TAIWANESE JOURNAL OF MATHEMATICS
Vol. 20, No. 5, pp. 1175–1184, October 2016
DOI: 10.11650/tjm.20.2016.6359
This paper is available online at `http://journal.tms.org.tw`

Answers to Kirk-Shahzad's Questions on Strong b-metric Spaces

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Abstract. In this paper, two open questions on strong b-metric spaces posed by Kirk and Shahzad [11, Chapter 12] are investigated. A counterexample is constructed to give a negative answer to the first question, and a theorem on the completion of a strong b-metric space is proved to give a positive answer to the second question.

1. Introduction and preliminaries

In 1993 Czerwik [4] introduced the notion of a b-metric which is a generalization of a metric with a view of generalizing the Banach contraction map theorem.

Definition 1.1. [4] Let X be a nonempty set and $d\colon X \times X \to [0, \infty)$ be a function such that for all $x, y, z \in X$,

(1) $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$;

(3) $d(x, z) \leq 2[d(x, y) + d(y, z)]$.

Then d is called a *b-metric* on X and (X, d) is called a *b-metric space*.

After that, in 1998, Czerwik [5] generalized this notion where the constant 2 was replaced by a constant $K \geq 1$, also with the name *b-metric*. The convergence, Cauchy sequence and completeness in b-metric spaces are defined as follows.

Definition 1.2. [5] Let (X, D, K) be a b-metric space.

(1) A sequence $\{x_n\}$ is called *convergent* to x, written $\lim_{n\to\infty} x_n = x$, if $\lim_{n\to\infty} D(x_n, x) = 0$.

(2) A sequence $\{x_n\}$ is called *Cauchy* if $\lim_{n,m\to\infty} D(x_n, x_m) = 0$.

Received May 31, 2015; Accepted April 27, 2016.
Communicated by Ruey-Lin Sheu.
2010 *Mathematics Subject Classification.* Primary: 47H10, 54H25; Secondary: 54D99, 54E99.
Key words and phrases. Strong b-metric, b-metric, Completion, Nadler's theorem.
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(3) (X, D, K) is called *complete* if every Cauchy sequence is a convergent sequence.

The same relaxation of the triangle inequality in definition of a b-metric was also discussed in 2003 by Fagin et al. [8], who called this new distance measure nonlinear elastic matching. The authors of that paper remarked that this measure had been used, for example, in [9] for trademark shapes and in [3] to measure ice floes. In 2009 Xia [13] used this semimetric distance to study the optimal transport path between probability measures.

In recent times, b-metric spaces and fixed point theory on b-metric spaces were studied by many authors [1], [7], [10], [11, Chapter 12], [12]. Some authors were also studied topological properties of b-metric spaces. An et al. [2] showed that every b-metric space with the topology induced by its convergence is a semi-metrizable space and thus many properties of b-metric spaces used in the literature are obvious. Then the authors proved the Stone-type theorem on b-metric spaces and obtained a sufficient condition for a b-metric space to be metrizable. Notice that a b-metric space is always understood to be a topological space with respect to the topology induced by its convergence and a b-metric need not be continuous [2, Examples 3.9 & 3.10]. This fact suggests a strengthening of the notion of b-metric spaces which remedies this defect. Recently Kirk and Shahzad introduced the notion of a strong b-metric space.

Definition 1.3. [11, Definition 12.7] Let X be a nonempty set, $K \geq 1$ and $D\colon X \times X \to [0, \infty)$ be a function such that for all $x, y, z \in X$,

(1) $D(x, y) = 0$ if and only if $x = y$;

(2) $D(x, y) = D(y, x)$;

(3) $D(x, z) \leq D(x, y) + KD(y, z)$.

Then D is called a *strong b-metric* on X and (X, D, K) is called a *strong b-metric space*.

Remark 1.4. [11, page 122] (1) Every strong b-metric is continuous.

(2) Every open ball $B(a, r) = \{x \in X : D(a, x) < r\}$ of a strong b-metric space (X, D, K) is open.

In [11, Chapter 12] Kirk and Shahzad surveyed b-metric spaces, strong b-metric spaces, and related problems. An interesting work was attracted many authors is to transform results of metric spaces to the setting of b-metric spaces. It is only fair to point out that some results seem to require the full use of the triangle inequality of a metric space. In this connection, Kirk and Shahzad [11, page 127] mentioned an interesting extension of

Nadler's theorem due to Dontchev and Hager [6]. Recall that for a metric space (X, d) and $A, B \subset X$, $x \in X$,

$$\mathrm{dist}(x, A) = \inf\{d(x, a) : a \in A\}, \quad \delta(A, B) = \sup\{\mathrm{dist}(x, A) : x \in B\}$$

and these notations are understood similarly on b-metric spaces.

Theorem 1.5. [11, Theorem 12.7] *Let (X, d) be a complete metric space, $T \colon X \to X$ be a map from X into a nonempty closed subset of X, and $x_0 \in X$ such that*

(1) $\mathrm{dist}(x_0, Tx_0) < r(1-k)$ *for some* $r > 0$ *and some* $k \in [0, 1)$*;*

(2) $\delta(Tx \cap B(x_0, r), Ty) \leq kd(x, y)$ *for all* $x, y \in B(x_0, r)$*.*

Then T has a fixed point in $B(x_0, r)$.

Based on the definition of $\delta(A, B)$ and the proof in [11, Theorem 12.7], Theorem 1.5(2) is implicitly understood as

$$\delta(Tx \cap B(x_0, r), Ty) \leq kd(x, y) \quad \text{for all } x, y \in B(x_0, r) \text{ and } Tx \cap B(x_0, r) \neq \emptyset.$$

The authors of [11] did not know whether Theorem 1.5 holds under the weaker strong b-metric assumption. Explicitly we have the following question.

Question 1.6. [11, page 128] Let (X, D, K) be a complete strong b-metric space, $T \colon X \to X$ be a map from X into a nonempty closed subset of X, and $x_0 \in X$ such that

(1) $\mathrm{dist}(x_0, Tx_0) < r(1-k)$ for some $r > 0$ and some $k \in [0, 1)$;

(2) $\delta(Tx \cap B(x_0, r), Ty) \leq kD(x, y)$ for all $x, y \in B(x_0, r)$ and $Tx \cap B(x_0, r) \neq \emptyset$.

Does the map T have a fixed point in $B(x_0, r)$?

Recall that a map $f \colon X \to Y$ from a b-metric space (X, D, K) into a b-metric space (Y, D', K') is called an *isometry* if $D'(f(x), f(y)) = D(x, y)$ for all $x, y \in X$. Also, a b-metric space (X^*, D^*, K^*) is called a *completion* of the b-metric space (X, D, K) if (X^*, D^*, K^*) is complete and there exists an isometry $f \colon X \to X^*$ such that $\overline{f(X)} = X^*$. A classical result is that every metric space is dense in a complete metric space. So it is interesting to ask whether this result holds or not in the setting of strong b-metric spaces.

Question 1.7. [11, page 128] Is every strong b-metric space dense in a complete strong b-metric space?

Kirk and Shahzad [11, page 128] commented that if the answer of Question 1.7 is positive then every contraction map $f\colon X \to X$ on a strong b-metric space X may be extended to a contraction map $f\colon X^* \to X^*$ on a complete strong b-metric space X^* which has a unique fixed point. Ostrowski's theorem [11, Theorem 12.6] then would provide a method for approximating this fixed point.

In this paper, the above two questions on strong b-metric spaces are investigated. A counterexample is constructed to give a negative answer to Question 1.6, and a theorem on the completion of a strong b-metric space is proved to give a positive answer to Question 1.7.

2. Main results

First, the following example gives a negative answer to Question 1.6.

Example 2.1. Let $X = \{1, 2, 3\}$, the function $D\colon X \times X \to [0, \infty)$ be defined by

$$\begin{aligned} &D(1,1) = D(2,2) = D(3,3) = 0, && D(1,2) = D(2,1) = 2, \\ &D(2,3) = D(3,2) = 1, && D(1,3) = D(3,1) = 6 \end{aligned}$$

and a map $T\colon X \to X$ be defined by $T1 = 2$, $T2 = 3$, $T3 = 1$. Then

(1) (X, D, K) is a complete strong b-metric space with $K = 4$.

(2) T and (X, D, K) satisfy all assumptions of Question 1.6 with $x_0 = 1$, $r = 6$, $k = 1/2$.

(3) T has no any fixed point.

Proof. (1) For all $x, y \in X$, it follows from definition of D that $D(x, y) = D(y, x)$, and $D(x, y) = 0$ if and only if $x = y$.

We also have

$$\begin{aligned} D(1,3) + KD(3,2) &= 6 + 4 \cdot 1 = 10 \geq 2 = D(1,2), \\ KD(1,3) + D(3,2) &= 4 \cdot 6 + 1 = 25 \geq 2 = D(1,2), \\ D(1,2) + KD(2,3) &= 2 + 4 \cdot 1 = 6 = D(1,3), \\ KD(1,2) + D(2,3) &= 4 \cdot 2 + 1 = 9 \geq 6 = D(1,3), \\ D(2,1) + KD(1,3) &= 2 + 4 \cdot 6 = 26 \geq 1 = D(2,3), \\ KD(2,1) + D(1,3) &= 4 \cdot 2 + 6 = 14 \geq 1 = D(2,3). \end{aligned}$$

By the above, D is a strong b-metric on X. Since X is finite and discrete, X is complete. So (X, D, K) is a complete strong b-metric space with $K = 4$.

(2) Since $TX = X$, TX is a nonempty closed subset of X. We have

$$\mathrm{dist}(x_0, Tx_0) = \mathrm{dist}(1, T1) = \mathrm{dist}(1, \{2\}) = D(1,2) = 2$$

and $r(1-k) = 6(1-\frac{1}{2}) = 3$. This proves that $\mathrm{dist}(x_0, Tx_0) < r(1-k)$.

We also have $B(x_0, r) = B(1,6) = \{1,2\}$. We will show that $\delta(Tx \cap B(x_0,r), Ty) \leq kD(x,y)$ for all $x, y \in B(x_0, r)$ and $Tx \cap B(x_0, r) \neq \emptyset$ as follows.

If $x = y = 1$ then $Tx \cap B(x_0, r) = \{2\}$ and

$$\delta(Tx \cap B(x_0,r), Ty) = \delta(\{2\}, \{2\}) = D(2,2) = 0 \leq kD(x,y).$$

If $x = y = 2$ then $Tx \cap B(x_0, r) = \emptyset$. If $x = 1$, $y = 2$ then

$$\delta(Tx \cap B(x_0,r), Ty) = \delta(\{2\}, \{3\}) = D(2,3) = 1 = \frac{1}{2}D(1,2) = kD(x,y).$$

If $x = 2$, $y = 1$ then $Tx \cap B(x_0, r) = \emptyset$.

By the above calculations we find that $\delta(Tx \cap B(x_0,r), Ty) \leq kD(x,y)$ for all $x, y \in B(x_0, r)$ and $Tx \cap B(x_0, r) \neq \emptyset$.

(3) By definition of T we see that T has no any fixed point. □

Next, the following theorem is a positive answer to Question 1.7.

Theorem 2.2. *Let (X, D, K) be a strong b-metric space. Then*

(1) *(X, D, K) has a completion (X^*, D^*, K).*

(2) *The completion of (X, D, K) is unique in the sense that if (X_1^*, D_1^*, K_1) and (X_2^*, D_2^*, K_2) are two completions of (X, D, K) then there is a bijective isometry $\varphi\colon X_1^* \to X_2^*$ which restricts to the identity on X.*

Proof. Put

$$\mathcal{C} = \{\{x_n\} : \{x_n\} \text{ is a Cauchy sequence in } (X, D, K)\}.$$

Define a relation $\sim$ on $\mathcal{C}$ as follows

$$\{x_n\} \sim \{y_n\} \quad \text{if and only if } \lim_{n\to\infty} D(x_n, y_n) = 0 \text{ for all } \{x_n\}, \{y_n\} \in \mathcal{C}.$$

The relation $\sim$ obviously satisfies reflexivity and symmetry. If $\{x_n\} \sim \{y_n\}$ and $\{y_n\} \sim \{z_n\}$ then $\lim_{n\to\infty} D(x_n, y_n) = \lim_{n\to\infty} D(y_n, z_n) = 0$. Since

$$0 \leq D(x_n, z_n) \leq D(x_n, y_n) + KD(y_n, z_n)$$

for all n, $\lim_{n\to\infty} D(x_n, z_n) = 0$. Thus $\{x_n\} \sim \{z_n\}$. Therefore, the relation $\sim$ is an equivalent relation on $\mathcal{C}$.

Denote $X^* = \{x^* = [\{x_n\}] : \{x_n\} \in \mathcal{C}\}$ where $x^* = [\{x_n\}]$ is the equivalence class of $\{x_n\}$ under the relation $\sim$, and define the function $D^*\colon X^* \times X^* \to \mathbb{R}$ by

$$D^*(x^*, y^*) = \lim_{n\to\infty} D(x_n, y_n). \tag{2.1}$$

We see that, for all n, m,

$$\begin{aligned} D(x_n, y_n) &\leq KD(x_n, x_m) + D(x_m, y_n) \\ &\leq KD(x_n, x_m) + D(x_m, y_m) + KD(y_m, y_n). \end{aligned}$$

This implies that

$$D(x_n, y_n) - D(x_m, y_m) \leq K\left[D(x_n, x_m) + D(y_m, y_n)\right]. \tag{2.2}$$

Also

$$\begin{aligned} D(x_m, y_m) &\leq KD(x_m, x_n) + D(x_n, y_n) \\ &\leq KD(x_n, x_m) + D(x_n, y_n) + KD(y_n, y_m). \end{aligned}$$

Therefore,

$$D(x_m, y_m) - D(x_n, y_n) \leq K\left[D(x_n, x_m) + D(y_m, y_n)\right]. \tag{2.3}$$

It follows from (2.2) and (2.3) that

$$|D(x_m, y_m) - D(x_n, y_n)| \leq K\left[D(x_n, x_m) + D(y_m, y_n)\right]. \tag{2.4}$$

Letting $n, m \to \infty$ in (2.4) we get $\lim_{n,m\to\infty} |D(x_m, y_m) - D(x_n, y_n)| = 0$, i.e., $\{D(x_n, y_n)\}$ is a Cauchy sequence in $\mathbb{R}$. Thus $\lim_{n\to\infty} D(x_n, y_n)$ exists.

Moreover, if $\{x_n\} \sim \{z_n\}$ and $\{y_n\} \sim \{w_n\}$ then

$$\lim_{n\to\infty} D(x_n, z_n) = \lim_{n\to\infty} D(y_n, w_n) = 0. \tag{2.5}$$

We see that

$$\begin{aligned} D(x_n, y_n) &\leq KD(x_n, z_n) + D(z_n, y_n) \\ &\leq KD(x_n, z_n) + D(z_n, w_n) + KD(w_n, y_n). \end{aligned}$$

This implies that

$$D(x_n, y_n) - D(z_n, w_n) \leq KD(x_n, z_n) + KD(w_n, y_n).$$

Similarly,

$$D(z_n, w_n) - D(x_n, y_n) \leq KD(z_n, x_n) + KD(y_n, w_n).$$

Therefore,

$$|D(x_n, y_n) - D(z_n, w_n)| \leq KD(x_n, z_n) + KD(w_n, y_n). \tag{2.6}$$

Letting $n, m \to \infty$ in (2.6) and using (2.5) we get $\lim_{n\to\infty} |D(x_n, y_n) - D(z_n, w_n)| = 0$. Thus $\lim_{n\to\infty} D(x_n, y_n) = \lim_{n\to\infty} D(z_n, w_n)$. Therefore, the function D^* is well-defined.

In the next, we shall prove that (X^*, D^*, K) is a strong b-metric space. For all $x^*, y^*, z^* \in X^*$ we have

1. $D^*(x^*, y^*) = \lim_{n\to\infty} D(x_n, y_n) \geq 0$ since $D(x_n, y_n) \geq 0$ for all n.
2. $D^*(x^*, y^*) = 0$ if and only if $\lim_{n\to\infty} D(x_n, y_n) = 0$, that is, $\{x_n\} \sim \{y_n\}$. This is equivalent to $x^* = y^*$.
3. $D^*(x^*, y^*) = \lim_{n\to\infty} D(x_n, y_n) = \lim_{n\to\infty} D(y_n, x_n) = D^*(y^*, x^*)$ since $D(x_n, y_n) = D(y_n, x_n)$ for all n.
4. $D^*(x^*, z^*) = \lim_{n\to\infty} D(x_n, z_n) \leq \lim_{n\to\infty} [D(x_n, y_n) + KD(y_n, z_n)] = D^*(x^*, y^*) + KD^*(y^*, z^*)$.

So (X^*, D^*, K) is a strong b-metric space.

For each $x \in X$, put $f(x) = [\{x, x, x, \ldots\}] \in X^*$. We see that f is an isometry from (X, D, K) into (X^*, D^*, K) since $D^*(f(x), f(y)) = \lim_{n\to\infty} D(x, y) = D(x, y)$ for all $x, y \in X$.

Next, we will prove that $f(X)$ is dense in X^*. If $x^* = [\{x_n\}] \in X^*$ then $\lim_{n,m\to\infty} D(x_n, x_m) = 0$. For each $i \in \mathbb{N}$, there exists n_0^i such that $D(x_n, x_m) \leq 1/i$ for all $n, m \geq n_0^i$. This implies that

$$0 \leq D^*(f(x_{n_0^i}), x^*) = \lim_{n\to\infty} D(x_{n_0^i}, x_n) \leq \frac{1}{i}.$$

So $\lim_{i\to\infty} D^*(f(x_{n_0^i}), x^*) = 0$. This proves that $\lim_{i\to\infty} f(x_{n_0^i}) = x^*$, that is, $f(X)$ is dense in X^*.

Next, we will prove that (X^*, D^*, K) is complete. Let $\{x_n^*\}$ be a Cauchy sequence in X^*, where $x_n^* = [\{x_i^n\}_i]$ for some $\{x_i^n\}_i \in \mathcal{C}$. Then

$$\lim_{n,m\to\infty} D^*(x_n^*, x_m^*) = 0. \tag{2.7}$$

Note that the open ball $B(x_n^*, \frac{1}{Kn})$ is open by Remark 1.4(2). From the fact that $f(X)$ is dense in X^*, for each n there exists $y_n \in X$ such that

$$D^*(f(y_n), x_n^*) < \frac{1}{Kn}. \tag{2.8}$$

By (2.8), for all n, m, we have

$$
\begin{aligned}
D(y_n, y_m) &= D^*(f(y_n), f(y_m)) \\
&\le KD^*(f(y_n), x_n^*) + D^*(x_n^*, f(y_m)) \\
&\le KD^*(f(y_n), x_n^*) + D^*(x_n^*, x_m^*) + KD^*(x_m^*, f(y_m)) \\
&< \frac{1}{n} + D^*(x_n^*, x_m^*) + \frac{1}{m}.
\end{aligned} \tag{2.9}
$$

Letting $n, m \to \infty$ in (2.9) and using (2.7) we get

$$
\lim_{n,m\to\infty} D(y_n, y_m) = 0. \tag{2.10}
$$

Thus $\{y_n\}$ is a Cauchy sequence in (X, D, K). Put $y^* = [\{y_n\}] \in X^*$. From (2.8) we have

$$
\begin{aligned}
D^*(x_n^*, y^*) &\le KD^*(x_n^*, f(y_n)) + D^*(f(y_n), y^*) \\
&< K\frac{1}{Kn} + \lim_{m\to\infty} D(y_n, y_m) \\
&= \frac{1}{n} + \lim_{m\to\infty} D(y_n, y_m).
\end{aligned} \tag{2.11}
$$

Letting $n \to \infty$ in (2.11) and using (2.10) we have $\lim_{n\to\infty} D^*(x_n^*, y^*) = 0$, that is, $\lim_{n\to\infty} x_n^* = y^*$ in (X^*, D^*, K). Therefore, (X^*, D^*, K) is complete.

Finally, we prove the uniqueness of the completion. Let (X_1^*, D_1^*, K_1) and (X_2^*, D_2^*, K_2) be two completions of (X, D, K). For each $x_1^* \in X_1^*$, there exists $\{x_n\} \subset X$ such that $\lim_{n\to\infty} f_1(x_n) = x_1^*$ where $f_1\colon X \to X_1^*$ is an isometry. Since $\{f_1(x_n)\}$ is convergent, $\{f_1(x_n)\}$ is a Cauchy sequence in X_1^*. Since f_1 is an isometry, $\{x_n\}$ is a Cauchy sequence in X. Note that there exists $f_2\colon X \to X_2^*$ which is also an isometry. Then $\{f_2(x_n)\}$ is a Cauchy sequence in X_2^* and thus there exists $x_2^* \in X_2^*$ such that $\lim_{n\to\infty} f_2(x_n) = x_2^*$. Define $\varphi\colon X_1^* \to X_2^*$ by $\varphi(x_1^*) = x_2^*$. We will show that $\varphi\colon X_1^* \to X_2^*$ is a bijective isometry. Indeed, if $y_2^* \in X_2^*$ then $y_2^* = \lim_{n\to\infty} f_2(y_n)$ for some $\{y_n\} \subset X$. Since $\{f_2(y_n)\}$ is convergent, $\{f_2(y_n)\}$ is a Cauchy sequence in X_2^*. Since f_2 is an isometry, $\{y_n\}$ is a Cauchy sequence in X. Also, f_1 is an isometry, $\{f_1(y_n)\}$ is a Cauchy sequence in X_1^*. Then there exists $y_1^* = \lim_{n\to\infty} f_1(y_n)$. Therefore, $y_2^* = \varphi(y_1^*)$. This proves that φ is bijective. Moreover, for every $x^*, y^* \in X_1^*$ with $x^* = \lim_{n\to\infty} f_1(x_n)$ and $y^* = \lim_{n\to\infty} f_1(y_n)$, by using the continuity of D_1^* and D_2^*, we have

$$
\begin{aligned}
D_1^*(x^*, y^*) &= \lim_{n\to\infty} D_1^*(f_1(x_n), f_1(y_n)) = \lim_{n\to\infty} D(x_n, y_n) \\
&= \lim_{n\to\infty} D_2^*(f_2(x_n), f_2(y_n)) = D_2^*(\varphi(x^*), \varphi(y^*)).
\end{aligned}
$$

This implies that $\varphi\colon X_1^* \to X_2^*$ is a bijective isometry which restricts to the identity on X. $\square$

Finally, the following example shows that techniques used in the proof of Theorem 2.2 may not be applied to b-metric spaces.

Example 2.3. Let $X = \left\{0, 1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots\right\}$ and

$$D(x,y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y \in \{0,1\}, \\ |x-y| & \text{if } x \neq y \in \{0\} \cup \left\{\frac{1}{2n} : n = 1, 2, \ldots\right\}, \\ 4 & \text{otherwise.} \end{cases}$$

Then D is a b-metric on X [2, Example 3.9]. Put $x_n = 1$, $y_n = \frac{1}{2n}$, $z_n = 1$ and $w_n = 0$ for all n. Then $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, $\{w_n\}$ are Cauchy sequences and $\{x_n\} \sim \{z_n\}$ and $\{y_n\} \sim \{w_n\}$. However,

$$\lim_{n\to\infty} D(x_n, y_n) = \lim_{n\to\infty} D\left(1, \frac{1}{2n}\right) = 4 \neq 1 = D(1,0) = \lim_{n\to\infty} D(z_n, w_n).$$

This shows that the formula (2.1) is not well-defined for the above b-metric D.

Though the above example shows that techniques used in the proof of Theorem 2.2 may not be applied to b-metric spaces we do not know whether Theorem 2.2 fully extends to b-metric spaces or not. So we conclude with the following question.

Question 2.4. Does every b-metric space have a completion?

Acknowledgments

The authors are greatly indebted to the referee for his/her helpful suggestions.

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