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General Decay for a Viscoelastic Wave Equation with Density and Time Delay Term in $\mathbb{R}^n$

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Abstract. A linear viscoelastic wave equation with density and time delay in the whole space $\mathbb{R}^n$ ($n \geq 3$) is considered. In order to overcome the difficulties in the non-compactness of some operators, we introduce some weighted spaces. Under suitable assumptions on the relaxation function, we establish a general decay result of solution for the initial value problem by using energy perturbation method. Our result extends earlier results.

1. Introduction

In this paper, we study the following Cauchy problem with a time delay term in the internal feedback:

$$\begin{aligned} u_{tt}(x,t) - \phi(x)\left(\Delta u(x,t) - \int_0^t g(t-s)\Delta u(x,s)\,ds\right) \\ + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = 0, \end{aligned} \tag{1.1}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \mathbb{R}^n, \tag{1.2}$$

$$u_t(x,t-\tau) = f_0(x,t-\tau), \quad x \in \mathbb{R}^n,\ 0 < t < \tau, \tag{1.3}$$

where $u_0(x)$, $u_1(x)$ and $f_0(x,t-\tau)$ are given initial data belonging to appropriate spaces. The function $g(t)$ is the relaxation function. The coefficient $\phi(x) := (\rho(x))^{-1}$ represents the speed of sound at the point $x \in \mathbb{R}^n$ and the function $\rho(x)$ is the density. The constants μ_1 and μ_2 are two real numbers and $\tau > 0$ denotes the time delay.

Equation (1.1) with the memory term $\int_0^t g(t-s)\Delta u(s)\,ds$ can be regarded as a viscoelastic wave equation with a perturbation, and it can be also regarded as an elastoplastic flow equation with some kind of memory effect. A more general equation of (1.1) without delay term reads

$$u_{tt} - \phi(x)\left(\Delta u - \int_0^t g(t-s)\Delta u(s)\,ds\right) = 0. \tag{1.4}$$

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When $\rho(x) = 1$, i.e., $\phi(x) = 1$, there are so many researchers studied the initial boundary value problem of (1.4) in a bounded domain, the main results are mainly concerned with global existence, stability and long-time dynamics, and many results may be found in the literature. For general decay results, we refer the reader to Cao and Yao [2], Messaoudi [13–15], Messaoudi and Al-Gharabli [16], Messaoudi and Soufyane [17], Mustafa and Messaoudi [18], Said-Houari, Messaoudi and Guesmia [25], Tatar [26], Wu [27] and so on. For Cauchy problem, Kafini and Messaoudi [8] studied the following problem

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(x,s)\,ds = 0, \quad x \in \mathbb{R}^n,\ t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \mathbb{R}^n. \end{cases}$$

They proved the energy decays polynomially for compactly supported initial data u_0, u_1 and for an exponentially decaying relaxation function $g(t)$. In [7], the same author investigated

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(x,s)\,ds + u_t = |u|^{p-1}u, \quad x \in \mathbb{R}^n,\ t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \mathbb{R}^n, \end{cases}$$

and proved a blow-up result of the problem, and this result extended the one of [31].

When the density $\rho(x) \neq 1$, Karachalios and Stavrakakis [10] considered the following semilinear hyperbolic initial value problem

$$\begin{cases} u_{tt} - \phi(x)\Delta u + \delta u_t + \lambda f(u) = \eta(x), \quad x \in \mathbb{R}^n,\ t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \mathbb{R}^n. \end{cases}$$

The authors proved local existence of solutions and established the existence of a global attractor in energy space $\mathcal{D}^{1,2}(\mathbb{R}^n) \times L^2_g(\mathbb{R}^n)$ by using the compactness of embedding $\mathcal{D}^{1,2}(\mathbb{R}^n) \subset L^2_g(\mathbb{R}^n)$ in the case where $(\phi(x))^{-1} := g(x) \in L^{n/2}(\mathbb{R}^n)$ and $n \geq 3$. Subsequently, Papadopoulos and Stavrakakis [24] studied a degenerate nonlocal quasilinear wave equation of Kirchhoff type with a weak dissipative term

$$\begin{cases} u_{tt} - \phi(x)\|\nabla u(t)\|^2\Delta u + \delta u_t = |u|^a u, \quad x \in \mathbb{R}^n,\ t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \mathbb{R}^n, \end{cases}$$

and proved global existence, energy decay and blow-up results of solutions in the case where $n \geq 3$, $\delta \geq 0$ and the positive function $(\phi(x))^{-1} := g(x)$ lies in $L^{n/2}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. In [6], Kafini considered the following initial-value problem

$$\begin{cases} \rho(x)u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(x,s)\,ds = 0, \quad x \in \mathbb{R}^n,\ t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \mathbb{R}^n, \end{cases}$$

and established general results of solutions in the case where $\rho(x)$ is continuous function lying in $L^{n/2}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Recently, Zennir [30] considered the following problem

$$\begin{cases} \rho(x)(|u_t|^{q-1}u_t)_t - M(\|\nabla u(t)\|^2)\Delta u + \int_0^t g(t-s)\Delta u(x,s)\,ds = 0, & x \in \mathbb{R}^n,\ t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \mathbb{R}^n, \end{cases}$$

and proved a general decay result of solutions for wider class of relaxation function $g(t)$ in the case where $q, n \geq 2$ and M is a positive C^1 function satisfying some suitable conditions. For more results in this respect, we also refer the reader to Cavalcanti et al. [3] and Zhou [32], and so on.

In recent years, many authors studied the wave equation with time delay effects. It is worth mentioning the work of Xu, Yung and Li [29]. In their paper, the authors studied the following closed loop system in one dimension

$$\begin{cases} u_{tt} - u_{xx} = 0, \quad x \in (0,1),\ t > 0, \\ u(0,t) = 0, \quad t > 0, \\ u_x(1,t) = -k\mu u_t(1,t) - k(1-\mu)u_t(1,t-\tau), \quad t \geq 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in (0,1), \\ u(1,t-\tau) = f(t-\tau), \quad t \in (0,\tau), \end{cases}$$

and proved that the system is exponentially stable if $\mu > 1/2$, and if $\mu < 1/2$, the system is unstable. When $\mu = 1/2$, they claimed that if $\tau \in (0,1)$ is rational, the system is unstable. If $\tau \in (0,1)$ is irrational, the system is asymptotically stable. In [19], Nicaise and Pignotti extended the results in [29] to higher dimensions and considered the following system

$$\begin{cases} u_{tt} - \Delta u = 0, \quad x \in \Omega,\ t > 0, \\ u(x,t) = 0, \quad x \in \Gamma_0,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau), \quad x \in \Gamma_1,\ t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \\ u(x,t-\tau) = f_0(x,t-\tau), \quad x \in \Gamma_1,\ t \in (0,\tau). \end{cases}$$

They proved the well-posedness of the problem by using semigroup method and then obtained that the energy decays exponentially by using an observability inequality. They also studied the case of internal feedbacks and obtained the exponential decay of energy. In both cases, the results hold for $0 < \mu_2 < \mu_1$. If $\mu_2 \geq \mu_1 > 0$, they obtained an explicit sequence of arbitrary small delays that destabilize the system. In [11], Kirane and Said-Houari studied the following viscoelastic wave equation with a delay term in internal feedbacks

$$u_{tt}(x,t) - \Delta u(x,t) + \int_0^t g(t-s)\Delta u(s)\,ds + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = 0, \quad x \in \Omega,\ t > 0,$$

where $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ is a regular and bounded domain and μ_1 and μ_2 are positive constants. They proved the global well-posedness of the initial boundary value problem by using some suitable assumptions on the relaxation function and some restrictions on the parameters μ_1 and μ_2. Furthermore, under the assumption $0 < \mu_2 \leq \mu_1$, they obtained a general decay result of the total energy to the system. But for the case $\mu_1 = 0$, they did not get the decay property of the energy. Based on the results in [11], Liu [12] extended the results to a system with time-dependent delay. Dai and Yang [4] considered the same equation as in [11] and solved the open problem proposed by Kirane and Said-Houari. In this paper, the authors proved the global existence of solutions without restrictions of $\mu_1, \mu_2 > 0$ and $\mu_2 \leq \mu_1$, and obtained an exponential decay result of energy in the case $\mu_1 = 0$. Recently, Kafini, Messaoudi and Nicaise [9] considered a nonlinear damped second-order evolution equation with delay of the form

$$u_{tt} + Au + G(u_t) + \mu G(u_t(t-\tau)) = F(u),$$

and proved the energy blows up in finite time under some suitable assumptions. Recently, Nicaise and Pignotti [21] considered (nonlinear) abstract evolution equations with constant time delay of the form

$$\begin{cases} U_t(t) = \mathcal{A}U(t) + F(U(t)) + k\mathcal{B}U(t-\tau), \\ U(0) = U_0, \quad \mathcal{B}U(t-\tau) = f(t), \end{cases}$$

where $\mathcal{B}$ is a bounded operator. They obtained that the operator associated to the part without delay generates a strongly continuous semigroup which is exponentially stable. They also proved that the model with delay remains exponentially stable under a smallness condition on the time delay feedback. For more results concerning the different boundary conditions under appropriate assumptions on μ_1 and μ_2, one can refer to Datko, Lagnese and Polis [5], Nicaise and Pignotti [20], Nicaise, Valein and Fridman [23], Nicaise and Valein [22], Wu [28], and the references therein. It is remarkable that all above results concerning with wave equation with delay were established in a bounded domain.

Equation (1.1) is a viscoelastic wave equation with density and a time delay term in the internal feedback. To the best of our knowledge, the general rate of decay for Cauchy problem (1.1)–(1.3) were not previously considered. So the objective of the present work is to establish the stability of initial value problem (1.1)–(1.3). Obviously, $\rho(x)$ can not be a constant here. The main contribution of this work is threefold:

(i) Since the non-compactness of some operators in unbounded domain, especially, the Poincaré inequality and some Sobolev embedding inequalities are not valid in the whole space, we introduce some weighted spaces as in [6, 10] to overcome these difficulties.

(ii) We establish a general rate of energy decay for solutions and extend the previous results in [4,8,11], where the exponential decay and polynomial decay are only special cases.

(iii) We consider the constants μ_1 and μ_2 are real numbers without restrictions of $\mu_1, \mu_2 > 0$ and $\mu_2 \leq \mu_1$, and establish the energy decay in the case $0 < |\mu_2| < \mu_1$ and in the case $\mu_1 = 0$, $|\mu_2| > 0$.

The outline of this paper is as follows. In Section 2, we give some preparations. In Section 3, we state our main results. In Section 4, we prove our main results. The conclusion and open problems will be given in Section 5.

2. Space setting and assumptions

In this section, we give the space setting and some assumptions. For convenience, we use the standard notations of Lebesgue integral and Sobolev spaces as

$$L^q(\mathbb{R}^n) \quad (1 \leq q \leq \infty) \quad \text{and} \quad H^1(\mathbb{R}^n).$$

In addition, $\|\cdot\|_B$ denotes the norm in the space B, we write $\|u\|$ instead of $\|u\|_2$ when $q = 2$ for simplify.

2.1. Space setting

As in [10], we introduce the weighted spaces $\mathcal{D}^{1,2}(\mathbb{R}^n)$ and $L^p_\rho(\mathbb{R}^n)$ for our system. First we assume the density $\rho(x)\colon \mathbb{R}^n \to \mathbb{R}$ $(n \geq 3)$ satisfies the following conditions.

(A) $\rho(x) > 0$, $\rho \in C^{0,\gamma}(\mathbb{R}^n)$ with $\gamma \in (0,1)$ and $\rho \in L^{n/2}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.

Now we define the weighted spaces $\mathcal{D}^{1,2}(\mathbb{R}^n)$ and $L^p_\rho(\mathbb{R}^n)$, $(1 < p < \infty)$.

(1) The space $\mathcal{D}^{1,2}(\mathbb{R}^n)$ is defined to be the closure of $C_0^\infty(\mathbb{R}^n)$ functions with respect to which norm

$$\mathcal{D}^{1,2}(\mathbb{R}^n) = \{u \in L^{2n/(n-2)}(\mathbb{R}^n) : \nabla u \in L^2(\mathbb{R}^n)\},$$

equipped with the norm $\|u\|_{\mathcal{D}^{1,2}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\nabla u|^2\,dx$.

(2) We introduce the weighted space $L^2_\rho(\mathbb{R}^n)$ to be defined the closure of $C_0^\infty(\mathbb{R}^n)$ functions with respect to the inner product

$$(u,v)_{L^2_\rho(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \rho uv\,dx,$$

and we know that $L^2_\rho(\mathbb{R}^n)$ is a separable Hilbert space and $\|u\|^2_{L^2_\rho(\mathbb{R}^n)} = (u,u)_{L^2_\rho(\mathbb{R}^n)}$.

(3) If u is a measurable function on $\mathbb{R}^n$, we define

$$\|u\|^p_{L^p_\rho(\mathbb{R}^n)}(\mathbb{R}^n) = \left(\int_{\mathbb{R}^n} \rho|u|^p\, dx\right)^{1/p} \quad \text{for } 1 < p < \infty,$$

and let $L^p_\rho(\mathbb{R}^n)$ consist of all u for which $\|u\|_{L^p_\rho(\mathbb{R}^n)} < \infty$.

From [6, 10], we can get the following lemma.

Lemma 2.1. *Assume the function ρ satisfies* (A), *then for any $u \in \mathcal{D}^{1,2}(\mathbb{R}^n)$,*

$$\|u\|_{L^q_\rho} \le \|\rho\|_{L^s}\|\nabla u\|$$

with $s = 2n/(2n - qn + 2q)$ and $2 \le q \le 2n/(n-2)$.

Corollary 2.2. *For $q = 2$, we have*

$$\|u\|_{L^2_\rho} \le \|\rho\|_{L^{n/2}}\|\nabla u\|.$$

If $\rho \in L^{n/2}(\mathbb{R}^n)$, we have

$$\|u\|_{L^2_\rho} \le c_*\|\nabla u\|, \tag{2.1}$$

where $c_* > 0$ is a constant.

2.2. Assumptions on relaxation function

We assume the relaxation function g satisfies the following conditions:

(G1) $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a differentiable function such that

$$g(0) > 0, \quad 1 - \int_0^\infty g(s)\, ds = l > 0.$$

(G2) There exists a nonincreasing differentiable function $\zeta(t)\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\int_0^\infty \zeta(s)\, ds = +\infty, \quad g'(t) \le -\zeta(t)g(t) \quad \text{for } t \ge 0.$$

3. Main results

In this section, we shall give the main results of the present work. First we can establish the global existence and uniqueness of problem (1.1)–(1.3), which is given by the following theorem.

Theorem 3.1. *Assume that* (A) *and* (G1) *hold. Then for any initial data* $u_0 \in \mathcal{D}^{1,2}(\mathbb{R}^n)$, $u_1 \in L^2_\rho(\mathbb{R}^n)$ *and* $f_0(x,t) \in L^2(\mathbb{R}^n \times (-\tau,0))$*, problem* (1.1)–(1.3) *has a unique solution such that for any* $T > 0$,

$$u \in C(0,T;\mathcal{D}^{1,2}(\mathbb{R}^n)) \quad \textit{and} \quad u_t \in C(0,T;L^2_\rho(\mathbb{R}^n)).$$

Remark 3.2. We can divide into two steps to prove the theorem. First we can prove the global existence of the problem by using Faedo-Galerkin method restricted on $B_R \times (0.T)$ satisfying the boundary condition $u = 0$ in $\partial B_R \times (0.T)$, where B_R is the ball with a radius of R. One can refer to Kirane and Said-Houari [11] and Dai and Yang [4] and so on. The next step is to extend the solutions to the whole space $\mathbb{R}^n$, we can employ the method developed by Babin and Vishik [1]. We can also refer to Karachalios and Stavrakakis [10], and hence we omit the detail proof here.

We introduce the modified energy functional to problem (1.1)–(1.3) by

$$\begin{aligned} E(t) = {} & \frac{1}{2}\|u_t(t)\|^2_{L^2_\rho} + \frac{1}{2}\left(1 - \int_0^t g(s)\,ds\right)\|\nabla u(t)\|^2 + \frac{1}{2}(g \circ \nabla u)(t) \\ & + \frac{\xi}{2}\int_{t-\tau}^t \int_{\mathbb{R}^n} \rho(x) e^{\sigma(s-t)} u_s^2(x,s)\,dxds, \end{aligned} \tag{3.1}$$

where σ and ξ are two positive constants to be determined later, and

$$(g \circ \nabla u)(t) = \int_0^t g(t-s)\|\nabla u(t) - \nabla u(s)\|^2\,ds.$$

The main result of the present work is to establish the general decay rate of the energy, which is given by the following theorem.

Theorem 3.3. *Assume the assumptions* (G1)–(G2) *hold. Let* $u_0 \in \mathcal{D}^{1,2}(\mathbb{R}^n)$, $u_1 \in L^2_\rho(\mathbb{R}^n)$ *and* $f_0(x,t) \in L^2(\mathbb{R}^n \times (-\tau,0))$*. Then we have:*

(i) *If* $0 < |\mu_2| < \mu_1$*, then there exist two constants* $\beta > 0$ *and* $\gamma > 0$ *such that the energy* $E(t)$ *defined by* (3.1) *satisfies*

$$E(t) \le \beta \exp\left(-\gamma \int_0^t \zeta(s)\,ds\right) \quad \textit{for all } t \ge 0. \tag{3.2}$$

(ii) *If* $\mu_1 = 0$, $0 < |\mu_2| < a$ *and* $\zeta(t) > \zeta_0$*, where the constants* $a > 0$ *and* $\zeta_0 > 0$ *are defined in* (4.21) *and* (4.25)*, respectively. Then there exists a constant* $\gamma' > 0$ *such that the energy* $E(t)$ *defined by* (3.1) *satisfies*

$$E(t) \le E(t_0) \exp\left(-\gamma' \int_{t_0}^t \zeta(s)\,ds\right) \quad \textit{for all } t \ge t_0. \tag{3.3}$$

4. General decay of energy

In this section, we shall establish the general decay property of the solution for problem (1.1)–(1.3). For this purpose we need the following technical lemmas.

Lemma 4.1. *Under the assumptions of Theorem* 3.3, *the modified energy functional defined by* (3.1) *satisfies for any* $t \geq 0$,

$$\begin{aligned} E'(t) \leq & \left(\frac{|\mu_2|}{2} - \mu_1 + \frac{\xi}{2}\right) \|u_t(t)\|^2_{L^2_\rho} + \left(\frac{|\mu_2|}{2} - \frac{\xi}{2}e^{-\sigma\tau}\right) \int_{\mathbb{R}^n} \rho(x) u_t^2(t-\tau)\, dx \\ & + \frac{1}{2}(g' \circ \nabla u)(t). \end{aligned} \tag{4.1}$$

Proof. Taking the derivative of $E(t)$, we have

$$\begin{aligned} E'(t) =& \int_{\mathbb{R}^n} \rho(x) u_{tt} u_t \, dx - \frac{1}{2} g(t) \|\nabla u\|^2 + \left(1 - \int_0^t g(s)\, ds\right) \int_{\mathbb{R}^n} \nabla u \cdot \nabla u_t \, dx \\ & + \frac{1}{2}(g' \circ \nabla u) + \int_{\mathbb{R}^n} \nabla u_t(t) \int_0^t g(t-s)(\nabla u(s) - \nabla u(t))\, ds dx + \frac{\xi}{2}\|u_t\|^2_{L^2_\rho} \\ & - \frac{\xi}{2} e^{-\sigma\tau} \int_{\mathbb{R}^n} \rho(x) u_t^2(t-\tau)\, dx - \frac{\sigma\xi}{2} \int_{t-\tau}^t \int_{\mathbb{R}^n} \rho(x) e^{-\sigma(t-s)} u_s^2(x,s)\, dx ds. \end{aligned}$$

By using equation (1.1) and integration by parts, we can easily get

$$\begin{aligned} E'(t) =& \frac{1}{2}(g' \circ \nabla u) - \frac{1}{2} g(t)\|\nabla u\|^2 - \mu_1 \int_{\mathbb{R}^n} \rho(x) u_t^2 \, dx + \frac{\xi}{2}\|u_t\|^2_{L^2_\rho} \\ & - \mu_2 \int_{\mathbb{R}^n} \rho(x) u_t u_t(t-\tau)\, dx - \frac{\xi}{2} e^{-\sigma\tau} \int_{\mathbb{R}^n} \rho(x) u_t^2(t-\tau)\, dx \\ & - \frac{\sigma\xi}{2} \int_{t-\tau}^t \int_{\mathbb{R}^n} \rho(x) e^{-\sigma(t-s)} u_s^2(x,s)\, dx ds, \end{aligned}$$

which, together with Young's inequality and Assumption (G1)–(G2), gives us

$$E'(t) \leq \left(\frac{|\mu_2|}{2} - \mu_1 + \frac{\xi}{2}\right) \|u_t\|^2_{L^2_\rho} + \left(\frac{|\mu_2|}{2} - \frac{\xi}{2} e^{-\sigma\tau}\right) \int_{\mathbb{R}^n} \rho(x) u_t^2(t-\tau)\, dx + \frac{1}{2}(g' \circ \nabla u).$$

Then the proof is complete. □

Lemma 4.2. *Under the assumptions of Theorem* 3.3, *let* (u, u_t) *be the solution of problem* (1.1)–(1.3). *The functional* $\Phi(t)$ *defined by*

$$\Phi(t) = \int_{\mathbb{R}^n} \rho u u_t \, dx,$$

satisfies that there exist three positive constants c_1, c_2 *and* c_3 *such that for any* $t > 0$,

$$\Phi'(t) \leq -\frac{l}{2}\|\nabla u(t)\|^2 + c_1 \|u_t(t)\|^2_{L^2_\rho} + c_2 \int_{\mathbb{R}^n} \rho u_t^2(t-\tau)\, dx + c_3 (g \circ \nabla u)(t). \tag{4.2}$$

Proof. We take the derivative of $\Phi(t)$ and use equation (1.1) to get

$$
\begin{aligned}
\Phi'(t) &= \int_{\mathbb{R}^n} \rho u_t^2\,dx + \int_{\mathbb{R}^n} \rho u_{tt} u\,dx \\
&= \int_{\mathbb{R}^n} \rho u_t^2\,dx + \int_{\mathbb{R}^n} \left(\Delta u - \int_0^t g(t-s)\Delta u(s)\,ds\right) u\,dx \\
&\quad + \int_{\mathbb{R}^n} \rho(-\mu_1 u_t - \mu_2 u_t(t-\tau))u\,dx \\
&= \int_{\mathbb{R}^n} \rho u_t^2\,dx - \int_{\mathbb{R}^n} |\nabla u|^2\,dx + \int_{\mathbb{R}^n} \nabla u(t) \int_0^t g(t-s)\nabla u(s)\,dsdx \\
&\quad - \mu_1 \int_{\mathbb{R}^n} \rho u u_t\,dx - \mu_2 \int_{\mathbb{R}^n} \rho u u_t(t-\tau)\,dx \\
&= \int_{\mathbb{R}^n} \rho u_t^2\,dx + \left(\int_0^t g(s)\,ds - 1\right)\|\nabla u\|^2 - \mu_1 \int_{\mathbb{R}^n} \rho u u_t\,dx \\
&\quad + \int_{\mathbb{R}^n} \nabla u(t) \int_0^t g(t-s)(\nabla u(s) - \nabla u(t))\,dsdx - \mu_2 \int_{\mathbb{R}^n} \rho u u_t(t-\tau)\,dx.
\end{aligned}
\tag{4.3}
$$

By using Young's and Hölder's inequalities, we arrive at for any $\epsilon > 0$,

$$
\begin{aligned}
\int_{\mathbb{R}^n} \nabla u(t) \int_0^t g(t-s)(\nabla u(s) - \nabla u(t))\,dsdx &\le \epsilon \int_{\mathbb{R}^n} |\nabla u|^2\,dx + \frac{1}{4\epsilon}\left(\int_0^t g(s)\,ds\right)(g \circ \nabla u) \\
&\le \epsilon\|\nabla u\|^2 + \frac{1-l}{4\epsilon}(g \circ \nabla u).
\end{aligned}
\tag{4.4}
$$

Similarly, by using (2.1), we obtain for any $\epsilon > 0$,

$$
\left|-\mu_1 \int_{\mathbb{R}^n} \rho u u_t\,dx\right| \le |\mu_1|\epsilon\|u\|^2_{L^2_\rho} + \frac{|\mu_1|}{4\epsilon}\|u_t\|^2_{L^2_\rho} \le |\mu_1|\epsilon c_*^2\|\nabla u\|^2 + \frac{|\mu_1|}{4\epsilon}\|u_t\|^2_{L^2_\rho},
\tag{4.5}
$$

$$
\left|-\mu_2 \int_{\mathbb{R}^n} u u_t(t-\tau)\,dx\right| \le |\mu_2|\epsilon c_*^2\|\nabla u\|^2 + \frac{|\mu_2|}{4\epsilon}\int_{\mathbb{R}^n} u_t^2(t-\tau)\,dx.
\tag{4.6}
$$

Inserting (4.4)–(4.6) into (4.3), using Assumption (G1) and taking $\epsilon > 0$ small enough, we can get (4.2) with

$$
c_1 := 1 + \frac{|\mu_1|}{4\epsilon}, \quad c_2 := \frac{|\mu_2|}{4\epsilon}, \quad c_3 := \frac{1-l}{4\epsilon}.
$$

The proof is therefore complete. □

Lemma 4.3. *Under the assumptions of Theorem* 3.3*, let* (u, u_t) *be the solution of problem* (1.1)–(1.3)*. The functional* $\Psi(t)$ *defined by*

$$
\Psi(t) = -\int_{\mathbb{R}^n} \rho u_t \int_0^t g(t-s)(u(t) - u(s))\,dsdx,
$$

satisfies that there exists a positive constant c_4 *such that for any* $\delta > 0$,

$$
\begin{aligned}
\Psi'(t) \le{}& \left(2\delta - \int_0^t g(s)\,ds\right)\|u_t(t)\|^2_{L^2_\rho} + [\delta + 2\delta(1-l)^2]\|\nabla u(t)\|^2 + c_4(g \circ \nabla u)(t) \\
&- \frac{g(0)c_*^2}{4\delta}(g' \circ \nabla u)(t) + \delta \int_{\mathbb{R}^n} \rho u_t^2(t-\tau)\,dx.
\end{aligned}
\tag{4.7}
$$

Proof. By using (1.1), it is easy to get

$$
\begin{aligned}
\Psi'(t) = &- \int_{\mathbb{R}^n} \rho u_{tt} \int_0^t g(t-s)(u(t)-u(s))\,ds dx - \int_0^t g(s)\,ds \cdot \|u_t\|_{L^2_\rho}^2 \\
&- \int_{\mathbb{R}^n} \rho u_t \int_0^t g'(t-s)(u(t)-u(s))\,ds dx \\
= &- \int_{\mathbb{R}^n} \Delta u \int_0^t g(t-s)(u(t)-u(s))\,ds dx \\
&+ \int_{\mathbb{R}^n} \left(\int_0^t g(s)\Delta u(s)\,ds\right)\left(\int_0^t g(t-s)(u(t)-u(s))\,ds\right) dx \\
&+ \mu_1 \int_{\mathbb{R}^n} \rho u_t \int_0^t g(t-s)(u(t)-u(s))\,ds dx \\
&+ \mu_2 \int_{\mathbb{R}^n} \rho u_t(t-\tau) \int_0^t g(t-s)(u(t)-u(s))\,ds dx \\
&- \int_{\mathbb{R}^n} \rho u_t \int_0^t g'(t-s)(u(t)-u(s))\,ds dx - \int_0^t g(s)\,ds \cdot \|u_t\|_{L^2_\rho}^2.
\end{aligned}
\tag{4.8}
$$

Using integration by parts, Young's inequality and Hölder's inequality, we have for any $\delta > 0$,

$$
\begin{aligned}
&\left| - \int_{\mathbb{R}^n} \Delta u \int_0^t g(t-s)(u(t)-u(s))\,ds dx \right| \\
&= \left| \int_{\mathbb{R}^n} \nabla u \int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\,ds dx \right| \\
&\le \delta \int_{\mathbb{R}^n} |\nabla u|^2\,dx + \frac{1}{4\delta}\left(\int_0^t g(s)\,ds\right)(g \circ \nabla u) \\
&\le \delta \|\nabla u\|^2 + \frac{1-l}{4\delta}(g\circ\nabla u),
\end{aligned}
\tag{4.9}
$$

$$
\begin{aligned}
&\left| \int_{\mathbb{R}^n} \left(\int_0^t g(t-s)(u(t)-u(s))\,ds\right)\cdot\left(\int_0^t g(t-s)\Delta u(s)\,ds\right) dx \right| \\
&= \left| - \int_{\mathbb{R}^n} \left(\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\,ds\right)\cdot\left(\int_0^t g(t-s)\nabla u(s)\,ds\right) dx \right| \\
&\le \delta \int_{\mathbb{R}^n} \left(\int_0^t g(t-s)\nabla u(s)\,ds\right)^2 dx + \frac{1}{4\delta}\int_{\mathbb{R}^n}\left(\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\,ds\right)^2 dx \\
&\le 2\delta \int_{\mathbb{R}^n} \left[\left(\int_0^t g(t-s)(\nabla u(s)-\nabla u(t))\,ds\right)^2 + \left(\int_0^t g(t-s)\nabla u(t)\,ds\right)^2\right] dx \\
&\quad + \frac{1}{4\delta}\int_{\mathbb{R}^n}\left(\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\,ds\right)^2 dx \\
&\le \left(2\delta + \frac{1}{4\delta}\right)\left(\int_0^t g(s)\,ds\right)(g\circ\nabla u) + 2\delta\left(\int_0^t g(s)\,ds\right)^2 \|\nabla u\|^2 \\
&\le \left(2\delta + \frac{1}{4\delta}\right)(1-l)(g\circ\nabla u) + 2\delta(1-l)^2\|\nabla u\|^2,
\end{aligned}
\tag{4.10}
$$

$$\left|\mu_1 \int_{\mathbb{R}^n} \rho u_t \int_0^t g(t-s)(u(t)-u(s))\, dsdx\right| \le \delta \|u_t\|^2_{L^2_\rho} + \frac{c_*^2}{4\delta}(g\circ \nabla u), \tag{4.11}$$

$$\left|\mu_2 \int_{\mathbb{R}^n} \rho u_t(t-\tau) \int_0^t g(t-s)(u(t)-u(s))\, dsdx\right| \le \delta \int_{\mathbb{R}^n} \rho u_t^2(t-\tau)\, dx + \frac{c_*^2}{4\delta}(g\circ \nabla u) \tag{4.12}$$

and

$$\left|-\int_{\mathbb{R}^n} \rho u_t \int_0^t g'(t-s)(u(t)-u(s))\, dsdx\right| \le \delta \|u_t\|^2_{L^2_\rho} - \frac{g(0)c_*^2}{4\delta}(g'\circ \nabla u). \tag{4.13}$$

Combining (4.9)–(4.13) with (4.8), we can obtain (4.7) with

$$c_4 := (1-l)\left[\left(2\delta + \frac{1}{4\delta}\right) + \frac{1}{4\delta}\right] + \frac{c_*^2}{2\delta}.$$

The proof of the lemma is done. □

Now we define the Lyapunov functional

$$\mathcal{L}(t) := E(t) + \varepsilon_1 \Phi(t) + \varepsilon_2 \Psi(t),$$

where ε_1 and ε_2 are positive constants.

Then we have the following lemma.

Lemma 4.4. *For $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ small enough, we have*

$$\frac{1}{2}E(t) \le \mathcal{L}(t) \le 2E(t). \tag{4.14}$$

Proof. By using Hölder's inequality, Young's inequality and (2.1), we can get for any $\delta > 0$,

$$\begin{aligned}
&|\mathcal{L}(t) - E(t)| \\
&\le \varepsilon_1 \int_{\mathbb{R}^n} |\rho u u_t|\, dx + \varepsilon_2 \int_{\mathbb{R}^n} \left|\rho u_t \int_0^t g(t-s)(u(t)-u(s))\, ds\right| dx \\
&\le \varepsilon_1 \left(\delta \|u_t\|^2_{L^2_\rho} + \frac{1}{4\delta}\|u\|^2_{L^2_\rho}\right) + \varepsilon_2 \left(\delta \|u_t\|^2_{L^2_\rho} + \frac{1}{4\delta}\int_0^t g(t-s)\|u(t)-u(s)\|^2_{L^2_\rho}\, ds\right) \\
&\le \varepsilon_1 \left(\delta \|u_t\|^2_{L^2_\rho} + \frac{c_*^2}{4\delta}\|\nabla u\|^2\right) + \varepsilon_2 \left(\delta \|u_t\|^2_{L^2_\rho} + \frac{c_*^2}{4\delta}(1-l)(g\circ \nabla u)\right) \\
&\le \delta(\varepsilon_1+\varepsilon_2)\|u_t\|^2_{L^2_\rho} + \frac{\varepsilon_1 c_*^2}{4\delta}\|\nabla u\|^2 + \frac{\varepsilon_2 c_*^2}{4\delta}(1-l)(g\circ \nabla u),
\end{aligned}$$

which gives us there exists a positive constant $\varepsilon > 0$ such that

$$|\mathcal{L}(t) - E(t)| \le \varepsilon E(t),$$

i.e.,

$$(1-\varepsilon)E(t) \le \mathcal{L}(t) \le (1+\varepsilon)E(t).$$

Noting that $\varepsilon > 0$ is small enough when ε_1 and ε_2 are small enough, we can get (4.14) when we choose $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ small enough. The proof is complete. □

Proof of Theorem 3.3. For any fixed $t_0 > 0$, we know that for any $t \geq t_0$,

$$\int_0^t g(s)\,ds \geq \int_0^{t_0} g(s)\,ds := g_0.$$

It follows from (4.1), (4.2) and (4.7) that for any $t \geq t_0$,

$$\begin{aligned}\mathcal{L}'(t) &= E'(t) + \varepsilon_1 \Phi'(t) + \varepsilon_2 \Psi'(t)\\ &\leq \left(\frac{|\mu_2|}{2} - \mu_1 + \frac{\xi}{2} + c_1\varepsilon_1 + \varepsilon_2(2\delta - g_0)\right)\|u_t(t)\|^2_{L^2_\rho}\\ &\quad + \left(\varepsilon_2(\delta + 2\delta(1-l)^2) - \frac{l\varepsilon_1}{2}\right)\|\nabla u(t)\|^2\\ &\quad + \left(\frac{|\mu_2|}{2} - \frac{\xi}{2}e^{-\sigma\tau} + c_2\varepsilon_1 + \varepsilon_2\delta\right)\int_{\mathbb{R}^n} \rho u_t^2(t-\tau)\,dx\\ &\quad + (c_3\varepsilon_1 + c_4\varepsilon_2)(g \circ \nabla u)(t) + \left(\frac{1}{2} - \frac{g(0)c_*^2}{4\delta}\varepsilon_2\right)(g' \circ \nabla u)(t).\end{aligned} \tag{4.15}$$

Case 1: $\mu_1 \neq 0$, $0 < |\mu_2| < \mu_1$.

Obviously, $e^{\sigma\tau}$ goes to 1 as $\sigma \to 0$. By using the continuity of the set of real numbers, then we can take σ so small that there exists a constant $\xi > 0$ such that

$$e^{\sigma\tau}|\mu_2| < \xi < \mu_1,$$

which gives us

$$\frac{|\mu_2|}{2} - \mu_1 + \frac{\xi}{2} < 0$$

and

$$\frac{|\mu_2|}{2} - \frac{\xi}{2e^{\sigma\tau}} < 0.$$

At this point we first choose $0 < \delta < g_0/2$ such that $2\delta - g_0 < 0$. For any fixed $\delta > 0$, we at last choose $\varepsilon_2 > 0$ and $\varepsilon_1 > 0$ small enough so that (4.14) remain valid, and further,

$$\varepsilon_2 < \min\left\{\frac{2\delta}{g(0)c_*^2}, \frac{1}{\delta}\left(\frac{\xi}{2e^{\sigma\tau}} - \frac{|\mu_2|}{2}\right)\right\}$$

and

$$\frac{2\varepsilon_2}{l}\left(\delta + 2\delta(1-l)^2\right) < \varepsilon_1 < \min\left\{\frac{\varepsilon_2}{c_1}(g_0 - 2\delta), \frac{\xi}{2c_2}e^{-\sigma\tau} - \frac{|\mu_2|}{2c_2} - \frac{\varepsilon_2\delta}{c_2}\right\},$$

which gives us

$$\frac{1}{2} - \frac{g(0)c_*^2\varepsilon_2}{4\delta} > 0, \quad \varepsilon_2\delta + \frac{|\mu_2|}{2} - \frac{\xi}{2e^{\sigma\tau}} < 0,$$
$$c_1\varepsilon_1 + \varepsilon_2(2\delta - g_0) < 0, \quad \varepsilon_2(\delta + 2\delta(1-l)^2) - \frac{l}{2}\varepsilon_1 < 0,$$

and

$$c_2\varepsilon_1 + \varepsilon_2\delta + \frac{|\mu_2|}{2} - \frac{\xi}{2}e^{-\sigma\tau} < 0.$$

From this it follows that there exist two positive constants γ_1 and γ_2 such that for any $t \ge t_0$,

$$\mathcal{L}'(t) \le -\gamma_1 E(t) + \gamma_2 (g \circ \nabla u)(t). \tag{4.16}$$

We multiply (4.16) by $\zeta(t)$ and use $\zeta(t)(g \circ \nabla u) \le -(g' \circ \nabla u) \le -2E'(t)$ to obtain

$$\begin{aligned}\zeta(t)\mathcal{L}'(t) &\le -\gamma_1 E(t)\zeta(t) + \gamma_2 \zeta(t)(g \circ \nabla u)(t) \\ &\le -\gamma_1 \zeta(t) E(t) - 2\gamma_2 E'(t),\end{aligned}$$

which implies

$$\zeta(t)\mathcal{L}'(t) + 2\gamma_2 E'(t) \le -\gamma_1 \zeta(t) E(t). \tag{4.17}$$

Let $\mathcal{E}(t) = \zeta(t)\mathcal{L}(t) + 2\gamma_2 E(t)$, then it is easy to get that $\mathcal{E}(t)$ is equivalent to the modified energy $E(t)$ by using (4.14), i.e., there exist two positive constants β_1 and β_2 such that

$$\beta_1 E(t) \le \mathcal{E}(t) \le \beta_2 E(t). \tag{4.18}$$

By using (4.17), (4.18) and $\zeta'(t) \le 0$, we infer that for any $t \ge t_0$,

$$\mathcal{E}'(t) \le -\gamma_1 \zeta(t) E(t) \le -\frac{\gamma_1}{\beta_2} \zeta(t) \mathcal{E}(t),$$

which gives us

$$\mathcal{E}(t) \le \mathcal{E}(t_0) \exp\left(-\frac{\gamma_1}{\beta_2} \int_{t_0}^t \zeta(s)\, ds \right).$$

Thus we have

$$E(t) \le \frac{\beta_2}{\beta_1} E(t_0) \exp\left(-\frac{\gamma_1}{\beta_2} \int_{t_0}^t \zeta(s)\, ds \right). \tag{4.19}$$

Therefore (3.2) follows by renaming the constants, and by the continuity and boundedness of $E(t)$.

Case 2: $\mu_1 = 0$, $|\mu_2| > 0$.

By using the same estimate as (4.15), we have for any $t \ge t_0$,

$$\begin{aligned}\mathcal{L}'(t) &= E'(t) + \varepsilon_1 \Phi'(t) + \varepsilon_2 \Psi'(t) \\ &\le \left(\frac{|\mu_2|}{2} + \frac{\xi}{2} + c_5 \varepsilon_1 + \varepsilon_2 (2\delta - g_0) \right) \|u_t(t)\|_{L^2_\rho}^2 \\ &\quad \left(\varepsilon_2 (\delta + 2\delta(1-l)^2) - \frac{l\varepsilon_1}{2} \right) \|\nabla u(t)\|^2 \\ &\quad + \left(\frac{|\mu_2|}{2} - \frac{\xi}{2} e^{-\sigma\tau} + c_5 \varepsilon_1 + \varepsilon_2 \delta \right) \int_{\mathbb{R}^n} \rho u_t^2 (t-\tau)\, dx \\ &\quad + (c_6 \varepsilon_1 + c_7 \varepsilon_2)(g \circ \nabla u)(t) + \left(\frac{1}{2} - \frac{g(0) c_*^2}{4\delta} \varepsilon_2 \right) (g' \circ \nabla u)(t),\end{aligned}$$

where c_i, $i = 5, 6, 7$, are positive constants.

At this point we first choose $\delta > 0$ so small that

$$\delta < \min\left\{\frac{g_0}{8}, \frac{g_0 l}{16c_5[1+2(1-l)^2]}\right\},$$

which yields

$$\frac{g_0}{8c_5} < \frac{g_0 - 4\delta}{2c_5}.$$

For fixed $\delta > 0$, we take ε_2 such that

$$0 < \varepsilon_2 < \frac{\delta}{2g(0)c_*^2}.$$

Then we know that

$$\frac{1}{2} - \frac{\varepsilon_2 g(0)c_*^2}{4\delta} > 0.$$

Next for any fixed $\delta > 0$ and $\varepsilon_2 > 0$, we choose $\varepsilon_1 > 0$ so small that (4.14) holds, and further

$$\max\left\{\frac{2}{l}[\varepsilon_2(\delta + 2\delta(1-l)^2)], \varepsilon_2\frac{g_0}{8c_5}\right\} < \varepsilon_1 < \min\left\{\frac{\delta_2(g_0 - 2\delta) - \varepsilon_2\delta}{2c_5}, \varepsilon_2\frac{g_0 - 4\delta}{2c_5}\right\},$$

which gives us

$$\varepsilon_2(g_0 - 2\delta) - \varepsilon_1 c_5 > \varepsilon_1 c_5 + \varepsilon_2\delta > 0 \tag{4.20}$$

and

$$\varepsilon_2(\delta + 2\delta(1-l)^2) - \frac{l\varepsilon_1}{2} < 0.$$

If we denote $\eta_1 := \varepsilon_2(g_0 - 2\delta) - \varepsilon_1 c_5$ and $\eta_2 := \varepsilon_1 c_5 + \varepsilon_2\delta$, it follows from (4.20) that $\eta_1 > \eta_2$. Note that $e^{\sigma\tau}$ goes to 1 as $\sigma \to 0$. Now we choose σ small enough so that there exists a positive constant ξ satisfies

$$2\eta_2 e^{\sigma\tau} < \xi < 2\eta_1.$$

Then we can get

$$2\eta_1 - \xi > 0 \quad \text{and} \quad \frac{\xi}{e^{\sigma\tau}} - 2\eta_2 > 0.$$

We select the constant μ_2 satisfies

$$|\mu_2| < \min\left\{2\eta_1 - \xi, \frac{\xi}{e^{\sigma\tau}} - 2\eta_2\right\} := a, \tag{4.21}$$

which implies

$$\frac{|\mu_2|}{2} + \frac{\xi}{2} < \eta_1 \quad \text{and} \quad \frac{\xi}{2e^{\sigma\tau}} - \frac{|\mu_2|}{2} > \eta_2.$$

From above it follows that

$$\frac{|\mu_2|}{2}+\frac{\xi}{2}+c_5\varepsilon_1+\varepsilon_2(2\delta-g_0)<0, \quad \varepsilon_2(\delta+2\delta(1-l)^2)-\frac{l\varepsilon_1}{2}<0,$$

and

$$\frac{1}{2}-\frac{g(0)c_*^2}{4\delta}\varepsilon_2>0, \quad \frac{|\mu_2|}{2}-\frac{\xi}{2}e^{-\sigma\tau}+c_5\varepsilon_1+\varepsilon_2\delta<0.$$

Therefore, there exist two positive constants γ_1 and γ_2 such that for any $t \geq t_0$,

$$\mathcal{L}'(t) \leq -\gamma_1 E(t)+\gamma_2(g\circ\nabla u)(t). \tag{4.22}$$

Since $\mu_1 = 0$, by (4.1), the energy $E(t)$ satisfies

$$E'(t) \leq \left(\frac{|\mu_2|}{2}+\frac{\xi}{2}\right)\|u_t(t)\|_{L^2_\rho}^2+\left(\frac{|\mu_2|}{2}-\frac{\xi}{2}e^{-\sigma\tau}\right)\int_{\mathbb{R}^n}\rho u_t^2(t-\tau)\,dx+\frac{1}{2}(g'\circ\nabla u)(t). \tag{4.23}$$

Multiplying (4.22) by $\zeta(t)$ and using Assumption (G2) and (4.23), we shall see that for any $t \geq t_0$,

$$\begin{aligned}\zeta(t)\mathcal{L}'(t) &\leq -\gamma_1\zeta(t)E(t)+\gamma_2\zeta(t)(g\circ\nabla u)(t)\\ &\leq -\gamma_1\zeta(t)E(t)-\gamma_2(g'\circ\nabla u)(t)\\ &\leq -\gamma_1\zeta(t)E(t)-2\gamma_2E'(t)+2\eta_1\gamma_2\|u_t(t)\|_{L^2_\rho}^2\\ &\leq -\gamma_1\zeta(t)E(t)-2\gamma_2E'(t)+4\eta_1\gamma_2E(t).\end{aligned} \tag{4.24}$$

If we assume

$$\zeta(t) > \frac{5\eta_1\gamma_2}{\gamma_1} := \zeta_0, \tag{4.25}$$

then we infer from (4.24) that there exists a positive constant γ_3 such that for any $t \geq t_0$,

$$\zeta(t)\mathcal{L}'(t)+2\gamma_2E'(t) \leq -\gamma_3E(t).$$

Therefore we can obtain (3.3) by using the similar analysis as (4.19). Then the proof of Theorem 3.3 is complete. □

Remark 4.5. By using the method in [12], our result can be easily extended to the case which the delay $\tau = \tau(t)$ is a function satisfying some suitable conditions, i.e., the delay is time-varying delay.

Remark 4.6. As in [13, 14, 18], we illustrate several rates of energy decay through the following examples.

1. If g decays exponentially, i.e., $\zeta(t) = a$, then (3.2) gives us

$$E(t) \leq \beta e^{-\gamma a t}.$$

2. If $\zeta(t) = a/(1+t)$, then (3.2) gives us
$$E(t) \leq \frac{\beta}{(1+t)^{\gamma a}}.$$

3. When $g(t) = ae^{-b(1+t)^{\alpha}}$ for $a, b > 0$ and $0 < \alpha \leq 1$, then we can choose $\zeta(t) = b\alpha(1+t)^{\alpha-1}$. Estimate (3.2) takes the form
$$E(t) \leq \beta \exp\left(-b\gamma(1+t)^{\alpha}\right).$$

4. If $g(t) = a\exp(-b\ln^{\alpha}(1+t))$ for $a, b > 0$ and $\alpha > 1$, we take $\zeta(t) = b\alpha\ln^{\alpha-1}(1+t)/(1+t)$. Estimate (3.2) takes the form
$$E(t) \leq \beta \exp\left(-\gamma b \ln^{\alpha}(1+t)\right).$$

5. Conclusion and open problems

In this work, we considered a linear viscoelastic wave equation with density and time delay in the whole space. In order to overcome the difficulties of the non-compactness of some operators in unbounded domains, we introduced some weighted spaces. Under suitable assumptions on the relaxation function, we established a general decay result of solution for the initial value problem by using the energy perturbation method. This study contains the exponential and polynomial rates as particular cases. Our result extends some recent works. Next we briefly discuss some possible extensions of our results and also state open problems on the subject.

(1) For the general decay result, it only valid for $0 < |\mu_2| < \mu_1$ and $\mu_1 = 0$, $|\mu_2| > 0$. Whether the stability property holds for $0 < \mu_1 = |\mu_2|$ is still open.

(2) One could address the problem when the real numbers μ_1 and μ_2 satisfies the condition $0 < \mu_1 < |\mu_2|$.

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