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Some Results on Skew Generalized Power Series Rings

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Abstract. Let R be a ring, $(S, \leq)$ a strictly ordered monoid and $\omega\colon S \to \operatorname{End}(R)$ a monoid homomorphism. The skew generalized power series ring $R[\![S,\omega]\!]$ is a common generalization of (skew) polynomial rings, (skew) power series rings, (skew) Laurent polynomial rings, (skew) group rings, and Mal'cev-Neumann Laurent series rings. In this paper, we continue the study of skew generalized power series ring $R[\![S,\omega]\!]$. It is shown that under suitable conditions, if R has a (flat) projective socle, then so does $R[\![S,\omega]\!]$. Necessary and sufficient conditions are obtained for $R[\![S,\omega]\!]$ to satisfy a certain ring property which is among being local, semilocal, semiperfect, semiregular, left quasi-duo, clean, exchange, right stable range one, projective-free, and I-ring, respectively.

1. Introduction

Throughout this paper all monoids and rings are with identity element that is inherited by submonoids and subrings and preserved under homomorphisms, but neither monoids nor rings are assumed to be commutative.

A partially ordered set $(S, \leq)$ is called *artinian* if every strictly decreasing sequence of elements of S is finite, and $(S, \leq)$ is called *narrow* if every subset of pairwise order-incomparable elements of S is finite. Thus, $(S, \leq)$ is artinian and narrow if and only if every nonempty subset of S has at least one but only a finite number of minimal elements. An *ordered monoid* is a pair $(S, \leq)$ consisting of a monoid S and an order $\leq$ on S such that for all $a, b, c \in S$, $a \leq b$ implies $ca \leq cb$ and $ac \leq bc$. An ordered monoid $(S, \leq)$ is said to be *strictly ordered* if for all $a, b, c \in S$, $a < b$ implies $ca < cb$ and $ac < bc$.

For a strictly ordered monoid S and a ring R, Ribenboim [31] defined the ring of generalized power series $R[\![S]\!]$ consisting of all maps from S to R whose support is artinian and narrow with the pointwise addition and the convolution multiplication. This construction provided interesting examples of rings (e.g., Elliott and Ribenboim, [4]; Ribenboim, [29, 30]) and it was extensively studied by many authors. In [16], R. Mazurek and

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M. Ziembowski, introduced a "twisted" version of the Ribenboim construction and studied when it produces a von Neumann regular ring. Now we recall the construction of the skew generalized power series ring introduced in [16]. Let R be a ring, $(S, \leq)$ a strictly ordered monoid, and $\omega\colon S \to \mathrm{End}(R)$ a monoid homomorphism. For $s \in S$, let ω_s denote the image of s under ω, that is $\omega_s = \omega(s)$. Let A be the set of all functions $f\colon S \to R$ such that the support $\mathrm{supp}(f) = \{s \in S : f(s) \neq 0\}$ is artinian and narrow. Then for any $s \in S$ and $f, g \in A$ the set

$$X_s(f,g) = \{(x,y) \in \mathrm{supp}(f) \times \mathrm{supp}(g) : s = xy\}$$

is finite. Thus one can define the product $fg\colon S \to R$ of $f, g \in A$ as follows:

$$fg(s) = \sum_{(u,v) \in X_s(f,g)} f(u)\omega_u(g(v)),$$

(by convention, a sum over the empty set is 0). With pointwise addition and multiplication as defined above, A becomes a ring, called *the ring of skew generalized power series* with coefficients in R and exponents in S (one can think of a map $f\colon S \to R$ as a formal series $\sum_{s\in S} r_s s$, where $r_s = f(s) \in R$) and denoted either by $R[\![S^{\leq}, \omega]\!]$ or by $R[\![S, \omega]\!]$ (see [14–16, 26, 27]).

The skew generalized power series construction embraces a wide range of classical ring-theoretic extensions, including skew polynomial rings, skew power series rings, skew Laurent polynomial rings, skew group rings, Mal'cev-Neumann Laurent series rings and of course the "untwisted" versions of all of these (for details see Section 2). Hence it can be applied to unify various results known for particular extensions. We would like to stress that using this general approach, in this paper we not only unified the already known theorems, but also obtained many new results, for several constructions simultaneously.

In this paper, we show that, if R has a projective socle, then so does $R[\![S, \omega]\!]$, where $(S, \leq)$ is a positively quasitotally ordered monoid and $\omega\colon S \to \mathrm{Aut}(R)$ a monoid homomorphism. Furthermore, in the case where S is positively ordered, we characterize when $R[\![S, \omega]\!]$ is local, semilocal, semiperfect, semiregular, left quasi-duo, clean, exchange, I-ring, and projective-free, respectively. In particular, we prove that $R[\![S, \omega]\!]$ is isomorphic to a full matrix ring over a local ring if and only if the ring R is isomorphic to a full matrix ring over a local ring. Also, we prove that several properties, including the semiboolean, right stable range one and 2-good property, transfer between R and the extension $R[\![S, \omega]\!]$. As an application, we provide (apparently) new examples of the aforementioned ring constructions.

2. Skew generalized power series rings over a ring with projective socle

Throughout this paper, all rings are associative. For a ring R, we denote by $U(R)$ and $J(R)$ the multiplicative group of units, and the Jacobson radical of R, respectively. For a nonempty subset X of R, $r_R(X)$ (resp. $\ell_R(X)$) is used for the right (resp. left) annihilator of X over R. We will denote by $\mathrm{End}(R)$ the monoid of ring endomorphisms of R, and by $\mathrm{Aut}(R)$ the group of ring automorphisms of R. The left socle of R will be symbolized by $\mathrm{Soc}(R)$. Also, we use $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{Q}$ and $\mathbb{R}$ for the set of integers, the set of positive integers, the field of rational numbers and the field of real numbers, respectively. We will use the symbol 1 to denote the identity element of the monoid S, the ring R, and the ring $R[\![S,\omega]\!]$, as well as the trivial monoid homomorphism $1\colon S \to \mathrm{End}(R)$ that sends every element of S to the identity endomorphism.

To each $r \in R$ and $s \in S$, we associate elements $c_r, e_s \in R[\![S,\omega]\!]$ defined by

$$c_r(x) = \begin{cases} r & x = 1, \\ 0 & x \in S \setminus \{1\}, \end{cases} \qquad e_s(x) = \begin{cases} 1 & x = s, \\ 0 & x \in S \setminus \{s\}. \end{cases}$$

It is clear that $r \mapsto c_r$ is a ring embedding of R into $R[\![S,\omega]\!]$ and $s \mapsto e_s$ is a monoid embedding of S into the multiplicative monoid of the ring $R[\![S,\omega]\!]$, and $e_s c_r = c_{\omega_s(r)} e_s$.

The construction of skew generalized power series rings generalizes some classical ring constructions such as polynomial rings ($S = \mathbb{N} \cup \{0\}$ with usual addition, and trivial $\leq$ and ω), monoid rings (trivial $\leq$ and ω), skew polynomial rings ($S = \mathbb{N} \cup \{0\}$ with usual addition and trivial $\leq$), skew Laurent polynomial rings ($S = \mathbb{Z}$ with usual addition and trivial $\leq$), skew monoid rings (trivial $\leq$), skew power series rings ($S = \mathbb{N} \cup \{0\}$ with usual addition and usual order), skew Laurent series rings ($S = \mathbb{Z}$ with usual addition and usual order) (see [32]), the Mal'cev-Neumann construction ($(S,\leq)$ a totally ordered group and trivial ω); (see [2, p. 528]), the Mal'cev-Neumann construction of twisted Laurent series rings ($(S,\leq)$ a totally ordered group); (see [8, p. 242]), and generalized power series rings $R[\![S]\!]$ (trivial ω; see [31, Section 4]), twisted generalized power series rings (see [16] and [12, Section 2]).

Recall from [14] that an ordered monoid $(S,\leq)$ is called *quasitotally ordered* (and that $\leq$ is a quasitotal order on S) if $\leq$ can be refined to an order $\preceq$ with respect to which S is a strictly totally ordered monoid. The class of quasitotally ordered monoids is quite large and important. For example, this class includes the linearly ordered monoids, submonoids of a free group, and torsion-free nilpotent groups (see [25, Lemma 13.1.6 and Corollary 13.2.8]). Also, every commutative, torsion-free, and cancellative ordered monoid is quasitotally ordered monoid (e.g., see [28, 3.3]). We say that an ordered monoid $(S,\leq)$ is positively ordered if 1 is the minimal element of S.

A ring R is called a *left PS-ring* if the left socle, $\mathrm{Soc}({}_RR)$ is projective. These rings were studied by Gordon in [6] and Nicholson and Watters in [23]. The concept of a PS-ring is not left-right symmetric by [23, Example 2.7]. The class of rings with projective socles includes all semiprime rings, nonsingular rings, V-rings (i.e., rings all of whose simple right modules are injective) and PP rings (i.e., rings all of whose principal left ideals are projective). In particular every Baer ring (i.e., rings in which every left (or right) annihilator is generated by an idempotent) is a PS-ring. Nicholson and Watters in [23] proved that R has a projective socle if and only if full matrix ring $M_n(R)$ has a projective socle by showing somewhat more. They proved that having a projective socle is a Morita invariant. In [23, Theorem 3.1], they showed that if R has a projective socle, then so does the polynomial ring $R[x]$ (and the power series ring $R[\![x]\!]$), but the converse is false. In [13], Liu Zhongkui and Li Fang showed that the commutative PS-ring condition is preserved by generalized power series ring $R[\![S,\leq]\!]$, where $(S,\leq)$ is a positively strictly totally ordered monoid. The motivation of this section is to investigate the PS property of the skew generalized power series rings.

The following characterization of a left PS-ring involving maximal left ideals has been presented by Nicholson and Watters in [23, Theorem 2.4]. This lemma plays a fundamental role to achieve our aim in this section.

Lemma 2.1. *The following conditions are equivalent for a ring R:*

(1) *R is a left PS-ring.*

(2) *If I is a maximal left ideal of R, then $r_R(I) = eR$, where $e^2 = e \in R$.*

Theorem 2.2. *Let R be a ring and $(S,\leq)$ be a positively quasitotally ordered monoid. Assume that $\omega\colon S \to \mathrm{Aut}(R)$ is a monoid homomorphism. If R is a right PS-ring, then $R[\![S,\omega]\!]$ is a right PS-ring.*

Proof. We adapt the proof of [13, Theorem 4]. Let I be a maximal right ideal of $A = R[\![S,\omega]\!]$. By the right-sides version of Lemma 2.1, it is enough to show that $\ell_A(I) = Ae$ for some idempotent $e \in A$. By hypothesis, the order $(S,\leq)$ can be refined to a strict total order $\preceq$ on S. For every $0 \neq f \in A$, since $\mathrm{supp}(f)$ is a nonempty artinian and narrow subset of S, the set of minimal elements of $\mathrm{supp}(f)$ is finite and nonempty. Thus there exists a unique minimal element of $\mathrm{supp}(f)$ under the total order $\preceq$, which will be denoted by $\pi(f)$.

For every $s \in S$, set $I_s = \{f(s) \mid f \in I, \pi(f) = s\}$, and $I^* = \bigcup_{s\in S} I_s$. Let J be the right ideal of R generated by I^*. If $J = R$, then there exist $s_1,\dots,s_n \in S$, $f_1,\dots,f_n \in I$, and $r_1,\dots,r_n \in R$, such that

$$1 = f_1(s_1)r_1 + \cdots + f_n(s_n)r_n$$

and $\pi(f_i) = s_i$, $f_i(s_i) \in I_{s_i}$, $i = 1, 2, \ldots, n$. Since ω_{s_i} is an automorphism, there exists a_i in R such that $\omega_{s_i}(a_i) = r_i$, for $i = 1, 2, \ldots, n$. So

$$1 = f_1(s_1)\omega_{s_1}(a_1) + \cdots + f_n(s_n)\omega_{s_n}(a_n). \tag{2.1}$$

Clearly we can assume that $f_i(s_i)\omega_{s_i}(a_i) \neq 0$, for $i = 1, 2, \ldots, n$. Thus $(f_i c_{a_i})(s_i) \neq 0$. For any $t \prec s_i$, if $(f_i c_{a_i})(t) = f_i(t)\omega_t(a_i) \neq 0$, then $f_i(t) \neq 0$, a contradiction with $\pi(f_i) = s_i$. Hence $\pi(f_i c_{a_i}) = s_i$. Suppose that $h \in \ell_A(I)$ and $h \neq 0$. Let $\pi(h) = t$. Since $f_i c_{a_i} \in I$, we have $h(f_i c_{a_i}) = 0$. Thus

$$0 = (hf_i c_{a_i})(ts_i) = \sum_{(u,v) \in X_{ts_i}(h, f_i c_{a_i})} h(u)\omega_u(f_i(v)\omega_v(a_i)).$$

Since t and s_i are the minimal elements of $\mathrm{supp}(h)$ and $\mathrm{supp}(f_i c_{a_i})$, respectively, under the total order $\preceq$, if $(u, v) \in X_{ts_i}(h, f_i c_{a_i})$, then $t \preceq u$ and $s_i \preceq v$. If $t \prec u$, since $\preceq$ is a strict order, $ts_i \prec us_i \preceq uv = ts_i$, a contradiction. Thus $u = t$. Similarly, $v = s_i$. Hence

$$0 = \sum_{(u,v) \in X_{ts_i}(h, f_i c_{a_i})} h(u)\omega_u(f_i(v)\omega_v(a_i)) = h(t)\omega_t(f_i(s_i)\omega_{s_i}(a_i)).$$

Thus $\omega_t^{-1}(h(t))f_i(s_i)\omega_{s_i}(a_i) = 0$, for $i = 1, 2, \ldots, n$.

Multiplying (2.1) by $\omega_t^{-1}(h(t))$ on the left-hand side, we obtain

$$\omega_t^{-1}(h(t)) = \omega_t^{-1}(h(t))f_1(s_1)\omega_{s_1}(a_1) + \cdots + \omega_t^{-1}(h(t))f_n(s_n)\omega_{s_n}(a_n) = 0$$

which contradicts with $h(t) \neq 0$. Therefore $h = 0$, and so $\ell_A(I) = 0$.

Now suppose that $J \neq R$. We show that J is a right maximal ideal of R. Let $r \in R \backslash J$. If $c_r \in I$, then $r = c_r(1) \in I_1$, since $\pi(c_r) = 1$, and so $r \in J$, a contradiction. Thus $c_r \notin I$. So $A = I + c_r A$. It follows that there exist $f \in I$ and $g \in A$ such that $c_1 = f + c_r g$. Thus

$$1 = c_1(1) = f(1) + c_r g(1) = f(1) + rg(1).$$

If $f(1) = 0$, then $1 \in rR$ and so $R = J + rR$. If $f(1) \neq 0$, then $1 \in \mathrm{supp}(f)$. Since S is positively ordered, we have $\pi(f) = 1$. Thus $f(1) \in I_1 \subseteq J$, which implies that $R = J + rR$. Hence J is a right maximal ideal of R.

Since R is a right PS-ring, there exists an idempotent $e^2 = e$ of R such that $\ell_R(J) = Re$. We will show that $\ell_A(I) = Ac_e$. Clearly, c_e is an idempotent of A.

If $c_e I \nsubseteq I$, then $A = I + c_e I$. So there exist $f, g \in I$ such that $c_1 = f + c_e g$. Thus $1 = f(1) + eg(1)$. Since S is positively ordered and $eJ = 0$, it follows that $1 \in J$, a contradiction. Therefore $c_e I \subseteq I$.

Suppose that $f \in I$. Then $c_e f \in I$. If $c_e f \neq 0$, then set $\pi(c_e f) = t$. So $(c_e f)(t) \neq 0$. Hence $(c_e f)(t) = ef(t) \in I_t \subseteq J$. Thus $e(ef(t)) = ef(t) = 0$, which is a contradiction. Therefore $c_e f = 0$. This follows that $Ac_e \subseteq \ell_A(I)$.

Let $g \in \ell_A(I)$ and $g \neq 0$. Set $\pi(g) = s$. For any $a \in J$, there exist $s_1, \ldots, s_n \in S$, $f_1, \ldots, f_n \in I$, and $r_1, \ldots, r_n \in R$, such that $a = f_1(s_1)r_1 + \cdots + f_n(s_n)r_n$, and $\pi(f_i) = s_i$, $f_i(s_i) \in I_{s_i}$, $i = 1, 2, \ldots, n$. Since ω_{s_i} is an automorphism, there exists a_i in R such that $\omega_{s_i}(a_i) = r_i$, for $i = 1, 2, \ldots, n$. So

$$a = f_1(s_1)\omega_{s_1}(a_1) + \cdots + f_n(s_n)\omega_{s_n}(a_n). \tag{2.2}$$

Clearly we can assume that $f_i(s_i)\omega_{s_i}(a_i) \neq 0$, for $i = 1, 2, \ldots, n$. Thus $(f_i c_{a_i})(s_i) \neq 0$. For any $t \prec s_i$, if $(f_i c_{a_i})(t) = f_i(t)\omega_t(a_i) \neq 0$, then $f_i(t) \neq 0$, a contradiction with $\pi(f_i) = s_i$. Hence $\pi(f_i c_{a_i}) = s_i$.

Since $f_i c_{a_i} \in I$, we have $g(f_i c_{a_i}) = 0$. Therefore

$$0 = (g f_i c_{a_i})(s s_i) = \sum_{(u,v) \in X_{ss_i}(g, f_i c_{a_i})} g(u)\omega_u(f_i(v)\omega_v(a_i)).$$

Since s and s_i are the minimal elements of $\mathrm{supp}(g)$ and $\mathrm{supp}(f_i c_{a_i})$, respectively, under the total order $\preceq$, if $(u, v) \in X_{ss_i}(g, f_i c_{a_i})$, then $s \preceq u$ and $s_i \preceq v$. If $s \prec u$, since $\preceq$ is a strict order, $ss_i \prec us_i \preceq uv = ss_i$, a contradiction. Thus $u = s$. Similarly, $v = s_i$. Hence

$$0 = \sum_{(u,v) \in X_{ss_i}(g, f_i c_{a_i})} g(u)\omega_u(f_i(v)\omega_v(a_i)) = g(s)\omega_s(f_i(s_i)\omega_{s_i}(a_i)).$$

Since ω_s is an automorphism, there exists d_s such that $\omega_s(d_s) = g(s)$. Then $d_s f_i(s_i)\omega_{s_i}(a_i) = 0$, for $i = 1, 2, \ldots, n$. Hence (2.2) implies that $d_s a = 0$. This follows that $d_s \in \ell_R(J) = Re$. Therefore $d_s = d_s e$. Hence $g(s) = g(s)\omega_s(e)$.

We claim that for any $u \in \mathrm{supp}(g)$, $g(u) = g(u)\omega_u(e)$. Suppose that $u \in \mathrm{supp}(g)$. Assume that $g(v) = g(v)\omega_v(e)$ for any $v \in \mathrm{supp}(g)$ with $v \prec u$. We will show that $g(u) = g(u)\omega_u(e)$. Denote

$$g_u(x) = \begin{cases} g(x) & x \prec u, \\ 0 & u \preceq x. \end{cases}$$

Then $\pi(g - g_u) = u$. By hypothesis it is easy to see that $g_u = g_u c_e \in Ac_e \subseteq \ell_A(I)$. Thus $g - g_u \in \ell_A(I)$. By analogy with the proof above, it follows that $(g - g_u)(u) = (g - g_u)(u)\omega_u(e)$, which implies that $g(u) = g(u)\omega_u(e)$. Thus our claim holds. It follows that $g = gc_e \in Ac_e$. Therefore $\ell_A(I) = Ac_e$, and so $R[[S, \omega]]$ is a right PS-ring, and the proof is complete. □

Corollary 2.3. [13, Theorem 4] *Let R be a commutative ring and $(S, \leq)$ be a positively totally ordered monoid. If R is a PS-ring, then the ring of generalized power series $R[[S]]$ is a PS-ring.*

Corollary 2.4. *Let R be a right PS-ring and α be an automorphism of R. Then the skew power series ring $R[[x; \alpha]]$ is a right PS-ring.*

Any submonoid of the additive monoid $\mathbb{N} \cup \{0\}$ is called a *numerical monoid.*

Corollary 2.5. *Let S be a numerical monoid, $\leq$ the usual natural order of $\mathbb{N} \cup \{0\}$. Let R be a ring and $\omega\colon S \to \mathrm{Aut}(R)$ a monoid homomorphism. If R is a right PS-ring, then $R[\![S,\omega]\!]$ is a right PS-ring.*

Corollary 2.6. *Let S be a submonoid of $(\mathbb{N} \cup \{0\})^n$ $(n \geq 2)$, endowed with the order $\leq$ induced by the product order, or lexicographic order or reverse lexicographic order. Let R be a right PS-ring and $\omega\colon S \to \mathrm{Aut}(R)$ a monoid homomorphism. Then $R[\![S,\omega]\!]$ is a right PS-ring.*

Recall from [36] that R is a *left FS-ring*, if $\mathrm{Soc}({}_R R)$ is flat. This class includes all SF-rings (i.e., rings whose simple modules are flat). Obviously PS-rings are FS-rings. Also, the class of right FS-rings is closed under the polynomial extensions, direct products, and excellent extensions (for more details see [11] and [36]).

By combining [36, Proposition 8] and Theorem 2.2, we obtain the following.

Corollary 2.7. *Let R be a commutative ring and $(S, \leq)$ be a positively quasitotally ordered monoid. Assume that $\omega\colon S \to \mathrm{Aut}(R)$ is a monoid homomorphism. If R is a FS-ring, then $R[\![S,\omega]\!]$ is a right FS-ring.*

3. Skew generalized power series rings in some classes of rings

In this section, we study various properties and a variety of conditions and related properties that are inherited by the skew generalized powers series ring $R[\![S,\omega]\!]$ from the ring R. Recall that a ring R is *local* if $R/J(R)$ is a division ring, and R is *semilocal* if $R/J(R)$ is a semisimple ring. In [18], R. Mazurek and M. Ziembowski examined which conditions on a ring R and a strictly ordered monoid $(S, \cdot, \leq)$ are necessary and which are sufficient for the generalized power series ring $R[\![S]\!]$ to be semilocal right Bézout or semilocal right distributive. A ring R is said to be *matrix local* if $R/J(R)$ is a simple Artinian ring. A ring R is said to be *semiperfect* if R is a semilocal ring and all idempotents of the Artinian ring $R/J(R)$ can be lifted to idempotents of the ring R. Due to Nicholson, a ring R is said to be an *I-ring* if every right ideal of the ring R not contained in $J(R)$ contains a nonzero idempotent and all idempotents of the ring $R/J(R)$ can be lifted to idempotents of the ring R. Recall from [21] that a ring R is *semiregular* if $R/J(R)$ is a (von Neumann) regular ring and idempotents can be lifted modulo $J(R)$. According to Nicholson and Zhou [24], a ring R is called a *clean* (*uniquely clean*) ring if every element $r \in R$ can be written (uniquely) in the form $r = u + e$ where u is a unit in R and $e^2 = e \in R$.

In the proof of Theorems 3.2, 3.5 and Proposition 3.9, we will need the following lemma. Statements (1) and (3) of the lemma are proved in [17, Lemma 1.3].

Lemma 3.1. *Let R be a ring and $(S, \leq)$ be a positively strictly ordered monoid. Assume that $\omega \colon S \to \mathrm{End}(R)$ is a monoid homomorphism, and let $A = R[\![S, \omega]\!]$. Then*

(1) *$f \in J(A)$ if and only if $f(1) \in J(R)$.*

(2) *If $f \in A$ such that $f - f^2 \in \langle B \rangle$, then $f(1)$ is an idempotent of the ring R and $f - c_{f(1)} \in \langle B \rangle \subseteq J(A)$, where $\langle B \rangle$ is the ideal generated by set $B = \{g \in A \mid g(1) = 0\}$ in A.*

(3) *The factor ring $R/J(R)$ is naturally isomorphic to the factor ring $A/J(A)$.*

(4) *All idempotents of the factor ring $A/J(A)$ can be lifted to idempotents of the ring A if and only if all idempotents of the factor ring $R/J(R)$ can be lifted to idempotents of the ring R.*

Proof. (2) By part (1), $\langle B \rangle \subseteq J(A)$. The remaining assertions are directly verified.

(4) The result follows from parts (2) and (3). $\square$

Theorem 3.2. *Let R be a ring and $(S, \leq)$ be a positively strictly ordered monoid. Assume that $\omega \colon S \to \mathrm{End}(R)$ is a monoid homomorphism. Then*

(1) *$R[\![S, \omega]\!]$ is a local ring if and only if R is a local ring.*

(2) *$R[\![S, \omega]\!]$ is a semilocal ring if and only if R is a semilocal ring.*

(3) *$R[\![S, \omega]\!]$ is a matrix local ring if and only if R is a matrix local ring.*

(4) *$R[\![S, \omega]\!]$ is a semiperfect ring if and only if R is a semiperfect ring.*

(5) *$R[\![S, \omega]\!]$ is isomorphic to a full matrix ring over a local ring if and only if the ring R is isomorphic to a full matrix ring over a local ring.*

(6) *$R[\![S, \omega]\!]$ is an I-ring if and only if R is an I-ring.*

(7) *$R[\![S, \omega]\!]$ is semiregular if and only if R is semiregular.*

(8) *$R[\![S, \omega]\!]$ is a clean ring if and only if R is a clean ring.*

Proof. (1), (2) and (3) follow from the fact that the ring $R/J(R)$ is isomorphic to the ring $R[\![S, \omega]\!]/J(R[\![S, \omega]\!])$ by Lemma 3.1(3).

(4) This result is a direct consequence of (2) and Lemma 3.1(4).

(5) By [8, Theorem 23.10], the class of all matrix local semiperfect rings coincides with the class of all rings that are isomorphic to full matrix rings over local rings. Therefore, part (5) follows from (3) and (4).

(6) The result follows from [20, Proposition 1.4] and Lemma 3.1, since R is an I-ring if and only if $R/J(R)$ is an I-ring and all idempotents of the ring $R/J(R)$ can be lifted to idempotents of the ring R.

(7) This result is a consequence of Lemma 3.1(3) and Lemma 3.1(4).

(8) The result follows from [1, Proposition 7] and Lemma 3.1, since R is a clean ring if and only if $R/J(R)$ is a clean ring and all idempotents of the ring $R/J(R)$ can be lifted to idempotents of the ring R. □

Let R be a ring, and consider the multiplicative monoid $\mathbb{N}^{\geq 1}$, endowed with the usual order $\leq$. Then $A = R[\![\mathbb{N}^{\geq 1}]\!]$ is the *ring of arithmetical functions* with values in R, endowed with the Dirichlet convolution:

$$fg(n) = \sum_{d|n} f(d)g(n/d), \quad \text{for each } n \geq 1.$$

Corollary 3.3. *Let R be a ring. Then the ring of arithmetical functions $R[\![\mathbb{N}^{\geq 1}]\!]$ is a semiperfect (resp. semiregular) ring if and only if R is a semiperfect (resp. semiregular) ring.*

Let α and β be endomorphisms of R such that $\alpha\beta = \beta\alpha$. Assume that $S = (\mathbb{N} \cup \{0\}) \times (\mathbb{N} \cup \{0\})$ is endowed with the lexicographic order, or the reverse lexicographic order, or the product order of the usual order of $\mathbb{N} \cup \{0\}$, and define $\omega\colon S \to \operatorname{End}(R)$ a monoid homomorphism via $\omega(m,n) = \alpha^m\beta^n$ for any $m, n \in \mathbb{N} \cup \{0\}$. Then $R[\![S,\omega]\!] \cong R[\![x,y;\alpha,\beta]\!]$, in which $(ax^my^n)(bx^py^q) = a\alpha^m\beta^n(b)x^{m+p}y^{n+q}$ for any $m,n,p,q \in \mathbb{N} \cup \{0\}$.

Corollary 3.4. *Let α and β be endomorphisms of a ring R such that $\alpha\beta = \beta\alpha$. Then the ring $R[\![x,y;\alpha,\beta]\!]$ is a semiperfect (resp. semiregular) ring if and only if R is a semiperfect (resp. semiregular) ring.*

A ring R with unity is called *right (left) quasi-duo* if every maximal right (left) ideal of R is two-sided or, equivalently, every right (left) primitive homomorphic image of R is a division ring. Examples of right quasi-duo rings include, for instance, commutative rings, local rings, rings in which every nonunit has a (positive) power that is central, endomorphism rings of uniserial modules, and power series rings and rings of upper triangular matrices over any of the above-mentioned rings (see [37]). But the n by n full matrix rings over right quasi-duo rings are not right quasi-duo (for more details see [9, 10, 37]).

A ring R is said to be *Dedekind finite* if $ab = 1$ implies $ba = 1$ for any $a, b \in R$, and R is *stably finite* if any matrix ring $M_n(R)$ is Dedekind-finite (for more details see [19]). Recall that a module ${}_RM$ has the (full) exchange property if for every module ${}_RA$ and any two decompositions $A = M' \oplus N = \bigoplus_{i\in I} A_i$ with $M' \cong M$, there exist submodules $A_i' \subseteq A_i$ such that $A = M' \oplus (\bigoplus_{i\in I} A_i')$. A module ${}_RM$ has the *finite exchange property* if the above condition is satisfied whenever the index set I is finite. Warfield [34] called a ring R an *exchange ring* if the left regular module ${}_RR$ has the finite exchange property and showed that this definition is left-right symmetric.

According to P. Vámos [33], a ring R is said to be 2-*good* if every element is the sum of two units. The ring of all $n \times n$ matrices over an elementary divisor ring is 2-good. A (right) self-injective Von Neumann regular ring is 2-good provided it has no 2-torsion. In [35], Yao Wang and Yanli Ren show that the 2-good property is preserved in extensions such as skew power series rings, full matrix rings, formal triangular matrix rings, upper triangular matrix rings, and trivial extension rings (for more details see [33, 35]).

Recall that a ring R is *semiboolean* if and only if $R/J(R)$ is Boolean and idempotents of R lift modulo $J(R)$. According to [24, Theorem 19], R is a Boolean ring if and only if R is uniquely clean and $J(R) = 0$. By Lemma 3.1(1), $R[\![S,\omega]\!]$ is not necessarily Boolean. But we will show that $R[\![S,\omega]\!]$ is semiboolean if and only if R is semiboolean.

Theorem 3.5. *Let R be a ring and $(S, \leq)$ be a positively strictly ordered monoid. Assume that $\omega\colon S \to \mathrm{End}(R)$ is a monoid homomorphism. Then*

(1) $R[\![S,\omega]\!]$ *is a left quasi-duo ring if and only if R is a left quasi-duo ring.*

(2) $R[\![S,\omega]\!]$ *is stably finite if and only if R is stably finite.*

(3) $R[\![S,\omega]\!]$ *is an exchange ring if and only if R is an exchange ring.*

(4) $R[\![S,\omega]\!]$ *is a 2-good if and only if R is a 2-good.*

(5) $R[\![S,\omega]\!]$ *is semiboolean if and only if R is semiboolean.*

Proof. We set $A = R[\![S,\omega]\!]$.

(1) The result follows from [37, Proposition 2.1] and Lemma 3.1(3), since a ring R is left quasi-duo if and only if $R/J(R)$ is.

(2) Let A be stably finite. Clearly, the subring R is also stably finite. Finally, suppose that R is stably finite. Consider the ideal $\langle B\rangle$ generated by set $B = \{g \in A \mid g(1) = 0\}$ in A. By Lemma 3.1(1), $\langle B\rangle \subseteq J(A)$. We have $A/\langle B\rangle \cong R$, so by [19, Lemma 2], the fact that R is stably finite implies that A is stably finite.

(3) The result follows from [22, Corollary 2.4], Lemma 3.1(3) and (4), since R is an exchange ring if and only if $R/J(R)$ is an exchange ring and all idempotents of the ring $R/J(R)$ can be lifted to idempotents of R.

(4) Let R be 2-good and $f \in A$. Write $f(1) = u+v$, where $u, v \in U(R)$. Set $g = f - c_v$. Then, by [16, Proposition 2.2], $g, c_v \in U(A)$. So A is 2-good. Conversely, if A is 2-good, then by analogy with the proof of (2), we can show that the ring R is a homomorphic

image of A. By [35, Proposition 2.15], every homomorphic image of a 2-good ring is again 2-good, and therefore, the result follows.

(5) The result follows from Lemma 3.1(3) and Lemma 3.1(4). $\square$

The following corollaries will give more examples of quasi-duo (resp. exchange) rings.

Corollary 3.6. *Let* $(S_1, \leq_1), \ldots, (S_n, \leq_n)$ *be strictly positively ordered monoids. Denote by* (lex $\leq$) *and* (relex $\leq$) *the lexicographic order, the reverse lexicographic order, respectively, on the ordered monoid* $S_1 \times \cdots \times S_n$*. If* R *is a ring and* $\omega\colon S_1 \times \cdots \times S_n \to \mathrm{End}(R)$ *is a monoid homomorphism, then the following statements are equivalent.*

(1) $R[\![S_1 \times \cdots \times S_n, \omega, \mathrm{lex} \leq]\!]$ *is a left quasi-duo (resp. exchange) ring;*

(2) $R[\![S_1 \times \cdots \times S_n, \omega, \mathrm{relex} \leq]\!]$ *is a left quasi-duo (resp. exchange) ring;*

(3) R *is a left quasi-duo (resp. exchange) ring.*

Corollary 3.7. *Let* R *be a left quasi-duo (resp. exchange) ring. Let* S *be any of the additive monoids* $\mathbb{Q}^+ = \{a \in \mathbb{Q} \mid a \geq 0\}$ *or* $\mathbb{R}^+ = \{a \in \mathbb{R} \mid a \geq 0\}$*, where* $\leq$ *is the usual order. Assume that* $\omega\colon S \to \mathrm{End}(R)$ *is a monoid homomorphism. Then the ring* $R[\![S, \omega]\!]$ *is left quasi-duo (resp. exchange) ring.*

Corollary 3.8. *Let* α *and* β *be endomorphisms of a ring* R *such that* $\alpha\beta = \beta\alpha$*. Then the ring* $R[\![x, y; \alpha, \beta]\!]$ *is a left quasi-duo (resp. exchange) ring if and only if* R *is a left quasi-duo (resp. exchange) ring.*

Two elements a and b in a ring R are said to be *conjugate* if there exists an invertible element $u \in R$ such that $b = uau^{-1}$.

Proposition 3.9. *Let* R *be a ring and* $(S, \leq)$ *be a positively strictly ordered monoid. Assume that* $\omega\colon S \to \mathrm{End}(R)$ *is a monoid homomorphism. Then any idempotent of* $R[\![S, \omega]\!]$ *is conjugate to an idempotent of* R*.*

Proof. Let $f^2 = f$ be an idempotent of $R[\![S, \omega]\!]$. Then $f = c_{f(1)} + g$, where $g = f - c_{f(1)}$. Since S is positively ordered, $f(1)$ is an idempotent of R. Hence $c_{f(1)}$ is an idempotent of $R[\![S, \omega]\!]$. By Lemma 3.1(1), we have $g \in J(R[\![S, \omega]\!])$. Now, [7, Corollary 11] implies that f and $c_{f(1)}$ are conjugate. $\square$

Due to P. M. Cohn [2], a ring R is said to be *projective-free* if every finitely generated projective left (equivalently right) R-module is free of unique rank. According to [2, Proposition 0.4.5], a ring is projective-free when it has invariant basis number (IBN for short) and every idempotent matrix is conjugate to a matrix of the form $\mathrm{diag}(1, \ldots, 1, 0, \ldots, 0)$.

Assume that S is a monoid and $\omega\colon S \to \mathrm{End}(R)$ a monoid homomorphism. For $s \in S$, let $\overline{\omega_s}\colon M_n(R) \to M_n(R)$ be the map obtained by applying ω_s to every entry of a given matrix in $M_n(R)$. We thereby obtain a monoid homomorphism $\overline{\omega}\colon S \to \mathrm{End}(M_n(R))$. Furthermore, with using a similar method as in [31, 4.3], we can show that the mapping $\Phi\colon M_n(R)[\![S,\overline{\omega}]\!] \to M_n(R[\![S,\omega]\!])$, given by $\Phi(f) = (f_{ij})$, where $f_{ij}(s) = (f(s))_{ij}$ for all $s \in S$ and $(f(s))_{ij}$ is the (i,j)-th entry of $f(s)$, is an isomorphism.

Theorem 3.10. *Let R be a projective-free ring and $(S, \leq)$ be a positively strictly ordered monoid. Assume that $\omega\colon S \to \mathrm{End}(R)$ is a monoid homomorphism. Then $R[\![S,\omega]\!]$ is a projective-free ring.*

Proof. We use the method employed in the proof of [7, Theorem 22]. Since R has the IBN property and is a homomorphic image of $R[\![S,\omega]\!]$, the ring $R[\![S,\omega]\!]$ has the IBN property as well. By [2, Proposition 0.4.5], it is enough to show that, for any $n \geq 1$, every idempotent matrix $e \in M_n(R[\![S,\omega]\!])$ is conjugate to a matrix of the form $\mathrm{diag}(1,\dots,1,0,\dots,0)$. Then by the above argument $T = M_n(R)[\![S,\overline{\omega}]\!]$ is isomorphic to the ring $M_n(R[\![S,\omega]\!])$. Thus, by Proposition 3.9, the idempotent e is conjugate in T to an idempotent of $M_n(R)$ which, in turn, is conjugate in $M_n(R) \subseteq T$ with an idempotent of the form $\mathrm{diag}(1,\dots,1,0,\dots,0)$, as R is projective-free, and the result follows. □

Corollary 3.11. *Let R be a projective-free ring and α be an endomorphism of R. Then skew power series ring $R[\![x;\alpha]\!]$ is a projective-free ring.*

Corollary 3.12. [3, Theorem 7] and [7, Theorem 22] *Let R be a projective-free ring. Then the power series ring $R[\![x]\!]$ is a projective-free ring.*

Corollary 3.13. *Let R be a projective-free ring. Then the ring of arithmetical functions $R[\![\mathbb{N}^{\geq 1}]\!]$ is a projective-free ring.*

Corollary 3.14. *Let S be a submonoid of $(\mathbb{N} \cup \{0\})^n$ $(n \geq 2)$, endowed with the order $\leq$ induced by the product order, or lexicographic order or reverse lexicographic order. Let R be a ring and $\omega\colon S \to \mathrm{End}(R)$ a monoid homomorphism. If R is projective-free ring, then $R[\![S,\omega]\!]$ is a projective-free ring.*

A ring R is said to have *right stable range one* if, whenever $aR + bR = R$, for $a, b \in R$, there exists $d \in R$ such that $a + bd \in U(R)$. The stable range one condition is especially interesting because of Evans' Theorem [5], which states that a module M cancels from direct sums whenever $\mathrm{End}_R(M)$ has stable range one.

Proposition 3.15. *Let R be a ring and $(S, \leq)$ be a positively strictly ordered monoid. Assume that $\omega\colon S \to \mathrm{End}(R)$ is a monoid homomorphism. Then $R[\![S,\omega]\!]$ has right stable range one if and only if R has right stable range one.*

Proof. Assume that $f, g \in A = R[\![S, \omega]\!]$ are such that $fA + gA = A$. Therefore $f(1)R + g(1)R = R$, and hence there exists $d \in R$ such that $f(1) + g(1)d \in U(R)$. Thus $f + gc_d \in U(A)$, by [16, Proposition 2.2]. Conversely, suppose that $a, b \in R$ are such that $aR + bR = R$. Thus $c_aA + c_bA = A$, and hence there exists $f \in A$ such that $c_a + c_bf \in U(A)$. Thus $a + bf(1) \in U(R)$, and the result follows. □

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