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Constructing Almost Peripheral and Almost Self-centered Graphs Revisited

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Abstract. The center and the periphery of a graph is the set of vertices with minimum resp. maximum eccentricity in it. A graph is almost self-centered (ASC) if it contains exactly two non-central vertices and is almost peripheral (AP) if all its vertices but one lie in the periphery. Answering a question from (Taiwanese J. Math. **18** (2014), 463–471) it is proved that for any integer $r \geq 1$ there exists an r-AP graph of order $4r-1$. Using this result it is proved that any graph G can be embedded into an r-AP graph by adding at most $4r-2$ vertices to G. A construction of ASC graphs from (Acta Math. Sin. (Engl. Ser.) **27** (2011), 2343–2350) is corrected and refined. Two new constructions of ASC graphs are also presented. Strong product graphs that are AP graphs are also characterized and it is shown that there are no strong product graphs that are ASC graphs. We conclude with some related open problems.

1. Introduction

If G is a graph, then the *distance* $d_G(u,v)$ between vertices u and v of G is the usual shortest-path distance. The *eccentricity* $\mathrm{ecc}_G(u)$ of the vertex u is $\max\{d_G(u,x) : x \in V(G)\}$. If G will be clear from the context we may shorten the notation to $d(u,v)$ and $\mathrm{ecc}(u)$, respectively. The minimum eccentricity and the maximum eccentricity over all vertices of G, respectively, are the *radius* $\mathrm{rad}(G)$ and the *diameter* $\mathrm{diam}(G)$ of G. The *center* $C(G)$ and the *periphery* $P(G)$ of G is the set of vertices of minimum, respectively maximum, eccentricity, their elements being called *central* resp. *peripheral vertices*. We refer to [15] for some general results on the structure of the center and the periphery of a graph and to [13] for some related extremality results.

Central and peripheral vertices are important, among others, in location theory and in the investigation of complex networks. If every vertex is central, in other words, when $C(G) = V(G)$ holds, then G is called a *self-centered (SC) graph* [2,8,14]. In this framework we also refer to related eccentric graphs [3] and eccentric digraphs [5]. Several classes of graphs closely related to self-centered graphs have also been introduced and studied. When

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$|C(G)| = |V(G)| - 2$ holds, that is, when G contains exactly two non-central vertices, G is called an *almost self-centered (ASC) graph* [1,9]. On the other hand, when $|P(G)| = |V(G)| - 1$ holds, that is, when all vertices but one are peripheral, G is called an *almost-peripheral (AP) graph* [10]. AP graphs recently found an application in the study of the so-called non-self-centrality of networks [17], while embeddings into ASC graphs were further investigated in [18]. (For some other recent studies of eccentricity see [4,7,12,16].) To summarize, a graph G is

- SC graph, if $|C(G)| = |V(G)|$,
- ASC graph, if $|C(G)| = |V(G)| - 2$, and
- AP graph, if $|P(G)| = |V(G)| - 1$.

Clearly, in an ASC graph the two non-central vertices are necessarily diametrical and hence peripheral, while in an AP graph the only non-peripheral vertex is a unique central vertex.

The rest of the paper is organized as follows. In the rest of the section additional definitions needed are given. In Section 2 constructions of AP graphs are considered while in Section 3 constructions of ASC graphs are treated. In Section 4, we propose several related open problems.

We say that G is an *r-SC graph* (resp. *r-ASC graph*, resp. *r-AP graph*) if G is an SC graph (resp. ASC graph, resp. AP graph) of radius r. A subgraph H of a graph G is *isometric* if $d_H(u,v) = d_G(u,v)$ holds for any $u,v \in V(H)$. The vertex deleted d-cube Q_d^-, $d \geq 1$, is obtained from the d-cube Q_d by removing one of its vertices. The *Cartesian product* $G \,\square\, H$ of graphs G and H is the graph with $V(G \,\square\, H) = V(G) \times V(H)$ and (g,h) is adjacent to (g',h') if and only if $gg' \in E(G)$ and $h = h'$, or $g = g'$ and $hh' \in E(H)$. The *strong product* $G \boxtimes H$ of graphs G and H is the graph obtained from the Cartesian product $G \,\square\, H$ by adding the edges between the vertices (g,h) and (g',h') for which $gg' \in E(G)$ and $hh' \in E(H)$ hold. Finally, for a positive integer n we will use the notation $[n] = \{1,2,\ldots,n\}$.

2. Constructing AP graphs

In [10] it was observed that Q_d^- is a $(d-1)$-AP graph for any $d \geq 2$. It was also demonstrated that for any integer $r \geq 2$ there exists an r-AP graph of order $4r+1$. Motivated by this result it was asked whether there exist r-AP graphs of order $n < 4r+1$ for any $r \geq 4$. We now reply the question as follows.

Theorem 2.1. *For any integer $r \geq 1$ there exists an r-AP graph of order $4r - 1$.*

Proof. P_3 and the vertex deleted 3-cube Q_3^- are required examples for $r = 1$ and $r = 2$, respectively. For an example that verifies the assertion for $r = 3$ consider the graph obtained from the 8-cycle on vertices $v_1, \ldots, v_8$ with natural adjacencies, and vertices x, y, z, where x is adjacent to v_1 and v_7, y to v_3 and v_5, and z to v_4 and v_8 (cf. [10, Figure 1]). It thus remains to prove the result for $r \geq 4$.

If $r \geq 4$, then let G_r be the graph constructed as follows. Its vertex set is

$$V(G_r) = \{v_1, \ldots, v_{2r+1}\} \cup \{u_1, \ldots, u_r\} \cup \{w_1, \ldots, w_{r-4}\} \cup \left\{v'_1, \ldots, v'_{r+1}\right\}.$$

In the particular case $r = 4$ we thus have $\{w_1, \ldots, w_{r-4}\} = \emptyset$. The edges of G_r are as follows. The vertices $v_1, \ldots, v_{2r+1}$ induce a cycle C (of length $2r + 1$). The vertices $v_1, \ldots, v_{r+1}, u_r, \ldots, u_1$ in the respective order induce another cycle C' (also of length $2r + 1$). For $i \in [r - 4]$, the vertex w_i is adjacent to u_{i+1} and to u_{i+2}. Finally, v'_1 is adjacent to u_1 and to v_{2r+1}, while the vertex v'_{r+1} is adjacent to u_r and to v_{r+2}. The graph G_r is schematically shown in Figure 2.1.

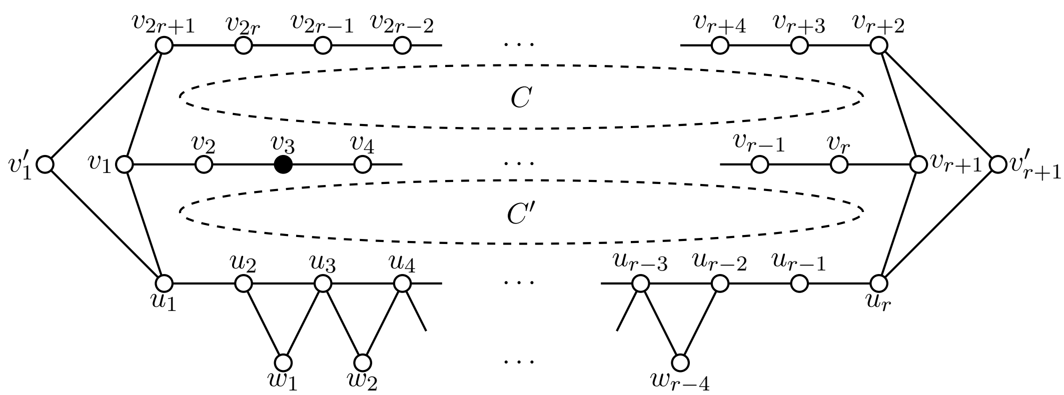

Figure 2.1: Graph G_r.

We claim that G_r is an r-AP graph. For this sake note first that both cycles C and C' are isometric subgraphs. In addition, the same assertion also holds for the cycle

$$C'' = u_1, u_2, \ldots, u_r, v'_{r+1}, v_{r+2}, \ldots, v_{2r+1}, v'_1, u_1$$

which is of length $2r + 2$. From these facts it quickly follows that the eccentricity of every vertex of C'' is $r + 1$. In particular, if $2 \leq i \leq r + 1$, then the vertex opposite to v_{r+i} on C'' is the vertex u_{i-1} and hence $d(v_{r+i}, u_{i-1}) = r + 1$. But then also $d(w_{i-2}, v_{r+i}) = r + 1$ for $3 \leq i \leq r - 2$. Therefore, also $\text{ecc}(w_i) = r + 1$, $i \in [r - 4]$. It remains to consider the vertices v_i, $i \in [r + 1]$. We first observe that $d(v_1, v'_{r+1}) = d(v_2, v'_{r+1}) = r + 1$ and that $d(v_r, v'_1) = d(v_{r+1}, v'_1) = r + 1$. Next, for $4 \leq i \leq r - 1$, $d(v_i, w_{r-i}) = r + 1$. Finally, $\text{ecc}(v_3) = r$, so we conclude that $C(G_r) = \{v_3\}$ and $P(G_r) = V(G_r) \setminus \{v_3\}$.

Since $|V(G_r)| = (2r + 1) + r + (r - 4) + 2 = 4r - 1$, the argument is complete. □

If G is a graph and r is a positive integer, then the r-AP index of G is defined [10] as follows:

$$\mathrm{AP}_r(G) = \min\{|V(H)| - |V(G)| : H \text{ is } r\text{-AP graph, } G \text{ induced in } H\}.$$

Using Theorem 2.1 we get:

Theorem 2.2. *If* $r \geq 1$ *and* G *is an arbitrary graph, then* $\mathrm{AP}_r(G) \leq 4r - 2$.

Proof. Let G be a graph and $r \geq 1$. Let H_r be an r-AP graph of order $4r-1$. The graph H_r exists by Theorem 2.1. Let $C(H_r) = \{u\}$ and let x be an arbitrary peripheral vertex of H_r. We now construct the graph $\widehat{G_r}$ as follows. First set $V(\widehat{G_r}) = (V(H_r) \setminus \{x\}) \cup V(G)$. The edge set of $\widehat{G_r}$ consists of $E(G)$, all the edges of H_r not incident to x, and all the edges ww', where $w \in V(G)$ and w' is a neighbor of x in H_r. In other words, $\widehat{G_r}$ is obtained from H_r by replacing x with G and connecting all the vertices of G to all the neighbors of x.

We now consider the eccentricities of the vertices of $\widehat{G_r}$. To simplify the notation, we will refer to them as vertices from $V(G)$ and from $V(H_r)$. Let first $w \in V(G)$. Then for any other vertex $w' \in V(G)$, $d_{\widehat{G_r}}(w, w') \leq 2$. In addition, for $w'' \in V(H_r)$, $d_{\widehat{G_r}}(w, w'') = d_{H_r}(x, w'')$. Consequently, $\mathrm{ecc}_{\widehat{G_r}}(w) = r + 1$. Let next $w \in V(H_r)$. Then for any $w' \in V(H_r)$, $d_{\widehat{G_r}}(w, w') = d_{H_r}(w, w')$, and for any $w' \in V(G)$, $d_{\widehat{G_r}}(w, w') = d_{H_r}(w, x)$. It follows that $\mathrm{ecc}_{\widehat{G_r}}(w) = \mathrm{ecc}_{H_r}(w)$. But then $\widehat{G_r}$ is an r-AP graph with $C(\widehat{G_r}) = \{u\}$. □

The construction from the proof of Theorem 2.2 improves the construction from [10, Theorem 2.3]. The latter construction would lead to a weaker conclusion asserting that $\mathrm{AP}_r(G) \leq 4r - 1$.

Remark 2.3. The upper bound for $\mathrm{AP}_r(G)$ in Theorem 2.2 is not sharp in general. For example, it was proved in [10, Theorem 3.1] that $\mathrm{AP}_2(G) \leq 5$ for an arbitrary graph G with at least two vertices and that equality holds precisely for complete graphs.

Different graph operations are often useful to construct families of graphs with given properties. For the AP graphs, the strong product appears to be such. Before characterizing AP strong product, we observe the following.

Lemma 2.4. *If* G *and* H *are connected graphs and* $(g, h) \in V(G \boxtimes H)$, *then*

$$\mathrm{ecc}_{G \boxtimes H}((g, h)) = \max\{\mathrm{ecc}_G(g), \mathrm{ecc}_H(h)\}.$$

Proof. We can argue as follows:

$$\begin{aligned}
\mathrm{ecc}_{G \boxtimes H}((g, h)) &= \max\left\{d_{G \boxtimes H}((g, h), (g', h')) : (g', h') \in V(G \boxtimes H)\right\} \\
&= \max\left\{\max\left\{d_G(g, g'), d_H(h, h')\right\} : (g', h') \in V(G \boxtimes H)\right\} \\
&= \max\left\{\max\left\{d_G(g, g') : g' \in V(G)\right\}, \max\left\{d_H(h, h') : h' \in V(H)\right\}\right\} \\
&= \max\{\mathrm{ecc}_G(g), \mathrm{ecc}_H(h)\},
\end{aligned}$$

where in the second equality we have used the well-known fact that for any connected graphs G and H, $d_{G \boxtimes H}((g,h),(g',h')) = \max\{d_G(g,g'), d_H(h,h')\}$, see [6, Proposition 5.4]. □

Proposition 2.5. *Let G and H be connected graphs and $r \geq 1$. Then $G \boxtimes H$ is an r-AP graph if and only if G and H are r-AP graphs.*

Proof. Suppose first that G and H are r-AP graphs. Let $C(G) = \{g\}$ and $C(H) = \{h\}$. Then by Lemma 2.4, $\mathrm{ecc}_{G \boxtimes H}((g,h)) = \max\{r,r\} = r$. Similarly, for any $h' \in V(H)$, $h' \neq h$, $\mathrm{ecc}_{G \boxtimes H}((g,h')) = \max\{r, r+1\} = r+1$ and for any $g' \in V(G)$, $g' \neq g$, $\mathrm{ecc}_{G \boxtimes H}((g',h)) = \max\{r+1, r\} = r+1$. Finally, if $g' \in V(G)$, $g' \neq g$, and $h' \in V(H)$, $h' \neq h$, then $\mathrm{ecc}_{G \boxtimes H}((g',h')) = \max\{r+1, r+1\} = r+1$. Hence $G \boxtimes H$ is an r-AP graph with $C(G \boxtimes H) = \{(g,h)\}$.

Conversely, suppose that $G \boxtimes H$ is an r-AP graph. Let $C(G \boxtimes H) = \{(g,h)\}$, so that $\mathrm{ecc}_{G \boxtimes H}((g,h)) = r$ while the eccentricity of any other vertex of $G \boxtimes H$ is $r+1$. By Lemma 2.4 and by the commutativity of the strong product we may without loss of generality assume that $\mathrm{ecc}_G(g) = r$ and $\mathrm{ecc}_H(h) = s$, where $r \geq s$. Then for any vertex $g' \in V(G)$, $g' \neq g$, we have $\mathrm{ecc}_{G \boxtimes H}((g',h)) = r+1 = \max\{\mathrm{ecc}_G(g'), s\}$, hence $\mathrm{ecc}_G(g') = r+1$. This implies that G is an r-AP graph. Similarly, for any vertex $h' \in V(H)$, $h' \neq h$, we must have $\mathrm{ecc}_H(h') = r+1$. But this implies (since $s \leq r$) that actually $s = r$ holds which in turn implies that also H is an r-AP graph. □

To conclude the section we note that no non-trivial Cartesian product graph is an AP graph. Indeed, assume on the contrary that $G \,\Box\, H$ is an AP graph, where both G and H are connected graphs on at least two vertices. Let $C(G \,\Box\, H) = \{(g,h)\}$ and let $\mathrm{ecc}_G(g) = a$ and $\mathrm{ecc}_H(h) = b$. Then, since both G and H have at least two vertices, G contains a vertex of eccentricity $a+1$, and H contains a vertex of eccentricity $b+1$. But then the set of eccentricities of $G \,\Box\, H$ contains $a+b$, $(a+1)+b = a+(b+1)$, and $(a+1)+(b+1)$, so $G \,\Box\, H$ cannot be an AP graph.

3. Constructing ASC graphs

In this section we are interested in constructions of ASC graphs. We first correct a related result from the literature. Then we continue with two novel constructions of ASC graphs from some specific graphs and discuss the corresponding graphs with exactly two different eccentricities. At the end of the section we observe that there are no non-trivial strong or Cartesian product graphs that are ASC graphs.

In [9, Theorem 2.3] the following result was stated.

Let G be an r-SC graph, u an arbitrary vertex of G, and X the set of eccentric vertices of u. Let H be a graph obtained from G by joining a new vertex x to all vertices of X. If the subgraph of G induced by X is of diameter at most 2, then H is an r-ASC graph.

To see that H need not be an r-ASC graph, consider first the sporadic example shown on the left-hand side of Figure 3.1. The graph G from the figure is a 2-SC graph, and u' is the only diametrical vertex of u, so that $X = \{u'\}$. The graph H is thus obtained from G by connecting the new vertex x to u'. But now in H, the vertex x has three diametrical vertices, so H is not an ASC graph.

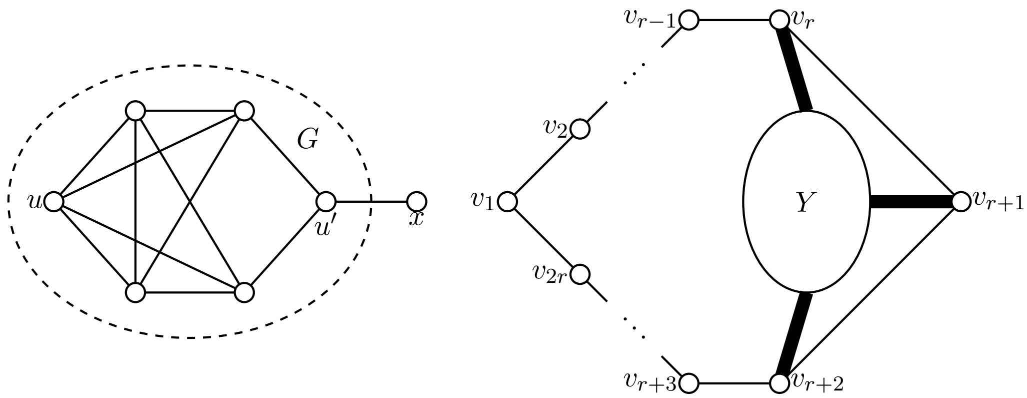

Figure 3.1: Sporadic example and graphs Y_r.

More generally, consider the following infinite family. If Y is an arbitrary graph and $r \geq 3$, then construct the graph Y_r as follows. Take the disjoint union of Y and the cycle C_{2r} whose consecutive vertices are $v_1, v_2, \ldots, v_{2r}$, and connect every vertex of Y with v_r, v_{r+1} and v_{r+2}. In other words, take the join between Y and the subgraph of C_{2r} induced by the vertices v_r, v_{r+1} and v_{r+2}. The construction is shown on the right-hand side of Figure 3.1, where the thick lines represent the joints between Y and vertices v_r, v_{r+1} and v_{r+2}, respectively.

Note that the cycle C_{2r} is an isometric subgraph of Y_r and that v_1 is the unique eccentric vertex for any vertex y of Y. Note also that $d_{Y_r}(y, v_1) = r$. Consecutively, Y_r is an r-SC graph. Select now u to be an arbitrary vertex of Y (or the vertex v_{r+1}), then the set of diametrical vertices of u consists of the single vertex v_1. Hence the graph H from the above statement is obtained from Y_r by attaching a pendant vertex x to v_1. But now $\mathrm{ecc}_H(x) = r+1$ with $\mathrm{Ecc}_H(x) = \{v_{r+1}\} \cup V(Y)$, (The notation $\mathrm{Ecc}_G(x)$ stands for the set of eccentric vertices of x in G, that is, vertices y with $d_G(x, y) = \mathrm{ecc}_G(x)$.) so H is not an r-ASC graph.

We now correct [9, Theorem 2.3] as follows. We include its complete proof for the sake of completeness and to point out the missing point in the original argument.

Theorem 3.1. *Let G be an r-SC graph, u an arbitrary vertex of G, and X the set of eccentric vertices of u, where the subgraph of G induced by X is of diameter at most 2. Let H be the graph obtained from G by joining a new vertex x_0 to all vertices of X. Then H is an r-ASC graph if and only if $\bigcap_{x\in X} \mathrm{Ecc}_G(x) = \{u\}$.*

Proof. Note first that $d_H(x_0, u) = r + 1$. Consider next $w \in V(H) \setminus \{x_0, u\}$ and let x' be an arbitrary vertex from X. Since G is an r-SC graph, $d_G(w, x') \le r$ and consequently $d_H(x_0, w) \le r + 1$. We conclude that $\mathrm{ecc}_H(x_0) = r + 1$. By the same argument we also get that $\mathrm{ecc}_H(u) = r + 1$.

Consider next a vertex $w \in V(H) \setminus \{u, x_0\}$. Since G is an r-SC graph, there exists a vertex $w' \in V(G)$ such that $d_G(w, w') = r$. Select an arbitrary shortest (w, w')-path in H, denote it with P. If x_0 does not lie on P, then P is a path that completely lies in G and hence its length is r. Otherwise P passes x_0, that is, P is of the form $P'x'x_0x''P''$, where $x', x'' \in X$ and P' is a (w, x')-subpath in P and P'' is an (x'', w')-subpath in P. Note that x' and x'' are not adjacent, since otherwise P would not be a shortest path. In addition, since X induces a subgraph of G of diameter at most 2, there exists a vertex $y \in X$ adjacent to x' and x''. Then the path $P'x'yx''P''$ is a shortest (w, w')-path of length r in G. Therefore $d_H(w, w') = r$. Similarly, if $w'' \in V(G)$, $w'' \ne w, w'$, then $d_H(w, w'') \le r$.

It remains to consider $d_H(w, x_0)$, where $w \in V(G) \setminus \{u\}$. (This case is missing in the proof of [9, Theorem 2.3].) We claim that $d_H(w, x_0) \le r$ holds for all $w \in V(G)\setminus\{u\}$ if and only if $\bigcap_{x\in X} \mathrm{Ecc}_G(x) = \{u\}$. Suppose first that $d_H(w, x_0) \le r$ holds for all $w \in V(G)\setminus\{u\}$. Then $d_H(w, x') \le r - 1$ holds for one vertex $x' \in X$. Consequently, $\bigcap_{x\in X} \mathrm{Ecc}_G(x) = \{u\}$. Conversely, suppose that $\bigcap_{x\in X} \mathrm{Ecc}_G(x) = \{u\}$. Since $d_H(x_0, u) = r+1$, it follows that for each vertex $w \in V(G) \setminus \{u\}$, there exists $x' \in X$ such that $d_H(x', w) \le r - 1$. Therefore, $d_H(w, x_0) \le r$ (for all $w \in V(G) \setminus \{u\}$). In conclusion, H is an r-ASC graph if and only if $\bigcap_{x\in X} \mathrm{Ecc}_G(x) = \{u\}$. □

We remark that all the examples of ASC graphs provided in [9] related to Theorem 3.1 fulfill the condition $\bigcap_{x\in X} \mathrm{Ecc}_H(x) = \{u\}$.

In the following theorem we give a novel construction of ASC graphs from a graph with special central structure by adding only one vertex.

Theorem 3.2. *Let $r \ge 2$ and let G be a connected graph with $\mathrm{rad}(G) = r - 1$ and $\mathrm{diam}(G) = r$. If there exists a vertex $u \in \bigcap_{\mathrm{ecc}_G(v)=r-1} \mathrm{Ecc}_G(v)$, such that $|\mathrm{Ecc}_G(u)| = 1$, then the graph H obtained by attaching a new pendant vertex x to u in G, is an r-ASC graph.*

Proof. Assume that $\mathrm{Ecc}_G(u) = \{w\}$. Note that $\mathrm{ecc}_G(u) = r$ for otherwise $\mathrm{ecc}_G(u) = r - 1$ would hold which in turn implies that $d_G(u, u) = 0 = r - 1$, which is not possible.

Then we immediately get $d_H(x,w) = r+1$. It follows that $\mathrm{ecc}_H(u) = r$ from the fact that x is pendant in H. Moreover, we have $\mathrm{ecc}_H(v) = r$ for any vertex $v \in V(G)$ with $\mathrm{ecc}_G(v) = r-1$ since $x \in \mathrm{Ecc}_H(v)$ with $d_H(v,x) = r$. For any vertex $y \in V(G) \setminus \{u,w\}$ different from any vertex v with $\mathrm{ecc}_G(v) = r-1$, we first find that $d_H(y,x) \le r$ since u is not an eccentric vertex of y in G. Further, there is at least one vertex $y' \in \mathrm{Ecc}_G(y)$ with $d_H(y,y') = d_G(y,y') = r$. Therefore we have $\mathrm{ecc}_H(y) = r$. Thus H is an r-ASC graph as desired. □

Note that the smallest example for Theorem 3.2 is provided by $G = P_3$ in which case the resulting 2-ASC graph is $H = P_4$. For another sporadic example consider the vertex deleted 3-cube Q_3^- (or any Q_d^-, $d \ge 3$, for that purpose) in which case the resulting 3-ASC graph is obtained from Q_3^- by attaching a new vertex to one of its vertices of degree 2.

A key property that we require from the graph G in Theorem 3.2 is that

$$|\{\mathrm{ecc}(u) : u \in V(G)\}| = 2. \tag{3.1}$$

Graphs fulfilling (3.1) form a wide generalization of ASC graphs, of AP graphs, as well as of the so-called WAP graphs that were introduced in [17] as the graphs in which all but two vertices lie in the periphery. The remaining two vertices of a WAP graph then both lie in the center from the following result due to Lesniak [11]: If G is a connected graph, then for each integer k in the range $\mathrm{rad}(G) < k \le \mathrm{diam}(G)$, the graph G contains at least two vertices of eccentricity k. Hence, if $|C(G)| = 1$ or 2, then the graph G fulfilling (3.1) is an $(r-1)$-AP and an $(r-1)$-WAP graph, respectively. And if $|C(G)| = |V(G)| - 2$, the graph G fulfilling (3.1) is an $(r-1)$-ASC graph.

To conclude the section we give an infinite family of graphs that fulfill condition (3.1). Let $r \ge 3$ and consider the graph as schematically shown in Figure 3.2.

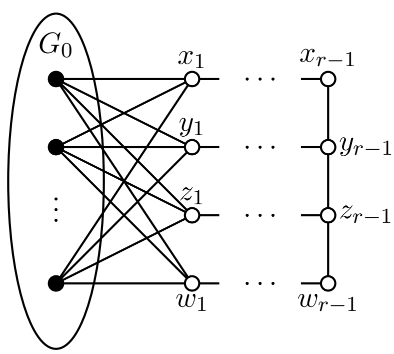

Figure 3.2: Graph G with only two eccentricities $r-1$ and r.

Here G_0 is an arbitrary graph, each of whose vertices is adjacent to any vertex from $\{x_1, y_1, z_1, w_1\}$. The black vertices have eccentricity $r-1$ and the white vertices have eccentricity r in G. It is then straightforward to verify that the graph G satisfies the conditions of Theorem 3.2, since the vertex z_{r-1} (y_{r-1}, resp.) can be viewed as u in the statement of Theorem 3.2, and x_1 (w_1, resp.) as w in the above proof.

After establishing Theorem 3.2, we naturally ask the following question: When an ASC graph can be constructed by adding only two pendant vertices to a given graph? As a special example, for a connected graph G of order n with two vertices u and v of degree $n-1$, we can get a 2-ASC graph by adding one pendant vertex to u and the other pendant vertex to v. In the following theorem we propose a more general related result for the graphs fulfilling (3.1).

Theorem 3.3. *Let $r \geq 3$, let G be an $(r-1)$-WAP graph with $C(G) = \{u, v\}$, and let H be the graph obtained from G by attaching a pendant vertex u' to u and a pendant vertex $v' \neq u'$ to v. If $d_G(u, v) = r-1$, then H is an r-ASC graph.*

Proof. Clearly, $\mathrm{ecc}_G(u) = \mathrm{ecc}_G(v) = r-1$. In addition, $\mathrm{ecc}_H(u') = \mathrm{ecc}_H(v') = r+1$ by the definition of a WAP graph, the construction of H from G, and the fact that $d_G(u, v) = r-1$. So it suffices to show that $\mathrm{ecc}_H(w) = r$ for any vertex $w \in V(G)$. Note that u' is a pendant vertex adjacent to u with $\mathrm{ecc}_H(u') = r+1$. It follows that $\mathrm{ecc}_H(u) = r$. Similarly, we have $\mathrm{ecc}_H(v) = r$. For any vertex $w \in V(G) \setminus \{u, v\}$, we observe that $d_G(u, w) \leq r-1$ and $d_G(v, w) \leq r-1$, that is, $d_H(u', w) \leq r$ and $d_H(v', w) \leq r$. Moreover, there is a vertex $w' \in \mathrm{Ecc}_G(w)$ with $d_G(w', w) = r$. Then we conclude that $\mathrm{ecc}_H(w) = r$. Thus H is an r-ASC graph as desired. □

In Figure 3.3 an example of an $(r-1)$-WAP graph which satisfies the condition of Theorem 3.3 is presented. Its central vertices are v_1 and w_r and are emphasized with filled vertices.

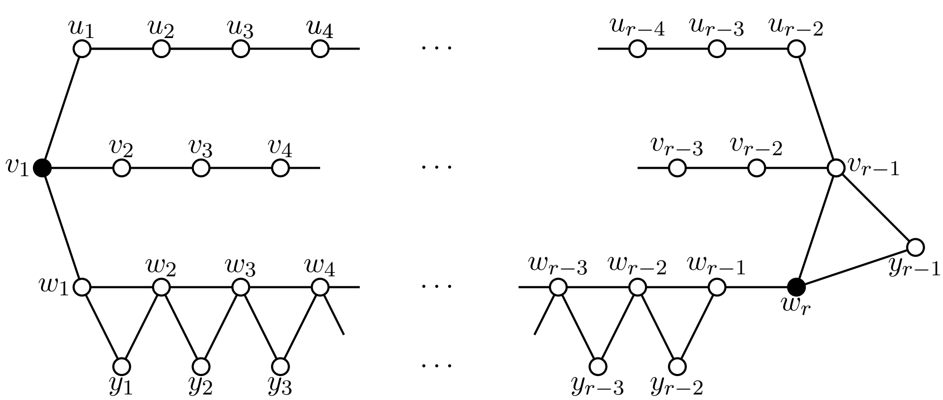

Figure 3.3: An $(r-1)$-WAP graph with $r \geq 3$.

We conclude the section with the following negative result.

Proposition 3.4. *Let G and H be connected graphs on at least two vertices. Then neither $G \boxtimes H$ nor $G \,\square\, H$ is an ASC graph.*

Proof. Suppose on the contrary that $G \boxtimes H$ is an r-ASC graph for some $r \geq 1$. Let (g, h) be one of the two peripheral vertices of $G \boxtimes H$ so that $\mathrm{ecc}_{G \boxtimes H}((g, h)) = r + 1$. Using the commutativity of the strong product we may without loss of generality assume that $\mathrm{ecc}_H(h) \leq \mathrm{ecc}_G(g) = r + 1$. If h' is another vertex of H, then $\mathrm{ecc}_{G \boxtimes H}((g, h')) \geq r + 1$ and thus necessarily $\mathrm{ecc}_{G \boxtimes H}((g, h')) = r + 1$. Hence, $G \boxtimes H$ is an r-ASC graph, it follows that for any $g' \neq g$, we have $\mathrm{ecc}_{G \boxtimes H}((g', h)) = r$ which (in view of Lemma 2.4) in turn implies that $\mathrm{ecc}_G(g') \leq r$. This would means that g is the unique vertex of G with the (maximum) eccentricity $r + 1$. We have a contradiction because a vertex of G that is diametrical to g must have eccentricity at least $r + 1$.

Suppose next that $G \,\square\, H$ is an r-ASC graph for some $r \geq 1$ and let (g, h) and (g', h') be the vertices of $G \,\square\, H$ with $d_{G \,\square\, H}((g, h), (g', h')) = r + 1$. Clearly, $g \neq g'$ and $h \neq h'$. But then $d_{G \,\square\, H}((g, h'), (g', h)) = r + 1$, a contradiction. □

4. Some open problems

In this section we propose some open problems for further research.

In view of Theorem 2.2 and Remark 2.3 we pose:

Problem 4.1. Let $r \geq 3$. Does there exist a graph G with $\mathrm{AP}_r(G) = 4r - 2$?

We further elaborate the above problem into:

Problem 4.2. If the answer to Problem 4.1 is positive, is K_1 a unique such graph? In addition, are complete graphs extremal with respect to AP_r in the class of all graphs on at least two vertices?

Moreover, determining the minimum order of r-AP graphs (for a given r) would be helpful for the above problems.

If G is a graph and r is a positive integer, then the r-ASC index of G is defined [9] as follows:

$$\mathrm{ASC}_r(G) = \min \{|V(H)| - |V(G)| : H \text{ is } r\text{-ASC graph, } G \text{ induced in } H\}.$$

Note that the graph G defined in Theorem 3.2 has $\mathrm{ASC}_r(G) = 1$. In view of it, we naturally give:

Problem 4.3. Characterize the graphs G with $\mathrm{ASC}_r(G) = 1$.

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