdots \phi(v_n)^{m'_{i_n}(k-1)} - 1 = 0.$$

So G is k-colourable if and only if there is a mapping $\phi\colon V(G) \to \{\omega_1, \omega_2, \ldots, \omega_k\}$ such that $\overline{P}_G(\phi) \neq 0$. Since in the polynomial $\overline{P}_G(x_1, x_2, \ldots, x_n)$, each variable x_i has degree at most $k-1$, we conclude that either $\overline{P}_G(x_1, x_2, \ldots, x_n)$ is a zero polynomial or there exists $\phi\colon V(G) \to \{\omega_1, \omega_2, \ldots, \omega_k\}$ such that $\overline{P}_G(\phi) \neq 0$. Therefore G is k-colourable if and only if $\overline{P}_G(x_1, x_2, \ldots, x_n)$ is not a zero polynomial.

Let $I' = \{(s_1, s_2, \ldots, s_n) : 0 \le s_j \le k-1\}$. Assume

$$\overline{P}_G(x_1, x_2, \ldots, x_n) = \sum_{(s_1, s_2, \ldots, s_n) \in I'} c'_{s_1, s_2, \ldots, s_n} x_1^{s_1} x_2^{s_2} \cdots x_n^{s_n}.$$

It follows from the definition that

$$c'_{s_1, s_2, \ldots, s_n} = \sum_{\substack{(i_1, i_2, \ldots, i_n) \in I \\ (r'_{i_1}, r'_{i_2}, \ldots, r'_{i_n}) = (s_1, s_2, \ldots, s_n)}} c_{i_1, i_2, \ldots, i_n}.$$

The polynomial $\overline{P}_G(x_1, x_2, \ldots, x_n)$ is not a zero polynomial if and only if for some $(s_1, s_2, \ldots, s_n) \in I'$, $c'_{s_1,s_2,\ldots,s_n} = \sum_{(i_1,i_2,\ldots,i_n)\in I,(r'_{i_1},r'_{i_2},\ldots,r'_{i_n})=(s_1,s_2,\ldots,s_n)} c_{i_1,i_2,\ldots,i_n} \neq 0$.

Similarly as before, $P_G(x_1, x_2, \ldots, x_n) = \sum w(D)$, where the summation is taken over all orientations D of G. An orientation D contributes 1 or -1 to the coefficient $c_{s_1,s_2,\ldots,s_n}$ if and only if D is an even or odd orientation of G with $r'_{d^+_D(v_j)} = s_j$ for $j = 1, 2, \ldots, n$.

Assume D is an even orientation with $r'_{d^+_D(v_j)} = s_j$ for $j = 1, 2, \ldots, n$. If D' is another orientation which also makes a contribution to the coefficient $c'_{s_1,s_2,\ldots,s_n}$, then for any vertex v_j of G, $d^+_{D'}(v_j) \equiv d^+_D(v_j) \pmod{k-1}$. Moreover, $d^+_{D'}(v_j) = 0$ if and only if $d^+_D(v_j) = 0$. This is equivalent to the condition that the edge set

$$E' = \{e : D \text{ and } D' \text{ orient } e \text{ in opposite directions}\}$$

induces a special modulo-$(k-1)$ Eulerian sub-digraph of D. So $\overline{P}_G$ is not a zero polynomial if and only if there is an orientation D of G such that $|\,\mathrm{EE}^s_{k-1}(D)| - |\,\mathrm{OE}^s_{k-1}(D)| \neq 0$. $\square$

Proof of Theorem 1.5. This proof is very much similar to that of Theorem 1.4, just use the graph polynomial for the signed graph (G, σ), which is similar to the proof of Theorem 1.2. We omit the details. $\square$

Proof of Theorem 1.6. Assume $k = 2q$ is an even integer and $\omega_1, \omega_2, \ldots, \omega_k$ are the k roots of $x^k - x$. Assume $\omega_k = 0$, $\omega_{k-1} = 1$ and for $i = 1, 2, \ldots, q-1$, $\omega_{2i-1} = -\omega_{2i}$.

Let (G, σ) be the signed graph, where G is obtained from the complete graph K_{2k} with vertex set $\{v_1, v_2, \ldots, v_k, u_1, u_2, \ldots, u_k\}$ by deleting the edge $v_k u_k$ and σ is the signature of G defined as follows:

$$\sigma_e = \begin{cases} 1 & \text{if } e \text{ is incident to } v_i \text{ and } e \neq u_i v_i \text{ (for } 1 \leq i \leq k), \\ & \text{or } e = u_{2i-1}u_{2i} \text{ and } 1 \leq i \leq q-1, \\ -1 & \text{otherwise.} \end{cases}$$

Let ϕ be the mapping defined as $\phi(v_i) = \phi(u_i) = \omega_i$ for $i = 1, 2, \ldots, k$. It is straightforward to verify (by using the definition of $P_{G,\sigma}$) that $P_{G,\sigma}(\phi) \neq 0$. So there is an orientation D of G with $|\,\mathrm{OE}^s_{k-1}(D)| \neq |\,\mathrm{EE}^s_{k-1}(D)|$. However, (G, σ) is not k-colourable. Indeed, if ψ is a k-colouring of G, then all the vertices $v_1, v_2, \ldots, v_k$ are coloured by distinct colours, as they induces a complete graph with all edges positive. So $v_1, v_2, \ldots, v_k$ use all the k colours. If $i \neq j$, the edge $u_i v_j$ is positive. Hence u_i, v_j are coloured by distinct colours. So $\psi(u_i) = \psi(v_i)$ for $i = 1, 2, \ldots, k$. If $\psi(u_i) = -\psi(u_j)$, then the edge $u_i u_j$ cannot be a negative edge. So in the subgraph of G induced by $\{u_1, u_2, \ldots, u_k\}$, there are at least q positive edges. But by our definition, this subgraph has only $q-1$ positive edges: $u_1u_2, u_3u_4, \ldots, u_{2q-3}u_{2q-2}$. This is a contradiction.

On the other hand, let (H, σ) be the signed graph, where $H = K_{2k}$ is a complete graph with vertex set $\{v_1, v_2, \ldots, v_k, u_1, u_2, \ldots, u_k\}$, and σ is defined as

$$\sigma_e = \begin{cases} -1 & \text{if } e = u_i v_i \text{ for } 1 \le i \le k, \\ 1 & \text{otherwise.} \end{cases}$$

Then $\psi(u_i) = \psi(v_i) = i$ for $1 \le i \le q$ and $\psi(u_{q+i}) = \psi(v_{q+i}) = -i$ is a k-colouring of (H, σ).

Now we show that H has no orientation D with $|\operatorname{OE}^s_{k-1}(D)| \neq |\operatorname{EE}^s_{k-1}(D)|$. Otherwise, there is a mapping $\phi\colon V(H) \to \{\omega_1, \omega_2, \ldots, \omega_k\}$ (recall that $\omega_k = 0$ and $\omega_{k-1} = 1$) such that $P_{G,\sigma}(\phi) \neq 0$. It follows from the definition that $\phi^{-1}(0)$ is edgeless, and hence $|\phi^{-1}(0)| \le 1$, and for any other ω_i, $\phi^{-1}(\omega_i)$ contains no positive edges, and hence contains at most two vertices. A contradiction. $\square$

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