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A Classification Theorem for Complete PMC Surfaces with Non-negative Gaussian Curvature in $M^n(c) \times \mathbb{R}$

Zhong Hua Hou* and Wang Hua Qiu

Abstract. Let $M^n(c)$ be an n-dimensional space form with constant sectional curvature c. Alencar-do Carmo-Tribuzy [5] classified all parallel mean curvature (abbrev. PMC) surfaces with non-negative Gaussian curvature K in $M^n(c) \times \mathbb{R}$ with $c < 0$. Later on, Fetcu-Rosenberg [28] generalized their results for $c \neq 0$. However, the classification to PMC surfaces in $M^n(c) \times \mathbb{R}$ with $K \equiv 0$ is still open. In this paper, we give a complete classification to the PMC surfaces in $M^n(c) \times \mathbb{R}$ with $K \equiv 0$ whose tangent plane spans the constant angle with factor $\mathbb{R}$.

1. Introduction

Let $M^n(c)$ be an n-dimensional space form with constant sectional curvature $c \neq 0$. In the past two decades, the submanifolds theory in product manifold $M^n(c) \times \mathbb{R}$ were widely studied. There have been lots of interesting and significant results (cf. [6, 10–12, 14, 18, 23–27, 29, 30] etc). For instance, Dillen etc in [19, 22] characterized surfaces with a canonical principal direction and in [20, 21] completely classified constant angle surfaces in $M^2(c) \times \mathbb{R}$. Abresch and Rosenberg [1] introduced a quadratic form

$$Q(X, Y) = 2Hh(X, Y) - c \langle X, \partial_t \rangle \langle Y, \partial_t \rangle,$$

on a surface Σ^2 with constant mean curvature (abbrev. CMC) immersed $M^2(c) \times \mathbb{R}$, where X, Y are tangent vectors on Σ^2 and ∂_t is the unit tangent vector to $\mathbb{R}$. Denote by $Q^{(2,0)}$ the $(2, 0)$-part of Q and it is proved to be holomorphic. Then they completely classified CMC surfaces with vanishing $Q^{(2,0)}$ as four classes in $M^2(c) \times \mathbb{R}$. The scholars call these kinds of surfaces *Abresch-Rosenberg surfaces*.

Alencar, do Carmo and Tribuzy [4] extended the quadratic form Q to immersed surface Σ^2 with parallel mean curvature vector (abbrev. PMC) in $M^n(c) \times \mathbb{R}$, which is defined by

$$Q(X, Y) = 2 \left\langle h(X, Y), \overrightarrow{H} \right\rangle - c \langle X, \partial_t \rangle \langle Y, \partial_t \rangle. \tag{1.1}$$

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*Corresponding author.

And they concluded that $Q^{(2,0)}$ vanishes for surfaces of genus zero, if $|dH| \leq g\left|Q^{(2,0)}\right|$, where g is a continuous, non-negative real function and $|dH|$ is the norm of the differential dH of the mean curvature H of Σ^2 in $M^2(c)\times\mathbb{R}$. We call the surfaces with $Q^{(2,0)}$ vanishing *Abresch-Rosenberg type surfaces.*

Batista [9] introduced a $(1,1)$-tensor S on a CMC surface in $M^2(c)\times\mathbb{R}$, which is given by

$$S = 2HA - c\langle T,\cdot\rangle T + \frac{c}{2}|T|^2 I - 2H^2 I,$$

where A is the shape operator, I the identity transform and T the tangent part to the surface of the unit vertical vector field ∂_t of $\mathbb{R}$. He proved that a complete surface in $M^2(c)\times\mathbb{R}$ with $S=0$ is an *Abresch-Rosenberg surface.* Fetcu and Rosebberg [28] defined a more general $(1,1)$-tensor S on immersed PMC surface Σ^2 in $M^n(c)\times\mathbb{R}$, say

$$S = 2\sum_\alpha H^\alpha A_\alpha - c\langle T,\cdot\rangle T + \frac{c}{2}|T|^2 I - 2H^2 I. \tag{1.2}$$

They showed that $|S|=0$ if and only if $Q^{(2,0)}=0$ (cf. Lemma 3.2 below). Therefore the surface with $|S|=0$ is an *Abresch-Rosenberg type surface.*

Alencar, do Carmo and Tribuzy [5] obtained a well-known Hopf theorem and classified the complete PMC surfaces immersed in $M^n(c)\times\mathbb{R}$.

Theorem 1.1. (cf. [5]) *Let $x\colon \Sigma^2 \to M^n(c)\times\mathbb{R}$ with $c\neq 0$ be a complete PMC surface. Then one of the following holds:*

(1) Σ^2 *is a minimal surface of a totally umbilical hypersurface of* $M^n(c)$*;*

(2) Σ^2 *is a CMC surface in a* 3*-dimensional totally umbilical or totally geodesic submanifold of* $M^n(c)$*;*

(3) Σ^2 *lies in* $M^4(c)\times\mathbb{R}$*.*

In [5], the following classification theorem is proved for $c<0$, and is generalized by Fetcu and Rosebberg [28] for $c\neq 0$.

Theorem 1.2. (cf. [5, 28]) *Let $x\colon \Sigma^2 \to M^n(c)\times\mathbb{R}$ with $c\neq 0$ be a complete non-minimal PMC surface with Gaussian curvature $K\geq 0$. Then one of the following holds:*

(1) $K\equiv 0$*;*

(2) Σ^2 *is a minimal surface of a totally umbilical hypersurface of* $M^n(c)$*;*

(3) Σ^2 *is a CMC surface in a totally umbilical* 3*-dimensional submanifold of* $M^n(c)$*;*

(4) Σ^2 *lies in* $M^4(c)\times\mathbb{R}\subset\mathbb{R}^6$ *and there exists a plane P such that the level lines of the height function $p\mapsto\langle x(p),\partial_t\rangle$ are curves lying in planes parallel to P.*

Then the authors in [5] proposed an interesting and open problem: *How to characterize those surfaces with $K\equiv 0$ in the above theorem.* It seems to be difficult even for the case of complete surfaces immersed in $M^2(c)\times\mathbb{R}$. In the present paper, we intend to solve this problem under the additional condition that $|T|$ is constant. Precisely, we proceed to prove the following

Main Theorem 1.3. *Let Σ^2 be a complete non-minimal PMC surface with non-negative Gaussian curvature in $M^n(c)\times\mathbb{R}$, $c\neq 0$. Then one of the following holds:*

(1) $|S|\equiv 0$, *the surface is an Abresch-Rosenberg type surface;*

(2) $|S|$ *is a nonzero constant and* $K\equiv 0$. *In addition, if* $|T|=\sin\theta$ *with constant* $\theta\in[0,\pi/2]$, *then*

  (i) $\theta=0$, Σ^2 *lies in* $M^n(c)$; *or*

  (ii) $\theta=\pi/2$, $\Sigma^2=\gamma\times\mathbb{R}$, *where γ is a curve of* $M^n(c)$; *or*

  (iii) $\theta\in(0,\pi/2)$ *and Σ^2 lies in* $M^4(c)\times\mathbb{R}\subset\mathbb{R}^6$ *with* $c>0$. *Up to an isometry of* $M^4(c)\times\mathbb{R}$, Σ^2 *is parameterized by*

$$x(u,v)=\left(\frac{\cos\theta}{b}\cos(bu),\frac{\cos\theta}{b}\sin(bu),\frac{\sin(av)}{a},\frac{\cos(av)}{a},\frac{2H}{ab},u\sin\theta\right),\tag{1.3}$$

where $b=\sqrt{c+c\cos^2\theta}$, $a=\sqrt{b^2+4H^2}$ *and H is the mean curvature of* Σ^2.

2. Preliminaries

Let Σ^2 be a surface in an $(n+1)$-dimensional Riemannian manifold $\overline{M}^{n+1}$. Choose a local orthonormal frame field $\{e_1,e_2,\ldots,e_{n+1}\}$ in $T\overline{M}^{n+1}$ along Σ^2 so that $\{e_1,e_2\}$ are tangent to Σ^2 and the others are normal to Σ^2. Denote the dual frame by $\{\omega^1,\omega^2,\ldots,\omega^{n+1}\}$. Let ∇ (resp. $\overline{\nabla}$) be the Riemannian connection of Σ^2 (resp. $\overline{M}^{n+1}$). We use the following convention on index ranges in the whole paper:

$$1\le i,j,k,l\le 2;\quad 3\le\alpha,\beta,\gamma\le n+1;\quad 1\le A,B,C\le n+1.$$

Let $\{\omega_B^A\}$ be the connection form. The second fundamental form h and the mean curvature vector $\overrightarrow{H}$ are defined by

$$h=\sum_{\alpha,i}\omega^i\otimes\omega_i^\alpha\otimes e_\alpha=\sum_{\alpha,i,j}h_{ij}^\alpha\omega^i\otimes\omega^j\otimes e_\alpha,\quad \overrightarrow{H}=\sum_\alpha H^\alpha e_\alpha,\tag{2.1}$$

where

$$\omega_i^\alpha=\sum_j h_{ij}^\alpha\omega^j,\quad h_{ij}^\alpha=h_{ji}^\alpha,\quad H^\alpha=\left(\sum_i h_{ii}^\alpha\right)/2.\tag{2.2}$$

The mean curvature H is defined to be

$$H = \left|\overrightarrow{H}\right| = \left[\sum_\alpha (H^\alpha)^2\right]^{1/2}. \tag{2.3}$$

The Gauss-Ricci equations are expressed as

$$R^i_{jkl} = \overline{R}^i_{jkl} + \sum_\alpha (h^\alpha_{ik}h^\alpha_{jl} - h^\alpha_{il}h^\alpha_{jk}), \quad R^\alpha_{\beta kl} = \overline{R}^\alpha_{\beta kl} + \sum_i (h^\alpha_{ik}h^\beta_{il} - h^\alpha_{il}h^\beta_{ik}). \tag{2.4}$$

We define the covariant differential of h by

$$\nabla h^\alpha_{ij} = \sum_k h^\alpha_{ijk}\omega^k = dh^\alpha_{ij} - \sum_k h^\alpha_{ik}\omega^k_j - \sum_k h^\alpha_{kj}\omega^k_i + \sum_\gamma h^\gamma_{ij}\omega^\alpha_\gamma. \tag{2.5}$$

Then the Codazzi equation is

$$h^\alpha_{ijk} - h^\alpha_{ikj} = \overline{R}^\alpha_{ikj}. \tag{2.6}$$

Let $\overline{M}^{n+1} = M^n(c)\times\mathbb{R}$ with $c \neq 0$. Naturally, the ambient space $M^n(c)\times\mathbb{R}$ is endowed with the metric

$$ds^2_{M^n(c)\times\mathbb{R}} = ds^2_{M^n(c)} + dt^2.$$

The induced metric on Σ^2 is denoted by $\langle\,,\,\rangle$. Let $\partial_t = T + N$, where T is the tangent part and N the normal part of ∂_t. Then

$$T = \sum_i T^i e_i, \quad T^i = \langle e_i, \partial_t\rangle; \quad N = \sum_\alpha N^\alpha e_\alpha, \quad N^\alpha = \langle e_\alpha, \partial_t\rangle. \tag{2.7}$$

For any $X \in \Gamma(T\overline{M}^{n+1})$, we denote $X|_{M^n(c)} = X - \langle X, \partial_t\rangle \partial_t$, which is the projection of X onto the factor $M^n(c)$. Then

$$e_i|_{M^n(c)} = e_i - T^i\partial_t, \quad e_\alpha|_{M^n(c)} = e_\alpha - N^\alpha\partial_t.$$

It follows that

$$\begin{aligned}\overline{R}^A_{BCD} = c\,\big(&\langle e_A|_{M^n(c)}, e_C|_{M^n(c)}\rangle \langle e_B|_{M^n(c)}, e_D|_{M^n(c)}\rangle \\ &- \langle e_A|_{M^n(c)}, e_D|_{M^n(c)}\rangle \langle e_B|_{M^n(c)}, e_C|_{M^n(c)}\rangle\big),\end{aligned}$$

from which we obtain

$$\overline{R}^i_{jkl} = c\left\{\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} + \delta_{il}T^jT^k + \delta_{jk}T^iT^l - \delta_{ik}T^jT^l - \delta_{jl}T^iT^k\right\}, \tag{2.8}$$

$$\overline{R}^\alpha_{ikj} = cN^\alpha(T^j\delta_{ik} - T^k\delta_{ij}), \quad \overline{R}^\alpha_{\beta kl} = 0. \tag{2.9}$$

3. Some lemmas

In this section, we introduce several lemmas needed for the proof of the Main Theorem.

The fundamental formulae of submanifolds are given by

$$\overline{\nabla}_{e_i} e_j = \nabla_{e_i} e_j + h(e_i, e_j), \tag{3.1}$$

$$\overline{\nabla}_{e_i} e_\alpha = -A_\alpha(e_i) + \nabla^\perp_{e_i} e_\alpha, \tag{3.2}$$

where A_α is the shape operator with respect to e_α, and

$$\nabla_{e_i} e_j = \omega_j^k(e_i) e_k, \quad \nabla^\perp_{e_i} e_\alpha = \omega_\alpha^\beta(e_i) e_\beta, \quad A_\alpha(e_i) = h_{ij}^\alpha e_j, \quad h(e_i, e_j) = h_{ij}^\alpha e_\alpha. \tag{3.3}$$

Lemma 3.1. (cf. [14, 33]) *Let Σ^2 be a surface in $M^n(c) \times \mathbb{R}$. Then we have*

$$\nabla T^i = \sum_{\alpha, j} N^\alpha h_{ij}^\alpha \omega^j, \tag{3.4}$$

$$\nabla^\perp N^\alpha = -\sum_{i,j} T^i h_{ij}^\alpha \omega^j. \tag{3.5}$$

Let Σ^2 be a PMC surface in $M^n(c) \times \mathbb{R}$. According to (1.2), the coefficients $\{S_{ij}\}$ of S are given by

$$S_{ij} = 2\sum_\alpha H^\alpha h_{ij}^\alpha - cT^iT^j + \frac{c}{2}|T|^2 \delta_{ij} - 2H^2\delta_{ij}, \tag{3.6}$$

for any i, j. The covariant derivatives $\{S_{ijk}\}$ of $\{S_{ij}\}$ are defined by

$$\sum_k S_{ijk}\omega^k = dS_{ij} - \sum_k S_{ik}\omega_j^k - \sum_k S_{kj}\omega_i^k. \tag{3.7}$$

It is known that $\nabla^\perp \overrightarrow{H} = \nabla^\perp H^\alpha e_\alpha$, where

$$\nabla^\perp H^\alpha = dH^\alpha + H^\beta \omega_\beta^\alpha = H^\alpha_{,k}\omega^k. \tag{3.8}$$

Then $\nabla^\perp \overrightarrow{H} = 0$ implies $H^\alpha_{,k} = 0$, for all k and α. Taking the covariant derivatives of both sides of equation (3.6) and using (3.4), we get

$$\begin{aligned} S_{ijk} &= 2\sum_\alpha H^\alpha h_{ijk}^\alpha - cT^i\nabla_{e_k}T^j - cT^j\nabla_{e_k}T^i + \frac{c}{2}\delta_{ij}\nabla_{e_k}(|T|^2) \\ &= 2\sum_\alpha H^\alpha h_{ijk}^\alpha - cT^i\sum_\alpha N^\alpha h_{jk}^\alpha - cT^j\sum_\alpha N^\alpha h_{ik}^\alpha + c\delta_{ij}\sum_{\alpha,l} N^\alpha T^l h_{lk}^\alpha. \end{aligned} \tag{3.9}$$

Lemma 3.2. (cf. [9, 28]) *Let Σ^2 be a complete PMC surface immersed $M^n(c) \times \mathbb{R}$. Then $|S| = 0$ if and only if $\left|Q^{(2,0)}\right| = 0$.*

Using the results in [39], the authors in [28] obtained the following Simons type equation for $|S|^2$:

Theorem 3.3. (cf. [28]) *Let Σ^2 be a PMC surface immersed $M^n(c) \times \mathbb{R}$. Then*

$$\frac{1}{2}\Delta(|S|^2) = |\nabla S|^2 + 2K\,|S|^2\,, \tag{3.10}$$

where $|\nabla S|^2 = \sum_{i,j,k} S_{ijk}^2$ and K is the Gaussian curvature of Σ^2.

At the end of this section, we prove the following lemma:

Lemma 3.4. *Let Σ^2 be a complete PMC surface in $M^n(c) \times \mathbb{R}$. Let $\{e_1, e_2\}$ be the local orthonormal tangent frame field on Σ^2 such that $\{S_{ij}\}$ is diagonalized. If $|\nabla S| = 0$, then $|S| = 0$ or $|S|$ is nonzero constant and $(\omega_j^i) = 0$.*

Proof. From (3.6), it is easy to see that S is symmetric and trace-free. Choose the orthonormal tangent frame filed $\{e_1, e_2\}$ on Σ^2 such that $S_{ij} = \mu_i\delta_{ij}$. Then $\mu_1 = -\mu_2 = \mu$. So $|S|^2 = 2\mu^2$.

$|\nabla S| = 0$ implies $S_{ijk} = 0$, for any i, j, k. Setting $i = j = 1$ and $i = 1$, $j = 2$ in (3.7), we obtain

$$\begin{aligned} 0 &= \sum_k S_{11k}\omega^k = d\mu - \sum_k S_{1k}\omega_1^k - \sum_k S_{k1}\omega_1^k = d\mu, \\ 0 &= \sum_k S_{12k}\omega^k = -\sum_k S_{1k}\omega_2^k - \sum_k S_{k2}\omega_1^k = 2\mu\omega_1^2, \end{aligned}$$

which imply $\mu = 0$ or μ is nonzero constant and $\omega_1^2 = 0$. The proof of Lemma 3.4 is completed. □

4. A classification theorem

In this section, we firstly prove a classification theorem. Then, we solve the problem proposed in [5] under the condition that $|T|$ is constant.

Theorem 4.1. *Let $x\colon \Sigma^2 \to M^n(c) \times \mathbb{R}$ be a complete PMC surface with Gaussian curvature $K \geq 0$. Then $|S| = 0$ or $|S|$ is nonzero constant and $K = 0$.*

Proof. According to (3.10) and the hypothesis, it follows that $\Delta\,|S|^2 \geq 0$. By a result of Huber [34], a complete surface with non-negative Gaussian curvature is a parabolic space. Therefore, $|S|^2$ is harmonic. One can immediately get $|\nabla S| = 0$ by (3.10) again. Note that $d\omega_2^1 = K\omega^1 \wedge \omega^2$. Following Lemma 3.4, we complete the proof. □

It is known that the surface with $|S|=0$ is an *Abresch-Rosenberg type surface.* In the sequel, we proceed to consider the rest case.

Denote $|T|=\sin\theta$ with $\theta\in[0,\pi/2]$. It is obvious that Σ^2 lies in $M^n(c)$ when $\theta=0$ and $\Sigma^2=\gamma\times\mathbb{R}$ in case $\theta=\pi/2$, where γ is a curve in $M^n(c)$. So we need only to treat the problem for $\theta\in(0,\pi/2)$.

Lemma 4.2. *Let $x\colon\Sigma^2\to M^n(c)\times\mathbb{R}$ with $c\neq 0$ be a complete non-minimal PMC surface with non-negative Gaussian curvature so that $|S|$ is nonzero constant. If $\theta\in(0,\pi/2)$ is constant, then Σ^2 lies in $M^4(c)\times\mathbb{R}$ and the mean curvature vector is orthogonal to ∂_t.*

Proof. According to Theorem 1.1 and the fact that θ is a non-zero constant, it follows that the surface Σ^2 lies in $M^4(c)\times\mathbb{R}$. By Lemma 3.4, $|S|$ is nonzero constant and

$$\omega_j^i=0, \tag{4.1}$$

under the chosen tangent frame $\{e_1,e_2\}$. We choose a local orthonormal normal frame field $\{e_3,e_4,e_5\}$ so that $\operatorname{span}\left\{\overrightarrow{H},N\right\}=\operatorname{span}\{e_3,e_4\}$. Then $\overrightarrow{H}$ and N can be decomposed as

$$\overrightarrow{H}=H^3e_3+H^4e_4,\quad N=N^3e_3+N^4e_4.$$

Using (3.4), we get

$$\nabla_{e_j}|T|^2=2\sum_{\alpha,i}N^\alpha T^i h_{ij}^\alpha=0, \tag{4.2}$$

for $j\in\{1,2\}$. From (4.2), we have a linear system of linear equations on $\{T^1,T^2\}$,

$$\left(\sum_\alpha N^\alpha h_{1j}^\alpha\right)T^1+\left(\sum_\alpha N^\alpha h_{2j}^\alpha\right)T^2=0,$$

for $j\in\{1,2\}$, which has non-zero solutions. Hence we obtain

$$\left(\sum_\alpha N^\alpha h_{11}^\alpha\right)\left(\sum_\alpha N^\alpha h_{22}^\alpha\right)-\left(\sum_\alpha N^\alpha h_{12}^\alpha\right)^2=0,$$

which is equivalent to

$$(N^3)^2\det(A_3)+(N^4)^2\det(A_4)+N^3N^4(h_{11}^3h_{22}^4+h_{11}^4h_{22}^3-2h_{12}^3h_{12}^4)=0. \tag{4.3}$$

Following (4.2) again, we get another system of linear equations on $\{N^3,N^4\}$,

$$\left(\sum_i T^ih_{ik}^3\right)N^3+\left(\sum_j T^jh_{jk}^4\right)N^4=0,$$

for $j \in \{1,2\}$, which has also non-zero solutions. And we also get

$$\sum_{i,j} T^i T^j (h^3_{i1} h^4_{j2} - h^4_{i1} h^3_{j2}) = 0. \tag{4.4}$$

Suppose that $\nabla^\perp \overrightarrow{H} = 0$. From the first equality in (3.8) we have, for every α,

$$dH^\alpha + H^\beta \omega^\alpha_\beta = 0. \tag{4.5}$$

Taking the exterior derivative on both-sides of (4.5) and using the structure equation, we obtain

$$\sum_{\beta,i} H^\beta (d\omega^\alpha_\beta - \omega^\gamma_\beta \wedge \omega^\alpha_\gamma) = \sum_{\beta,i,j} H^\beta R^\alpha_{\beta ij} \omega_i \wedge \omega_j = 0.$$

It follows from (2.4), (2.9) and the third equality in (3.3) that

$$\left[A_{\overrightarrow{H}}, A_\alpha\right] = A_{\overrightarrow{H}} A_\alpha - A_\alpha A_{\overrightarrow{H}} = \left(\sum_\beta H^\beta R^\alpha_{\beta ij}\right)_{2\times 2} = 0, \tag{4.6}$$

for any α. Setting $\alpha = 3, 4$ respectively, one has

$$H^3(A_3A_4 - A_4A_3) = H^4(A_3A_4 - A_4A_3) = 0.$$

Since $(H^3)^2 + (H^4)^2 > 0$, we obtain

$$A_3A_4 = A_4A_3. \tag{4.7}$$

According (4.7), one can obtain

$$h^3_{12}(h^4_{11} - h^4_{22}) = h^4_{12}(h^3_{11} - h^3_{22}), \tag{4.8}$$

which is equivalent with

$$h^3_{11}h^4_{12} - h^4_{11}h^3_{12} = h^3_{22}h^4_{12} - h^4_{22}h^3_{12}. \tag{4.9}$$

Define four normal vectors in the normal space span $\{e_3, e_4\}$ as follows:

$$A = \sum_{\alpha=3}^4 h^\alpha_{11} e_\alpha, \quad B = \sum_{\alpha=3}^4 h^\alpha_{22} e_\alpha, \quad C = \sum_{\alpha=3}^4 h^\alpha_{12} e_\alpha, \quad D = A - B = \sum_{\alpha=3}^4 (h^\alpha_{11} - h^\alpha_{22}) e_\alpha.$$

In order to prove $\overrightarrow{H} \perp \partial_t$, it suffices to prove $\left\langle N, \overrightarrow{H}\right\rangle = 0$. Now, we divide our proof in two cases.

Case 1. D is nonzero.
It is clear that C is parallel to D by (4.8). So we have $C = \lambda D$ for some function λ on Σ^2, which implies

$$h^\alpha_{12} = \lambda(h^\alpha_{11} - h^\alpha_{22}), \tag{4.10}$$

for $\alpha\in\{3,4\}$. Substituting (4.9) and (4.10) into (4.4), we have

$$\begin{aligned}0&=\left[(T^1)^2-(T^2)^2\right](h^3_{11}h^4_{12}-h^4_{11}h^3_{12})+T^1T^2(h^3_{11}h^4_{22}-h^4_{11}h^3_{22})\\&=\lambda\left[(T^1)^2-(T^2)^2\right](h^4_{11}h^3_{22}-h^3_{11}h^4_{22})+T^1T^2(h^3_{11}h^4_{22}-h^4_{11}h^3_{22})\\&=\left\{\lambda\left[(T^1)^2-(T^2)^2\right]-T^1T^2\right\}(h^4_{11}h^3_{22}-h^3_{11}h^4_{22}),\end{aligned}\tag{4.11}$$

which implies $\lambda\left[(T^1)^2-(T^2)^2\right]=T^1T^2$ or $h^4_{11}h^3_{22}=h^3_{11}h^4_{22}$. We discuss it in two subcases.

Subcase 1. $\lambda\left[(T^1)^2-(T^2)^2\right]=T^1T^2$.
If $T^1T^2\neq 0$, then $(T^1)^2\neq(T^2)^2$ and $\lambda\neq 0$. From (3.6) and (4.10), we have

$$\begin{aligned}S_{12}&=2\sum_\alpha H^\alpha h^\alpha_{12}-cT^1T^2=2\sum_\alpha H^\alpha\lambda(h^\alpha_{11}-h^\alpha_{22})-cT^1T^2\\&=4\lambda\left(\sum_\alpha H^\alpha h^\alpha_{11}-H^2\right)-cT^1T^2=0,\end{aligned}$$

from which we obtain

$$\begin{aligned}\mu=S_{11}&=2\sum_\alpha H^\alpha h^\alpha_{11}-c(T^1)^2+\frac{c}{2}|T|^2-2H^2\\&=2\left(\frac{cT^1T^2}{4\lambda}+H^2\right)+\frac{c}{2}\left[(T^2)^2-(T^1)^2\right]-2H^2\\&=\frac{c}{2\lambda}\left\{T^1T^2+\lambda\left[(T^2)^2-(T^1)^2\right]\right\}=0,\end{aligned}\tag{4.12}$$

that contradicts to the assumption $|S|\neq 0$. Therefore, $T^1T^2=0$. Without loss of generality, we assume $T^2=0$ and $T^1=\sin\theta$. It follows from (4.1) that

$$\nabla T^i=dT^i+\sum_k T^k\omega^i_k=T^1\omega^i_1=0,$$

for any i. Using (3.4), we obtain

$$\nabla_{e_j}T^i=\sum_\alpha N^\alpha h^\alpha_{ij}=0,\tag{4.13}$$

for any i,j. Setting $i=j$ and taking summation, we have $\left\langle N,\vec{H}\right\rangle=\sum_\alpha N^\alpha H^\alpha=0$.

Subcase 2. $h^4_{11}h^3_{22}=h^3_{11}h^4_{22}$.
In this case, A is parallel to B. Owing to the non-minimal property, A and B do not vanish simultaneously. Without loss of generality, we assume that $h^\alpha_{11}=\nu h^\alpha_{22}$ for $\alpha\in\{3,4\}$, where $\nu\neq -1$ is a function on Σ^2 since $H\neq 0$. Using (4.10), we obtain

$$h^\alpha_{11}=\frac{2\nu}{\nu+1}H^\alpha,\quad h^\alpha_{22}=\frac{2}{\nu+1}H^\alpha,\quad h^\alpha_{12}=\frac{2\lambda(\nu-1)}{\nu+1}H^\alpha,\tag{4.14}$$

which is equivalent with $A_\alpha=H^\alpha A_0$, where

$$A_0=\frac{2}{\nu+1}\begin{pmatrix}\nu&\lambda(\nu-1)\\\lambda(\nu-1)&1\end{pmatrix}.\tag{4.15}$$

By (4.14) and (4.15), we have

$$\det(A_\alpha) = (H^\alpha)^2 \det(A_0), \quad \det(A_0) = \frac{4}{(\nu+1)^2}\left[\nu - \lambda^2(\nu-1)^2\right]. \tag{4.16}$$

Substituting (4.14) and (4.16) into (4.3), we get

$$0 = (N^3H^3 + N^4H^4)^2 \det(A_0) = \left\langle N, \overrightarrow{H} \right\rangle^2 \det(A_0). \tag{4.17}$$

It follows that $\left\langle N, \overrightarrow{H} \right\rangle = 0$ or $\det(A_0) = 0$. In latter case, we have $\nu = \lambda^2(\nu-1)^2$. By applying (3.6) and (4.14), we obtain

$$S_{12} = 2\sum_\alpha H^\alpha h^\alpha_{12} - cT^1T^2 = 4\lambda H^2\left(\frac{\nu-1}{\nu+1}\right) - cT^1T^2 = 0. \tag{4.18}$$

Moreover, we have

$$\mu = S_{11} = 2H^2\left(\frac{\nu-1}{\nu+1}\right) - c(T^1)^2 + \frac{c}{2}|T|^2$$

which implies

$$c(T^1)^2 = \frac{c}{2}|T|^2 + 2H^2\left(\frac{\nu-1}{\nu+1}\right) - \mu, \quad c(T^2)^2 = \frac{c}{2}|T|^2 - 2H^2\left(\frac{\nu-1}{\nu+1}\right) + \mu. \tag{4.19}$$

Using (4.18) and (4.19), we obtain via a straightforward calculation

$$\frac{16\nu H^4}{(\nu+1)^2} = \frac{16\lambda^2(\nu-1)^2H^4}{(\nu+1)^2} = c^2(T^1)^2(T^2)^2 = \frac{c^2}{4}|T|^4 - \left[\mu - \frac{2(\nu-1)H^2}{\nu+1}\right]^2,$$

where we used $\nu = \lambda^2(\nu-1)^2$ in the first equality. Simplifying the above equation, we find

$$\frac{\nu-1}{\nu+1} = \frac{c^2|T|^4 - 16H^4 - 4\mu^2}{16\mu H^2}, \tag{4.20}$$

which is constant. Therefore, both T^1 and T^2 are constant by (4.19) and (4.20). By similar argument as in Subcase 1, we obtain (4.13), from which we obtain $\left\langle N, \overrightarrow{H} \right\rangle = 0$.

Case 2. D is a zero vector.
In this case we have $A = B = \overrightarrow{H}$. Applying (4.4), we have

$$\begin{aligned} 0 &= (T^1)^2(H^3h^4_{12} - H^4h^3_{12}) + (T^2)^2(h^3_{21}H^4 - h^4_{21}H^3) + T^1T^2(H^3H^4 - H^4H^3) \\ &= \left[(T^1)^2 - (T^2)^2\right](H^3h^4_{12} - H^4h^3_{12}), \end{aligned} \tag{4.21}$$

from which we get $(T^1)^2 = (T^2)^2$ or $H^3h^4_{12} = H^4h^3_{12}$. When $(T^1)^2 = (T^2)^2$, we can see that T^1 and T^2 are constant, which implies that $\left\langle N, \overrightarrow{H} \right\rangle = 0$.

Suppose that $H^3 h^4_{12} = H^4 h^3_{12}$. Then C is parallel to $\overrightarrow{H}$. Set $C = \rho \overrightarrow{H}$. Then $h^\alpha_{12} = \rho H^\alpha$ for $\alpha \in \{3,4\}$. It follows that $\det(A_3) = (H^3)^2(1-\rho^2)$ and $\det(A_4) = (H^4)^2(1-\rho^2)$. According to (4.3), we find

$$\begin{aligned} 0 &= (N^3H^3)^2(1-\rho^2) + (N^4H^4)^2(1-\rho^2) + 2N^3N^4H^3H^4(1-\rho^2) \\ &= (1-\rho^2)\left\langle N, \overrightarrow{H}\right\rangle^2, \end{aligned} \tag{4.22}$$

from which we have $\left\langle N, \overrightarrow{H}\right\rangle = 0$ or $\rho^2 = 1$. In case $\rho^2 = 1$, we have from (3.6) that

$$0 = S_{12} = 2\sum_\alpha H^\alpha h^\alpha_{12} - cT^1T^2 = 2\rho H^2 - cT^1T^2,$$

which implies

$$(cT^1T^2)^2 = (2\rho H^2)^2 = 4H^4. \tag{4.23}$$

According (4.23) and the fact that $(T^1)^2 + (T^2)^2 = \sin^2\theta$, it follows that T^1 and T^2 are constant, which means that $\left\langle N, \overrightarrow{H}\right\rangle = 0$.

Summarizing the above cases, we claim that $\overrightarrow{H}$ is orthogonal to N. Our statement is proved. □

Then we present an example that satisfies all the conditions described in Lemma 4.2.

Example 4.3. Define the map $x\colon \mathbb{R}^2 \to \mathbb{R}^6$ as follows:

$$\begin{aligned} x(u,v) &= (x^1, \ldots, x^6) \\ &= \left(\frac{\cos\theta}{b}\cos(bu), \frac{\cos\theta}{b}\sin(bu), \frac{\sin(av)}{a}, \frac{\cos(av)}{a}, \frac{2H}{ab}, u\sin\theta\right), \end{aligned} \tag{4.24}$$

where $b = \sqrt{c + c\cos^2\theta}$, $a = \sqrt{b^2 + 4H^2}$ and $H, c > 0$, θ are constant. Let $\Sigma^2 = x(\mathbb{R}^2)$. Then Σ^2 is a surface described as in Lemma 4.2.

Firstly, $\sum_{i=1}^5 (x^i)^2 = 1/c$, thus Σ^2 lies in $M^4(c) \times \mathbb{R}$. Let $\partial_t = (0, \ldots, 0, 1)$ and

$$\partial_u = x_u = (-\cos\theta\sin(bu), \cos\theta\cos(bu), 0, 0, 0, \sin\theta), \tag{4.25}$$

$$\partial_v = x_v = (0, 0, \cos(av), -\sin(av), 0, 0). \tag{4.26}$$

Then we have

$$|x_u|^2 = |x_u|^2 = 1, \quad \langle x_u, x_v\rangle = 0, \quad \omega^i_j = 0, \quad K = 0, \tag{4.27}$$

$$T = \langle \partial_t, x_u\rangle x_u + \langle \partial_t, x_v\rangle x_v = \sin\theta x_u, \quad |T|^2 = \sin^2\theta. \tag{4.28}$$

Let us consider $M^4(c)\times\mathbb{R}$ as a hypersurface in $\mathbb{R}^6$. Its normal vector $\overline{e}_6=(x^1,\ldots,x^5,0)$. Denote its unit normal vector by $e_6=\sqrt{c}\overline{e}_6$. Let D be Euclidean connection and $\overline{h}$ be the second fundamental form of $M^4(c)\times\mathbb{R}$ in $\mathbb{R}^6$. Then

$$\overline{h}(X,Y)=-c\left\langle X|_{M^4(c)},Y|_{M^4(c)}\right\rangle\overline{e}_6=c(\langle X,\partial_t\rangle\langle Y,\partial_t\rangle-\langle X,Y\rangle\overline{e}_6,\tag{4.29}$$

for any $X,Y\in\mathscr{X}(M^4(c)\times\mathbb{R})$. From (3.1), (4.29) and the third fact in (4.27), we have

$$h(X,Y)=D_XY-\nabla_XY-\overline{h}(X,Y)=D_XY+c\left\langle X|_{M^4(c)},Y|_{M^4(c)}\right\rangle\overline{e}_6,\tag{4.30}$$

for any tangent vectors $X,Y\in\mathscr{X}(\Sigma^2)$. Using (4.25), (4.26) and (4.30), we obtain

$$\begin{aligned}
h(\partial_u,\partial_u)&=x_{uu}+c\cos^2\theta\cdot\overline{e}_6\\
&=(-b\cos\theta\cos(bu),-b\cos\theta\sin(bu),0,0,0,0)\\
&\quad+c\cos^2\theta\left(\frac{\cos\theta}{b}\cos(bu),\frac{\cos\theta}{b}\sin(bu),\frac{\sin(av)}{a},\frac{\cos(av)}{a},\frac{2H}{ab},0\right)\\
&=c\cos\theta\left(-\frac{\cos(bu)}{b},-\frac{\sin(bu)}{b},\frac{\cos\theta\sin(av)}{a},\frac{\cos\theta\cos(av)}{a},\frac{2H\cos\theta}{ab},0\right),\\
h(\partial_u,\partial_v)&=x_{uv}+c\left\langle\partial_u|_{M^4(c)},\partial_v|_{M^4(c)}\right\rangle\overline{e}_6=0,\\
h(\partial_v,\partial_v)&=x_{vv}+c\overline{e}_6=(0,0,-a\sin(av),-a\cos(av),0,0)\\
&\quad+c\left(\frac{\cos\theta}{b}\cos(bu),\frac{\cos\theta}{b}\sin(bu),\frac{\sin(av)}{a},\frac{\cos(av)}{a},\frac{2H}{ab},0\right)\\
&=\left(\frac{c\cos\theta}{b}\cos(bu),\frac{c\cos\theta}{b}\sin(bu),\frac{c-a^2}{a}\sin(av),\frac{c-a^2}{a}\cos(av),\frac{2cH}{ab},0\right).
\end{aligned}\tag{4.31}$$

Thus

$$\overrightarrow{H}=\frac{1}{2}[h(\partial_u,\partial_u)+h(\partial_v,\partial_v)]=\frac{H}{a}\left(0,0,-2H\sin(av),-2H\cos(av),b,0\right),\tag{4.32}$$

and

$$\left\langle\overrightarrow{H},h(\partial_u,\partial_u)\right\rangle=\left\langle\overrightarrow{H},h(\partial_u,\partial_v)\right\rangle=0,\quad\left\langle\overrightarrow{H},h(\partial_v,\partial_v)\right\rangle=2H^2.\tag{4.33}$$

From (3.6) and (4.33), we get

$$\langle S\partial_u,\partial_u\rangle=-\frac{c}{2}\sin^2\theta-2H^2=-\langle S\partial_v,\partial_v\rangle,\quad\langle S\partial_u,\partial_v\rangle=0,$$

which imply

$$|S|^2=2\left(\frac{c}{2}\sin^2\theta+2H^2\right)^2,\tag{4.34}$$

which is a nonzero constant. From (4.25), (4.26) and (4.32), we have

$$\left\langle\overrightarrow{H},\partial_t\right\rangle=\left\langle\overrightarrow{H},\partial_u\right\rangle=\left\langle\overrightarrow{H},\partial_v\right\rangle=0.$$

Using (4.29), we get

$$\overline{h}(\partial_u,\overrightarrow{H})=\overline{h}(\partial_v,\overrightarrow{H})=0,$$

from which, together with (3.1), (3.2) and (4.33), we obtain

$$\begin{aligned}\nabla^{\perp}_{\partial_u}\overrightarrow{H}&=D_{\partial_u}\overrightarrow{H}+A_{\overrightarrow{H}}\partial_u-\overline{h}(\partial_u,\overrightarrow{H})=(\overrightarrow{H})_u+0+0=0,\\ \nabla^{\perp}_{\partial_v}\overrightarrow{H}&=D_{\partial_v}\overrightarrow{H}+A_{\overrightarrow{H}}\partial_v-\overline{h}(\partial_v,\overrightarrow{H})=-2H^2\partial_v+2H^2\partial_v+0=0,\end{aligned}\tag{4.35}$$

which imply $\nabla^{\perp}\overrightarrow{H}=0$. Therefore, Σ^2 defined by (4.24) satisfies all conditions described in Lemma 4.2.

In the sequel, we proceed to characterize the surfaces in Lemma 4.2 and prove that the kind of surfaces in Example 4.3 is the only class, up to an isometry of $M^4(c)\times\mathbb{R}$.

Lemma 4.4. *Under the same conditions as in Lemma 4.2, Σ^2 lies in $M^4(c)\times\mathbb{R}$ with $c>0$ and $\overrightarrow{H}\perp N$. With respect to the normal frame field $\left\{e_3=\overrightarrow{H}/H, e_4=N/|N|, e_5\right\}$, the shape operators A_α and the connection form matrix (ω^A_B) are represented as*

$$A_3=\begin{pmatrix}0&0\\0&2H\end{pmatrix},\quad A_4=0,\quad A_5=\begin{pmatrix}\sqrt{c}\cos\theta&0\\0&-\sqrt{c}\cos\theta\end{pmatrix},\tag{4.36}$$

$$(\omega^A_B)=\begin{pmatrix}0&0&0&0&\sqrt{c}\cos\theta\,\omega^1\\0&0&2H\omega^2&0&-\sqrt{c}\cos\theta\,\omega^2\\0&-2H\omega^2&0&0&0\\0&0&0&0&-\sqrt{c}\sin\theta\,\omega^1\\-\sqrt{c}\cos\theta\,\omega^1&\sqrt{c}\cos\theta\,\omega^2&0&\sqrt{c}\sin\theta\,\omega^1&0\end{pmatrix}.\tag{4.37}$$

Proof. Owing to $\nabla^{\perp}\overrightarrow{H}=0$, we have

$$0=dH^\alpha+\sum_\beta H^\beta\omega^\alpha_\beta=H\omega^\alpha_3,$$

which implies $\omega^\alpha_3=\omega^\alpha_4=0$, for $\alpha\in\{3,4\}$. Thus by (3.5), we have

$$\sum_{i,j}T^i h^\alpha_{ij}\omega^j=-\nabla^{\perp}N^\alpha=-dN^\alpha-\sum_\beta N^\beta\omega^\alpha_\beta=-|N|\,\omega^\alpha_4=0,$$

which is equivalent with

$$T^1h^\alpha_{11}+T^2h^\alpha_{12}=0\quad\text{and}\quad -T^2h^\alpha_{11}+T^1h^\alpha_{12}=-2H^\alpha T^2,\tag{4.38}$$

for $\alpha\in\{3,4\}$. By Cramer's Rule, we obtain

$$A_3=\frac{2H}{|T|^2}\begin{pmatrix}(T^2)^2&-T^1T^2\\-T^1T^2&(T^1)^2\end{pmatrix},\tag{4.39}$$

and $A_4 = 0$ which is the desired form. From (3.6) and (4.39), we get

$$\mu = S_{11} = 2Hh^3_{11} - c(T^1)^2 + \frac{c}{2}|T|^2 - 2H^2 = \frac{1}{2}\left[(T^2)^2 - (T^1)^2\right]\left(c + \frac{4H^2}{|T|^2}\right), \tag{4.40}$$

$$0 = S_{12} = 2Hh^3_{12} - cT^1T^2 = -T^1T^2\left(c + \frac{4H^2}{|T|^2}\right), \tag{4.41}$$

which imply $T^1T^2 = 0$. Without loss of generality, we assume $T^2 = 0$ and $T^1 = |T|$. Then, A_3 becomes the desired form. From (4.6), we get $A_3A_5 = A_5A_3$, which implies $h^5_{12} = 0$. It follows that

$$0 = K = c\cos^2\theta + \sum_\alpha \det(A_\alpha) = c\cos^2\theta + \det(A_5) = c\cos^2\theta - (h^5_{11})^2, \tag{4.42}$$

which implies that $c > 0$. We can set $h^5_{11} = \sqrt{c}\cos\theta$ (one can change the direction of e_5 if necessarily). Consequently, A_5 has the desired form. At last, by (3.5), we obtain

$$\begin{aligned} \cos\theta\,\omega^5_4 &= dN^5 + \sum_\alpha N^\alpha\omega^5_\alpha = \nabla^\perp N^5 \\ &= -\sum_{i,j} T^i h^5_{ij}\omega^j = -T^1h^5_{11}\omega^1 = -\sqrt{c}\sin\theta\cos\theta\,\omega^1, \end{aligned} \tag{4.43}$$

which implies $\omega^5_4 = -\sqrt{c}\sin\theta\,\omega^1$. To summarize what we have proven, and applying $\omega^\alpha_i = \sum_j h^\alpha_{ij}\omega^j$, we can see that the connection form matrix is given by (4.37). This completes the proof of Lemma 4.4. □

In Lemma 4.4, we obtain the connection form of the surface and, following the existence and uniqueness theorem of submanifolds, the surface is unique (up to an isometry of $M^4(c)\times\mathbb{R}$) for a fixed θ, and given by Example 4.3. Next, we will give a way of looking for the surface of Example 4.3, which is similar to that in [20, 21].

Theorem 4.5. *Let $x\colon \Sigma^2 \to M^n(c)\times\mathbb{R}$ with $c \neq 0$ be a complete non-minimal PMC surface with non-negative Gaussian curvature and $|S| \neq 0$. If $\theta \in (0, \pi/2)$ is constant, then Σ^2 lies in $M^4(c)\times\mathbb{R}$ with $c > 0$. Up to an isometry of $M^4(c)\times\mathbb{R}$, the immersion is given by* (4.24).

Proof. By Lemma 4.4, Σ^2 lies in $M^4(c)\times\mathbb{R}$ with $c > 0$. Let $x(u,v) = (x^1,\dots,x^6)$ be the position vector satisfying $\sum_{i=1}^5 (x^i)^2 = 1/c$. We choose the same frame $\{e_1,\dots,e_5\}$ as in Lemma 4.4. Thus, one can take coordinates (u,v) on Σ^2 with $\partial_u = e_1$, $\partial_v = e_2$. Then we have

$$T = \sin\theta\,\partial_u, \quad \overrightarrow{H} = He_3, \quad N = \cos\theta\,e_4.$$

Regard $M^4(c)\times\mathbb{R}$ as a hypersurface in $\mathbb{R}^6$. The normal vector of $M^4(c)\times\mathbb{R}$ in $\mathbb{R}^6$ is $\overline{e}_6 = (x^1,\dots,x^5,0)$. Let $e_6 = \sqrt{c}\,\overline{e}_6$ and denote

$$e_\alpha = (\xi^\alpha_1,\dots,\xi^\alpha_6),$$

for $\alpha\in\{3,4,5\}$. Then $\{e_\alpha\}_{\alpha=3}^6$ forms a normal frame of Σ^2 in $\mathbb{R}^6$. We denote by D (resp. $D^\perp$) the Euclidean connection (resp. the normal connection). Then, for any $X\in\mathscr{X}(\Sigma^2)$, we have

(4.44)
$$D_X\overline{e}_6 = X|_{M^4(c)} = X-\langle X,\partial_t\rangle\,\partial_t,\quad D^\perp_X\overline{e}_6=\sum_{\alpha=3}^5\langle D_X\overline{e}_6,e_\alpha\rangle\,e_\alpha=-\cos\theta\,\langle X,T\rangle\,e_4,$$

from which one gets

$$A_{\overline{e}_6}X=-D_X\overline{e}_6+D^\perp_X\overline{e}_6=-X\big|_{M^4(c)}-\cos\theta\,\langle X,T\rangle\,e_4=-X+\langle X,T\rangle\,T. \tag{4.45}$$

By applying (4.45), we obtain

$$\begin{aligned}-\cos^2\theta\,\partial_u&=-\partial_u+\langle\partial_u,T\rangle\,T=A_{\overline{e}_6}(\partial_u)=-\partial_u\big|_{M^4(c)}-\cos\theta\sin\theta\,e_4,\\ -\partial_v&=-\partial_v+\langle\partial_v,T\rangle\,T=A_{\overline{e}_6}(\partial_v)=-\partial_v\big|_{M^4(c)}.\end{aligned}$$

It follows that

$$e_4=\frac{1}{\sin\theta\cos\theta}\left(-\partial_u\big|_{M^4(c)}+\cos^2\theta\,\partial_u\right),\quad \partial_v=\partial_v\big|_{M^4(c)},$$

from which we have, by rewriting them into components of e_4,

$$(x^j)_u=-\xi_j^4\cot\theta,\quad 1\le j\le 5, \tag{4.46}$$

$$(x^6)_u=\xi_6^4\tan\theta=\tan\theta\,\langle e_4,\partial_t\rangle=\sin\theta,\quad (x^6)_v=0. \tag{4.47}$$

From (4.47), we can take $x^6=u\sin\theta$. Using (4.29), we have

$$\overline{h}(X,e_\alpha)=c\,\langle X,T\rangle\,\langle e_\alpha,N\rangle\,\overline{e}_6=c\delta_{4\alpha}\cos\theta\,\langle X,T\rangle\,\overline{e}_6, \tag{4.48}$$

for any $\alpha\in\{3,4,5\}$ and $X\in\mathscr{X}(\Sigma^2)$. By the fundamental formulae of submanifolds, we get

$$\begin{aligned}D_Xe_\alpha&=-A_\alpha X+\nabla^\perp_Xe_\alpha+\overline{h}(X,e_\alpha)\\ &=-A_\alpha X+\sum_{\beta=3}^5\omega_\alpha^\beta(X)e_\beta+c\delta_{4\alpha}\cos\theta\,\langle X,T\rangle\,\overline{e}_6,\end{aligned} \tag{4.49}$$

from which we obtain

$$\begin{aligned}&D_{\partial_u}e_3=-A_3(\partial_u)=0,\quad D_{\partial_v}e_3=-A_3(\partial_v)=-2H\partial_v,\\ &D_{\partial_u}e_4=\omega_4^5(\partial_u)\,e_5+c\cos\theta\,\langle\partial_u,T\rangle\,\overline{e}_6\\ &\qquad=-\sqrt{c}\sin\theta\,e_5+c\sin\theta\cos\theta\,\overline{e}_6,\quad D_{\partial_v}e_4=0,\\ &D_{\partial_u}e_5=-A_5(\partial_u)+\omega_5^4(\partial_u)\,e_4=-\sqrt{c}\cos\theta\,\partial_u+\sqrt{c}\sin\theta\,e_4,\\ &D_{\partial_v}e_5=-A_5(\partial_v)+\omega_5^4(\partial_v)\,e_4=\sqrt{c}\cos\theta\,\partial_v.\end{aligned} \tag{4.50}$$

By rewriting (4.50) into components of e_α's, we have

$$(\xi_A^3)_u = 0, \quad (\xi_A^3)_v = -2H(x^A)_v, \tag{4.51}$$

$$(\xi_j^4)_u = -\sqrt{c}\sin\theta\,\xi_j^5 + c\sin\theta\cos\theta\, x^j, \tag{4.52}$$

$$(\xi_6^4)_u = -\sqrt{c}\sin\theta\,\xi_6^5, \tag{4.53}$$

$$(\xi_A^4)_v = 0, \tag{4.54}$$

$$(\xi_A^5)_u = -\sqrt{c}\cos\theta\,(x^A)_u + \sqrt{c}\sin\theta\,\xi_A^4, \tag{4.55}$$

$$(\xi_A^5)_v = \sqrt{c}\cos\theta\,(x^A)_v, \tag{4.56}$$

where $1 \le A \le 6$ and $1 \le j \le 5$. Substituting (4.46) into (4.55), we have

$$(\xi_j^5)_u = \sqrt{c}\cos\theta\cot\theta\,\xi_j^4 + \sqrt{c}\sin\theta\,\xi_j^4 = \frac{\sqrt{c}}{\sin\theta}\xi_j^4. \tag{4.57}$$

Taking the derivative of (4.52) with respect to u and using (4.46) and (4.57), we obtain

$$(\xi_j^4)_{uu} = -\sqrt{c}\sin\theta\,\frac{\sqrt{c}}{\sin\theta}\xi_j^4 + c\sin\theta\cos\theta\,(-\cot\theta\,\xi_j^4) = -c(1+\cos^2\theta)\xi_j^4. \tag{4.58}$$

By solving (4.58) and using (4.54), we get

$$\xi_j^4 = C_j^1\sin(bu) + C_j^2\cos(bu) + C_j^6, \tag{4.59}$$

where $b = \sqrt{c + c\cos^2\theta}$ and C_j^1, C_j^2, C_j^6 are constants. Combing (4.46) and (4.59), we have

$$(x^j)_u = -\cot\theta\left[C_j^1\sin(bu) + C_j^2\cos(bu) + C_j^6\right],$$

from which, we obtain

$$x^j = \frac{\cot\theta}{b}\left[C_j^1\cos(bu) - C_j^2\sin(bu)\right] - \cot\theta\,C_j^6 u + \psi_j(v), \tag{4.60}$$

where $\psi_j(v)$ is a function with respect to v, for all $1 \le j \le 5$.

From (4.29), we have $\overline{h}(\partial_v, \partial_v) = -c\overline{e}_6$. From

$$D_{\partial_v}\partial_v = \nabla_{\partial_v}\partial_v + \sum_{\alpha=3}^{5}\langle A_\alpha(\partial_v), \partial_v\rangle\, e_\alpha + \overline{h}(\partial_v, \partial_v) = 2He_3 - \sqrt{c}\cos\theta\, e_5 - c\overline{e}_6,$$

it follows that

$$(x^j)_{vv} = 2H\xi_j^3 - \sqrt{c}\cos\theta\,\xi_j^5 - cx^j. \tag{4.61}$$

Substituting (4.60) into the second equality of (4.51), we obtain

$$(\xi_j^3)_v = -2H(x^j)_v = -2H\psi_j'(v),$$

from which, combining with the first equality of (4.51), we get

$$\xi_j^3 = -2H\psi_j(v) + C_j^3, \tag{4.62}$$

where C_j^3 is a constant. According to (4.52) and (4.59), we have

$$\begin{aligned} \xi_j^5 &= \frac{1}{\sqrt{c}\,\sin\theta}\left[c\sin\theta\cos\theta\, x^j - (\xi_j^4)_u\right] \\ &= \frac{1}{\sqrt{c}\,\sin\theta}\left[c\sin\theta\cos\theta\, x^j - bC_j^1\cos(bu) + bC_j^2\sin(bu)\right]. \end{aligned} \tag{4.63}$$

Substituting (4.60), (4.62), (4.63) into (4.61), we obtain

$$\begin{aligned} \psi_j''(v) &= -4H^2\psi_j(v) + 2HC_j^3 - \cot\theta\left[c\sin\theta\cos\theta\, x^j - bC_j^1\cos(bu) + bC_j^2\sin(bu)\right] - cx^j \\ &= -4H^2\psi_j(v) + 2HC_j^3 + b\cot\theta\left[C_j^1\cos(bu) - C_j^2\sin(bu)\right] - b^2x^j \\ &= -(4H^2+b^2)\psi_j(v) + 2HC_j^3 + b^2\cot\theta\, C_j^6 u, \end{aligned}$$

which is equivalent with

$$\psi_j''(v) + a^2\psi_j(v) = 2HC_j^3 + b^2\cot\theta\, C_j^6 u, \tag{4.64}$$

where $a = \sqrt{4H^2+b^2}$. Since u, v are two independent parameters, we have from (4.64) that

$$C_j^6 = 0, \tag{4.65}$$

$$\psi_j''(v) + a^2\psi_j(v) = 2HC_j^3, \tag{4.66}$$

for any $1 \le j \le 5$. From (4.66), we have

$$\psi_j(v) = C_j^4\sin(av) + C_j^5\cos(av) + \frac{2H}{a^2}C_j^3, \tag{4.67}$$

where C_j^4, C_j^5 are constants. Putting (4.65), (4.67) into (4.60), we get

$$x^j = \varphi_j(u) + \psi_j(v), \tag{4.68}$$

for any $1 \le j \le 5$, where

$$\varphi_j(u) = \frac{\cot\theta}{b}\left[C_j^1\cos(bu) - C_j^2\sin(bu)\right]. \tag{4.69}$$

Next, let us characterize C_j^i for $1 \le i, j \le 5$. Denote $C^j = (C_1^j, \ldots, C_5^j)$, and suppose that

$$\begin{aligned} \varphi(u) &= \frac{\cot\theta}{b}\left[\cos(bu)C^1 - \sin(bu)C^2\right], \\ \psi(v) &= \sin(av)C^4 + \cos(av)C^5 + \frac{2H}{a^2}C^3. \end{aligned} \tag{4.70}$$

Then from (4.59), (4.62), (4.63) and (4.68), we have

$$\begin{aligned} &e_1 = x_u = (\varphi'(u), \sin\theta), && e_2 = x_v = (\psi'(v), 0), \\ &e_3 = (-2H\psi(v) + C^3, 0), && e_4 = (-\tan\theta\,\varphi'(u), \cos\theta), \\ &e_5 = (\sqrt{c}(\cos\theta\psi(v) - \cos^{-1}\theta\varphi(u)), 0), && e_6 = \sqrt{c}(\varphi(u) + \psi(v), 0). \end{aligned} \tag{4.71}$$

From $\langle e_1, e_1\rangle = 1$, we get

$$\frac{1}{2}\left(\left|C^1\right|^2 + \left|C^2\right|^2\right) + \frac{1}{2}\left(\left|C^2\right|^2 - \left|C^1\right|^2\right)\cos(2bu) + \left\langle C^1, C^2\right\rangle \sin(2bu) = \sin^2\theta,$$

which implies

$$\left|C^1\right|^2 = \left|C^2\right|^2 = \sin^2\theta, \quad \left\langle C^1, C^2\right\rangle = 0. \tag{4.72}$$

Similarly, by $\langle e_2, e_2\rangle = 1$, we have

$$\frac{1}{2}\left(\left|C^4\right|^2 + \left|C^5\right|^2\right) + \frac{1}{2}\left(\left|C^4\right|^2 - \left|C^5\right|^2\right)\cos(2av) + \left\langle C^4, C^5\right\rangle \sin(2av) = \frac{1}{a^2},$$

which implies

$$\left|C^4\right|^2 = \left|C^5\right|^2 = \frac{1}{a^2}, \quad \left\langle C^4, C^5\right\rangle = 0. \tag{4.73}$$

By $\langle e_1, e_6\rangle = 0$, we obtain

$$\begin{aligned} 0 = &\left[\left\langle C^1, C^4\right\rangle \sin(av) + \left\langle C^1, C^5\right\rangle \cos(av) + \frac{2H}{a^2}\left\langle C^1, C^3\right\rangle\right] \sin(bu) \\ &+ \left[\left\langle C^2, C^4\right\rangle \sin(av) + \left\langle C^2, C^5\right\rangle \cos(av) + \frac{2H}{a^2}\left\langle C^2, C^3\right\rangle\right] \cos(bu). \end{aligned}$$

It follows that

$$\left\langle C^1, C^\alpha\right\rangle = \left\langle C^2, C^\alpha\right\rangle = 0 \tag{4.74}$$

for any $\alpha \in \{3, 4, 5\}$. In the same way, we have from $\langle e_2, e_6\rangle = 0$ and $\langle e_3, e_6\rangle = 0$ that

$$\left\langle C^3, C^4\right\rangle = \left\langle C^3, C^5\right\rangle = 0, \quad \left|C^3\right|^2 = \frac{a^2}{b^2}. \tag{4.75}$$

Using (4.72) to (4.75), we conclude that $\left\{C^i\right\}_{i=1}^5$ is orthogonal with each other and $\left|C^1\right|^2 = \left|C^2\right|^2 = \sin^2\theta$, $\left|C^3\right|^2 = a^2/b^2$, $\left|C^4\right|^2 = \left|C^5\right|^2 = 1/a^2$. Through an orthogonal transformation, the surface is given by (4.24) and it is unique for a given θ, up to an isometry of $M^4(c) \times \mathbb{R}$. □

Remark 4.6. For any fixed $\theta \in (0, \pi/2)$ and given real number H, we construct a class of surfaces with constant mean curvature H, as described in Lemma 4.2. Therefore, we get a kind of surface which is not *Abresch-Rosenberg type surfaces.*

Proof of Main Theorem. Following Theorem 4.1, we obtain case (1) and case (2). In the latter case, $\theta = 0$ implies that Σ^2 lies in $M^n(c)$, and $\theta = \pi/2$ means that $\Sigma^2 = \gamma \times \mathbb{R}$ where γ is a curve of $M^n(c)$. When $\theta \in (0, \pi/2)$, we have from Theorem 4.5 that Σ^2 lies in $M^4(c) \times \mathbb{R}$ with $c > 0$ and is parameterized by (4.24), up to an isometry of $M^4(c) \times \mathbb{R}$. This completes the proof of Main Theorem. □

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References

[1] U. Abresch and H. Rosenberg, *A Hopf differential for constant mean curvature surfaces in* $\mathbb{S}^2 \times \mathbb{R}$ *and* $\mathbb{H}^2 \times \mathbb{R}$, Acta Math. **193** (2004), no. 2, 141–174. http://dx.doi.org/10.1007/bf02392562

[2] ———, *Generalized Hopf differentials*, Mat. Contemp. **28** (2005), 1–28.

[3] H. Alencar, M. do Carmo, I. Fernández and R. Tribuzy, *A theorem of H. Hopf and the Cauchy-Riemann inequality II*, Bull. Braz. Math. Soc. (N.S.) **38** (2007), no. 4, 525–532. http://dx.doi.org/10.1007/s00574-007-0058-x

[4] H. Alencar, M. do Carmo and R. Tribuzy, *A theorem of Hopf and the Cauchy-Riemann inequality*, Comm. Anal. Geom. **15** (2007), no. 2, 283–298. http://dx.doi.org/10.4310/cag.2007.v15.n2.a3

[5] ———, *A Hopf theorem for ambient spaces of dimensions higher than three*, J. Differential Geom. **84** (2010), no. 1, 1–17.

[6] ———, *Surfaces of* $M_k^2 \times \mathbb{R}$ *invariant under a one-parameter group of isometries*, Ann. Mat. Pura Appl. (4) **193** (2014), no. 2, 517–527. http://dx.doi.org/10.1007/s10231-012-0288-4

[7] C. P. Aquino, H. F. de Lima and E. A. Lima, Jr., *On the angle of complete CMC hypersurfaces in Riemannian product spaces*, Differential Geom. Appl. **33** (2014), 139–148. http://dx.doi.org/10.1016/j.difgeo.2014.02.006

[8] J. O. Baek, Q.-M. Cheng and Y. J. Suh, *Complete space-like hypersurfaces in locally symmetric Lorentz spaces*, J. Geom. Phys. **49** (2004), no. 2, 231–247. http://dx.doi.org/10.1016/s0393-0440(03)00090-1

[9] M. Batista, *Simons type equation in* $\mathbb{S}^2 \times \mathbb{R}$ *and* $\mathbb{H}^2 \times \mathbb{R}$ *and applications*, Ann. Inst. Fourier (Grenoble) **61** (2011), no. 4, 1299–1322.
http://dx.doi.org/10.5802/aif.2641

[10] M. Batista, M. P. Cavalcante and D. Fetcu, *Constant Mean Curvature Surfaces in* $\mathbb{M}^2(c) \times \mathbb{R}$ *and Finite Total Curvature*, arXiv:1402.1231.

[11] M. do Carmo, *Some recent developments on Hopf's holomorphic form*, Results Math. **60** (2011), no. 1-4, 175–183. http://dx.doi.org/10.1007/s00025-011-0150-9

[12] M. do Carmo and I. Fernández, *A Hopf theorem for open surfaces in product spaces*, Forum Math. **21** (2009), no. 6, 951–963.
http://dx.doi.org/10.1515/forum.2009.047

[13] B.-Y. Chen, *Riemannian submanifolds: A Survey*, in: *Handbook of Differential Geometry, Vol. I*, 187–418, North-Holland, Amsterdam, 2000.
http://dx.doi.org/10.1016/s1874-5741(00)80006-0

[14] H. Chen, G. Y. Chen and H. Z. Li, *Some pinching theorems for minimal submanifolds in* $\mathbb{S}^m(1) \times \mathbb{R}$, Sci. China Math. **56** (2013), no. 8, 1679–1688.
http://dx.doi.org/10.1007/s11425-012-4556-y

[15] S. Y. Cheng and S. T. Yau, *Hypersurfaces with constant scalar curvature*, Math. Ann. **225** (1977), no. 3, 195–204. http://dx.doi.org/10.1007/bf01425237

[16] S. S. Chern, *On surfaces of constant mean curvature in a three-dimensional space of constant curvature*, in: *Geometric Dynamics* (Rio de Janeiro, 1981), 104–108, Lecture Notes in Math. **1007**, Springer, Berlin, 1983.
http://dx.doi.org/10.1007/bfb0061413

[17] S. S. Chern, M. do Carmo and S. Kobayashi, *Minimal submanifolds of a sphere with second fundamental form of constant length*, in: *Functional Analysis and Related Fields*, 59–75, Springer, New York, 1970.
http://dx.doi.org/10.1007/978-3-642-49908-1_2

[18] B. Daniel, *Isometric immersions into* $\mathbb{S}^n \times \mathbb{R}$ *and* $\mathbb{H}^n \times \mathbb{R}$ *and applications to minimal surfaces*, Trans. Amer. Math. Soc. **361** (2009), no. 12, 6255–6282.
http://dx.doi.org/10.1090/s0002-9947-09-04555-3

[19] F. Dillen, J. Fastenakels and J. Van der Veken, *Surfaces in* $\mathbb{S}^2 \times \mathbb{R}$ *with a canonical principal direction*, Ann. Global Anal. Geom. **35** (2009), no. 4, 381–396.
http://dx.doi.org/10.1007/s10455-008-9140-x

[20] F. Dillen, J. Fastenakels, J. Van der Veken and L. Vrancken, *Constant angle surfaces in* $\mathbb{S}^2 \times \mathbb{R}$, Monatsh. Math. **152** (2007), no. 2, 89–96. http://dx.doi.org/10.1007/s00605-007-0461-9

[21] F. Dillen and M. I. Munteanu, *Constant angle surfaces in* $\mathbb{H}^2 \times \mathbb{R}$, Bull. Braz. Math. Soc. (N.S.) **40** (2009), no. 1, 85–97. http://dx.doi.org/10.1007/s00574-009-0004-1

[22] F. Dillen, M. I. Munteanu and A. I. Nistor, *Canonical coordinates and principal directions for surfaces in* $\mathbb{H}^2 \times \mathbb{R}$, Taiwanese J. Math. **15** (2011), no. 5, 2265–2289.

[23] J. M. Espinar, J. A. Gálvez and H. Rosenberg, *Complete surfaces with positive extrinsic curvature in product spaces*, Comment. Math. Helv. **84** (2009), no. 2, 351–386. http://dx.doi.org/10.4171/cmh/165

[24] J. M. Espinar and H. Rosenberg, *Complete constant mean curvature surfaces and Bernstein type theorems in* $M^2 \times \mathbb{R}$, J. Differential Geom. **82** (2009), no. 3, 611–628.

[25] ———, *Complete constant mean curvature surfaces in homogeneous spaces*, Comment. Math. Helv. **86** (2011), no. 3, 659–674. http://dx.doi.org/10.4171/cmh/237

[26] M. J. Ferreira and R. Tribuzy, *Parallel mean curvature surfaces in symmetric spaces*, Ark. Mat. **52** (2014), no. 1, 93–98. http://dx.doi.org/10.1007/s11512-012-0170-z

[27] D. Fetcu, C. Oniciuc and H. Rosenberg, *Biharmonic submanifolds with parallel mean curvature in* $\mathbb{S}^n \times \mathbb{R}$, J. Geom. Anal. **23** (2013), no. 4, 2158–2176. http://dx.doi.org/10.1007/s12220-012-9323-3

[28] D. Fetcu and H. Rosenberg, *A note on surfaces with parallel mean curvature*, C. R. Math. Acad. Sci. Paris **349** (2011), no. 21-22, 1195–1197. http://dx.doi.org/10.1016/j.crma.2011.10.012

[29] ———, *Surfaces with parallel mean curvature in* $\mathbb{S}^3 \times \mathbb{R}$ *and* $\mathbb{H}^3 \times \mathbb{R}$, Michigan Math. J. **61** (2012), no. 4, 715–729. http://dx.doi.org/10.1307/mmj/1353098510

[30] ———, *On complete submanifolds with parallel mean curvature in product spaces*, Rev. Mat. Iberoam. **29** (2013), no. 4, 1283–1306. http://dx.doi.org/10.4171/rmi/757

[31] Y. Fu and A. I. Nistor, *Constant angle property and canonical principal directions for surfaces in* $\mathbb{M}^2(c) \times \mathbb{R}_1$, Mediterr. J. Math. **10** (2013), no. 2, 1035–1049. http://dx.doi.org/10.1007/s00009-012-0219-z

[32] H. Hopf, *Differential Geometry in the Large*, Lectures Notes in Mathematics **1000**, Springer-Verlag, Berlin, 1983.
http://dx.doi.org/10.1007/978-3-662-21563-0

[33] Z. H. Hou and W.-H. Qiu, *Submanifolds with parallel mean curvature vector field in product spaces*, Vietnam J. Math. **43** (2015), no. 4, 705–723.
http://dx.doi.org/10.1007/s10013-015-0130-6

[34] A. Huber, *On subharmonic functions and differential geometry in the large*, Comment. Math. Helv. **32** (1957), no. 1, 13–72. http://dx.doi.org/10.1007/bf02564570

[35] H. F. de Lima and E. A. Lima, Jr., *Generalized maximum principles and the unicity of complete spacelike hypersurfaces immersed in a Lorentzian product space*, Beitr. Algebra Geom. **55** (2014), no. 1, 59–75.
http://dx.doi.org/10.1007/s13366-013-0137-7

[36] K. Nomizu and B. Smyth, *A formula of Simons' type and hypersurfaces with constant mean curvature*, J. Differential Geom. **3** (1969), 367–377.

[37] J. Simons, *Minimal varieties in riemannian manifolds*, Ann. of Math. (2) **88** (1968), no. 1, 62–105. http://dx.doi.org/10.2307/1970556

[38] B. Smyth, *Submanifolds of constant mean curvature*, Math. Ann. **205** (1973), no. 4, 265–280. http://dx.doi.org/10.1007/bf01362697

[39] S. T. Yau, *Harmonic functions on complete Riemannian manifolds*, Comm. Pure Appl. Math. **28** (1975), no. 2, 201–228.
http://dx.doi.org/10.1002/cpa.3160280203

Zhong Hua Hou and Wang Hua Qiu
Institute of Mathematics, Dalian University of Technology, Dalian 116024, Liaoning, P. R. China
E-mail address: zhhou@dlut.edu.cn, qiuwh_15@163.com