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ON FUNCTIONS STARLIKE WITH RESPECT TO SYMMETRIC AND CONJUGATE POINTS

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Abstract. A class $S_s^*(\alpha,\beta)$ of functions f, regular and univalent in $D = \{z : |z| < 1\}$ given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and satisfying the condition

$$\left|\frac{zf'(z)}{f(z)-f(-z)} - 1\right| < \beta \left|\frac{\alpha z f'(z)}{f(z)-f(-z)} + 1\right|,$$

$z \in D, 0 \le \alpha \le 1, 0 < \beta \le 1$ is introduced and studied. An analogous class $S_c^*(\alpha,\beta)$ is also examined.

1. Introduction

Let S be the class of functions f, regular ard univalent in $D = \{z : |z| < 1\}$ given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

Let S^* be the subclass of S consisting of functions starlike in D. It is well known [4] that $f \in S^*$ if and only if Re $\{zf'(z)/f(z)\} > 0$ for $z \in D$.

Let S_s^* be the subclass of S consisting of functions given by (1.1) satisfying Re $\{(zf'(z)/(f(z) - f(-z))\} > 0$ for $z \in D$. These functions are called starlike with respect to symmetric points and were introduced by Sakaguchi[5]. Recently ELAshwa and Thomas [2] have obtained various results concerning functions in S_s^* and two other classes namely the class S_c^* of functions starlike

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with respect to conjugate points and the class S^*_{sc} of functions starlike with respect to symmetric conjugate points.

In this paper,we introduce the class $S^*_s(\alpha,\beta)$ of functions f, regular and univalent in D given by (1.1) and satisfying the condition

$$\left|\frac{zf'(z)}{f(z)-f(-z)}-1\right|<\beta\left|\frac{\alpha zf'(z)}{f(z)-f(-z)}+1\right|$$

$z\in D, 0\le\alpha\le 1, 0<\beta\le 1$.

S^*_s(1,1) is precisely the class S^*_s. In this paper we obtain coefficient estimates for functions in the class $S^*_s(\alpha,\beta)$. We also obtain a sufficient condition for a function to belong to the class $S^*_s(\alpha,\beta)$.

We also consider the class $S^*_c(\alpha,\beta)$ of functions f,regular in D with $f(0)=0$ and $f'(0)=1$ and satisfying

$$\left|\frac{zf'(z)}{f(z)+\overline{f(\overline{z})}}-1\right|<\beta\left|\frac{\alpha zf'(z)}{f(z)+\overline{f(\overline{z})}}+1\right|$$

with $0\le\alpha\le 1, 0<\beta\le 1$ and $z\in D$.

The class S^*_c (1,1) is precisely the class S^*_c. We analogously obtain coefficient estimates for functions in the class $S^*_c(\alpha,\beta)$.

2. Coefficient Estimates

We need a lemma of Lakshminarasimhan [3].

Lemma 2.1. *Let $H(z)$ be analytic in D and satisfy the condition*

$$\left|\frac{1-H(z)}{1+\alpha H(z)}\right|<\beta \tag{2.1}$$

$z\in D, 0\le\alpha\le 1, 0<\beta\le 1$ with $H(0)=1$. Then we have

$$H(z)=\frac{1-z\phi(z)}{1+\alpha z\phi(z)} \tag{2.2}$$

where $\phi(z)$ is analytic in D and $|\phi(z)|\le\beta$ for $z\in D$. Conversely any function $H(z)$ given by (2.2) *above is analytic in D and satisfies* (2.1).

We now prove a lemma, which is used to obtain the coefficient estimates for functions in the class $S^*_s(\alpha,\beta)$ and $S^*_c(\alpha,\beta)$.

Lemma 2.2. *Let f and g belong to S and satisfy*

$$\left|\frac{zf'(z)}{g(z)} - 1\right| < \beta\left|\frac{\alpha z f'(z)}{g(z)} + 1\right| \tag{2.3}$$

$0 \le \alpha \le 1, 0 < \beta \le 1$ *and* $z \in D$, *with* f *given by* (1.1), *and* $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$. *Then for* $n \ge 2$

$$|na_n - b_n|^2 \le 2(\alpha\beta^2 + 1) \sum_{k=1}^{n-1} k|a_k|\ |b_k|\ (|a_1| = |b_1| = 1). \tag{2.4}$$

Proof. We use the method of Clunie and-keogh [1] and Thomas [6]. By Lemma 2.1 we have

$$\frac{zf'(z)}{g(z)} = \frac{1 - z\phi(z)}{1 + \alpha z\phi(z)},$$

$\phi(z)$ is analytic in D and $|\phi(z)| \le \beta$ for $z \in D$. Then

$$zf'(z) = g(z)\left[\frac{1 - z\phi(z)}{1 + \alpha z\phi(z)}\right]$$

(or)

$$[\alpha z f'(z) + g(z)]z\phi(z) = g(z) - zf'(z).$$

Now if

$$\psi(z) = z\phi(z) = \sum_{n=1}^{\infty} t_n z^n,$$

then

$$|\psi(z)| \le \beta|z| \text{ for } z \in D.$$

Therefore

$$\begin{aligned}&\left[\alpha z + z + \propto \sum_{n=2}^{\infty} na_n z^n + \sum_{n=2}^{\infty} b_n z^n\right]\ \left[\sum_{n=1}^{\infty} t_n z^n\right] \\ &= \sum_{n=2}^{\infty} b_n z^n - \sum_{n=2}^{\infty} na_n z^n.\end{aligned} \tag{2.5}$$

Equating the coefficient of z^n in (2.5), we have

$$b_n - na_n = (\alpha + 1)t_{n-1} + (\alpha 2a_2 + b_2)t_{n-2} + \ldots + (\alpha(n-1)a_{n-1} + b_{n-1})t_1.$$

Thus the coefficient combination on the right side of (2.5) depends only upon the coefficients combination $(\alpha 2a_2 + b_2), \ldots (\alpha(n-1)a_{n-1} + b_{n-1})$ on the left side.

Hence for $n \geq 2$ we can write

$$\begin{aligned}&\left[(\alpha+1)z + \sum_{k=2}^{n-1}(\alpha k a_k + b_k)z^k\right]\psi(z)\\ &= \sum_{k=2}^{n}(b_k - k a_k)z^k + \sum_{k=n+1}^{\infty} c_k z^k \text{ (say).}\end{aligned} \tag{2.6}$$

Squaring the moduli of both sides of (2.6) and integrating along $|z| = r < 1$ and on using the fact that $|\psi(z)| \leq \beta|z|$, we obtain

$$\begin{aligned}&\sum_{k=2}^{n}|ka_k - b_k|^2 r^{2k} + \sum_{k=n+1}^{\infty}|c_k|^2 r^{2k}\\ &< \beta^2\left[(\alpha+1)^2 r^2 + \sum_{k=2}^{n-1}|\alpha k a_k + b_k|^2 r^{2k}\right].\end{aligned}$$

Letting $r \to 1$ on the left side of this inequality, we obtain

$$\sum_{k=2}^{n}|ka_k - b_k|^2 < \beta^2(1+\alpha)^2 + \beta^2\sum_{k=2}^{n-1}|\alpha k a_k + b_k|^2.$$

This implies that

$$\begin{aligned}|na_n - b_n|^2 &\leq \beta^2(1+\alpha)^2 + \beta^2\sum_{k=2}^{n-1}|aka_k + b_k|^2 - \sum_{k=2}^{n-1}|ka_k - b_k|^2\\ &\leq \beta^2(1+\alpha)^2 + (\alpha^2\beta^2 - 1)\sum_{k=2}^{n-1}k^2|a_k|^2 + (\beta^2 - 1)\sum_{k=2}^{n-1}|b_k|^2\\ &\quad + 2\alpha\beta^2\sum_{k=2}^{n-1}k|a_k b_k| + 2\sum_{k=2}^{n-1}k|a_k|\ \ |b_k|\end{aligned} \tag{2.7}$$

(or)

$$|na_n - b_n|^2 \leq 2\alpha\beta^2\sum_{k=1}^{n-1}k|a_k|\ |b_k| + 2\sum_{k=1}^{n-1}k|a_k|\ |b_k|$$

$(|a_1| = |b_1| = 1)$, since $0 \leq \alpha \leq 1, 0 < \beta \leq 1$.

Theorem 2.1. *Let f and g belong to S and be given as in Lemma 2.2. Then for $n \geq 2$*

$$|na_n - b_n|^2 \leq 2(\alpha\beta^2 + 1)CA(1 - 1/n, f)^{1/2}A(1 - 1/n, g)^{1/2}$$

where $A(r,f)$ denotes the area enclosed by $f(|z|=r)$ and where C is a constant.

Proof. We have by (2.4) of lemma (2.2)

$$|na_n - b_n|^2 \le 2(\alpha\beta^2+1)\sum_{k=1}^{n-1} k|a_k|\,|b_k|\ (|a_1|=|b_1|=1).$$

The Cauchy-Schwarz inequality gives for $0<r<1$

$$\begin{aligned}|na_n-b_n|^2 &\le 2\alpha\beta^2\left(\sum_{k=1}^{n-1}k|a_k|^2\right)^{1/2}\left(\sum_{k=1}^{n-1}k|b_k|^2\right)^{1/2}\\ &\quad +2\left(\sum_{k=1}^{n-1}k|a_k|^2\right)^{1/2}\left(\sum_{k=1}^{n-1}k|b_k|^2\right)^{1/2}\\ &\le \frac{2\alpha\beta^2}{r^{2n}}\left(\sum_{k=1}^{n-1}k|a_k|^2r^{2k}\right)^{1/2}\left(\sum_{k=1}^{n-1}k|b_k|^2r^{2k}\right)^{1/2}\\ &\quad +\frac{2}{r^{2n}}\left(\sum_{k=1}^{n-1}k|a_k|^2r^{2k}\right)^{1/2}\left(\sum_{k=1}^{n-1}k|b_k|^2r^{2k}\right)^{1/2}\\ &\le \frac{2\alpha\beta^2}{\pi r^{2n}}A(r,f)^{1/2}A(r,g)^{1/2}+\frac{2}{\pi r^{2n}}A(r,f)^{1/2}A(r,g)^{1/2},\end{aligned}$$

since $A(r,f)=\pi\sum\limits_{n-1}^{\infty} n|a_n|^2r^{2n}$.

Choosing $r=1-1/n$ for $n\ge 2$, the result follows.

Remark 2.1. When $\alpha=\beta=1$, we obtain Theorem 1 (i) of EL-Ashwah and Thomas [2].

Theorem 2.2. *Let $f\in S_s^*(\alpha,\beta)$ and be given by (1.1). Then*

(i) $m^2|a_{2m}|^2 \le 1/2(\alpha\beta^2+1)\ \left(\sum_{j=1}^{m}(2j-1)|a_{2j-1}|^2\right),\ m\ge 1, |a_1|=1$

(ii) $(m-1)^2|a_{2m-1}|^2 \le 1/2(\alpha\beta^2+1)\ \left(\sum_{j=1}^{m-1}(2j-1)|a_{2j-1}|^2\right),\ m\ge 2.$

Further, if $\alpha\beta<1$,

(iii) $m^2|a_{2m}|^2 \leq \dfrac{\beta^2-1}{4}\left(\sum_{j=1}^{m}|a_{2j-1}|^2\right) + \dfrac{\beta+1}{2}\left(\sum_{j=1}^{m}(2j-1)|a_{2j-1}|^2\right)$

for $m \geq 1, |a_1| = 1$ *and*

(iv) $(m-1)^2|a_{2m-1}|^2 \leq \dfrac{\beta^2-1}{4}\left(\sum_{j=1}^{m-1}|a_{2j-1}|^2\right)$

$$+\frac{\beta+1}{2}\left(\sum_{j=1}^{m-1}(2j-1)|a_{2j-1}|^2\right), \ m \geq 2.$$

The inequalities (i) *and* (ii) *are sharp.*

Proof. Since $f \in S_s^*(\alpha,\beta)$, by Lemma 2.1 we have $\frac{zf'(z)}{g(z)} = h(z)$, where g is an odd star like function with $g(z) = \frac{f(z)-f(-z)}{2}$ and $h(z) = \frac{1-z\phi(z)}{1+\alpha z\phi(z)}$, $\phi(z)$ analytic in D and $|\phi(z)| \leq \beta$ for $z \in D$. Thus, with $g(z) = z + \sum_{n=2}^{\infty} a_{2n-1}z^{2n-1}$ for $z \in D$, using (2.4) of Lemma 2.2 with b_n suitably chosen, the inequalities (i) and (ii) in the theorem follow. Indeed, when $\alpha\beta < 1$ using (2.7) of Lemma 2.2

$$|na_n - b_n|^2 \leq (\beta^2-1)\sum_{k=1}^{n-1}|b_k|^2 + 2(\beta+1)\sum_{k=1}^{n-1}k|a_k|\,|b_k|$$

and the inequalities (iii) and (iv) follow.

The inequalities (i) and (ii) are sharp as can be seen from the function $f(z) = 1/2(\alpha\beta^2+1)\frac{z}{1-z}$; we note that when $\alpha = \beta = 1$, inequalities (i) and (ii) give Theorem 2(i) and (ii) of EL-Ashwah and Thomas [2].

Theorem 2.3. *If* $f \in S_s^*(\alpha,\beta)$ *with* $\alpha\beta < 1$, *then* $a_n = 0(1/n)$ *as* $n \to \infty$.

Proof. We observe that when $\alpha\beta < 1$, for $f \in S_s^*(\alpha,\beta)$, $\frac{zf'(z)}{f(z)-f(-z)}$ is bounded. We first prove that

$$\Big(n-(1-(-1)^n)\Big)^2|a_n|^2 \leq 4(\beta+1)\sum_{k=1}^{n-1}k|a_k|^2 \quad (|a_1| = |b_1| = 1).$$

If $f \in S_s^*(\alpha,\beta)$ is given by (1.1), we have using Lemms 2.1

$$\frac{zf'(z)}{f(z)-f(-z)} = \frac{1-z\phi(z)}{1+\alpha z\phi(z)},$$

$\phi(z)$ is analytic in D and $|\phi(z)| \leq \beta$ for $z \in D$. Then

$$[\alpha zf'(z) + f(z) - f(-z)]z\phi(z) = [f(z)-f(-z)] - zf'(z).$$

Now if

$$\psi(z) = z\phi(z) = \sum_{n=0}^{\infty} t_n z^n,$$

then

$$|f(z)| \leq \beta|z| \text{ for } z \in D.$$

Therefore

$$\begin{aligned} &\left[\alpha z + \alpha \sum_{n=2}^{\infty} n a_n z^n + 2z + \sum_{n=2}^{\infty} a_n z^n (1-(-1)^n)\right] \left(\sum_{n=0}^{\infty} t_n z^n\right) \\ &\quad = \left[z + \sum_{n=2}^{\infty} ((1-(-1)^n) - n) a_n z^n\right]. \end{aligned} \tag{2.8}$$

Equating coefficients of z^n in (2.8), we have

$$\begin{aligned} \Big((1-(-1)^n) - n\Big) = {} & (2+\alpha) t_{n-1} + \Big(\alpha 2 a_2 + (1-(-1)^2)\Big) t_{n_2} + \dots \\ & + \Big(\alpha(n-1) a_{n-1} + (1-(-1)^{n-1})\Big) t_1. \end{aligned}$$

Thus the coefficient combination on the right side of (2.8) depends only upon the coefficient combination

$$(\alpha 2 a_2 + (1-(-1)^2), \dots (\alpha(n-1) a_{n-1} + (1-(-1)^{n-1}))$$

on the left side. Hence for $n \geq 2$ we can write

$$\begin{aligned} &\left[(\alpha+2) z + \sum_{k=2}^{n-1} (\alpha k + (1-(-1)^k)) a_k z^k\right] \psi(z) \\ &\quad = \sum_{k=2}^{n} ((1-(-1)^k) - k) a_k z^k + \sum_{k=n+1}^{\infty} c_k z^k \text{ (say).} \end{aligned} \tag{2.9}$$

Squaring the moduli of both sides of (2.9) and integrating along $|z| = r < 1$, we obtain on using the fact that $|\psi(z)| \leq \beta|z|$

$$\begin{aligned} &\sum_{k=2}^{n} (k - (1-(-1)^k))^2 |a_k|^2 r^{2k} + \sum_{k=n+1}^{\infty} |c_k|^2 r^{2k} \\ &\quad < \beta^2 \left[(\alpha+2)^2 r^2 + \sum_{k=2}^{n-1} (\alpha k + (1-(-1)^k))^2 |a_k|^2 r^{2k}\right]. \end{aligned}$$

Letting $r \to 1$ on the left side of the inequality we obtain

$$\sum_{k=2}^{n} (k - (1-(-1)^k))^2 |a_k|^2 < \beta^2 \left[(\alpha+2)^2 + \sum_{k=2}^{n-1} (\alpha k + (1-(-1)^k))^2\right].$$

This implies

$$\begin{aligned}(n-(1-(-1)^n))^2|a_n|^2 &< \beta^2(2+\alpha)^2+\beta^2\sum_{k=2}^{n-1}(\alpha k+(1-(-1)^k))^2|a_k|^2\\ &\quad -\sum_{k=2}^{n-1}(k-(1-(-1)^k))^2|a_k|^2\\ &\le \beta^2(2+\alpha)^2+(\alpha^2\beta^2-1)\sum_{k=2}^{n-1}k^2|a_k|^2\\ &\quad +(\beta^2-1)\sum_{k=2}^{n-1}(1-(-1)^k)^2|a_k|^2\\ &\quad +2\alpha\beta^2\sum_{k=2}^{n-1}k(1-(-1)^k)|a_k|^2\\ &\quad +2\sum_{k=2}^{n-1}k(1-(-1)^k)|a_k|^2\end{aligned} \tag{2.10}$$

(or)

$$\begin{aligned}(n-(1-(-1)^n))^2|a_n|^2 &\le 4\beta\sum_{k=1}^{n-1}k|a_k|^2+4\sum_{k=1}^{n-1}k|a_k|^2\\ &\le 4(\beta+1)\sum_{k=1}^{n-1}k|a_k|^2 \quad (|a_1|=|b_1|=1)\end{aligned} \tag{2.11}$$

since $\alpha\beta < 1$.

It remains to show that $a_n = 0(1/n)$ as $n \to \infty$. From (2.11) we have

$$(n-(1-(-1)^n))^2|a_n|^2 \le 4(\beta+1)\left(1+\sum_{k=2}^{n-1}k|a_k|^2\right). \tag{2.12}$$

Since $\frac{zf'(z)}{f(z)-f(-z)}$ is bounded, it follows that $f(z)$ is bounded. Now following Clunie and Keogh [1] we conclude that Δ, the area of the image of $f(z)$ is given by

$$\Delta = \pi\left(1+\sum_{k=2}^{\infty}k|a_k|^2\right), \tag{2.13}$$

and consequently, $\sum_{k=2}^{\infty}k|a_k|^2 < \infty$ and hence $r_n = \sum_{k=2}^{\infty}k|a_k|^2 \to 0$ as $n \to \infty$. Thus we have

$$\sum_{k=2}^{n-1} k|a_k|^2 = \sum_{k=2}^{n-1}(r_k - r_{k+1}) = r_2 - r_n = 0(1) \text{ as } n \to \infty. \tag{2.14}$$

Using (2.12) and (2.14), we have $a_n = 0(1/n)$ as $n \to \infty$.

3. Sufficient Condition

We obtain a sufficient condition for functions to belong to the class $S_s^*(\alpha, \beta)$.

Theorem 3.1. *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *be analytic in the unit disc* D. *If for* $0 \le \alpha \le 1, 1/2 < \beta \le 1$

$$\sum_{n=2}^{\infty}\left[\frac{(1+\beta\alpha)n}{\beta(2+\alpha)-1} + \frac{\beta(1-(-1)^n)-(1-(-1)^n)}{\beta(2+\alpha)-1}\right]|a_n| \le 1,$$

or equivalently,

$$\sum_{m=1}^{\infty}\left[\frac{(1+\beta\alpha)2m|a_{2m}|}{\beta(2+\alpha)-1} - \frac{(1+\beta\alpha)\ (2m+1)|a_{2m+1}| + 2(\beta-1)|a_{2m+1}|}{\beta(2+\alpha)-1}\right] \le 1, \tag{3.1}$$

then $f(z)$ belongs to the class $S_s^*(\alpha, \beta)$.

Proof. We use the method of Clvnic and Keogh [1]. Suppose that $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, then for $|z| < 1$

$$\begin{aligned}
&|zf'(z) - f(z) - f(-z)| - \beta|\alpha z f'(z) + f(z) - f(-z)| \\
&= \left|z + \sum_{n=2}^{\infty} n a_n z^n - 2z - \sum_{n=2}^{\infty}(1-(-1)^n)a_n z^n\right| \\
&\quad -\beta\left|\alpha z + \alpha\sum_{n=2}^{\infty} n a_n z^n + 2z + \sum_{n=2}^{\infty}(1-(-1)^n)a_n z^n\right| \\
&= \left|-z + \sum_{n=2}^{\infty} n a_n z^n - \sum_{n=2}^{\infty}(1-(-1)^n)a_n z^n\right| \\
&\quad -\beta\left|z(2+\alpha) + \alpha\sum_{n=2}^{\infty} n a_n z^n + \sum_{n=2}^{\infty}(1-(-1)^n)a_n z^n\right|
\end{aligned}$$

$$= \left|-z + \sum_{n=2}^{\infty}(n - (1 - (-1)^n))a_n z^n\right|$$
$$-\beta\left|z(2+\alpha) + \alpha\sum_{n=2}^{\infty}(n\alpha + (1 - (-1)^n))a_n z^n\right|$$
$$\leq \sum_{n=2}^{\infty}(n - (1 - (-1)^n))|a_n|r^n + r$$
$$-\beta\left[(2+\alpha)r - \sum_{n=2}^{\infty}(an + (1 - (-1)^n))|a_n|r^n\right]$$
$$< \left[\sum_{n=2}^{\infty}(n - (1 - (-1)^n)|a_n| + 1 - \beta(2+\alpha) + \sum_{n=2}^{\infty}\beta(\alpha n + (1 - (-1)^n))|a_n|\right] r$$
$$< \sum_{n=2}^{\infty}[(1+\alpha\beta)n + (\beta(1 - (-1)^n)) - (1 - (-1)^n)]\ |a_n| - (\beta(2+\alpha) - 1)]r$$
$$< \left[\sum_{m=1}^{\infty}(1+\beta\alpha)2m|a_{2m}| + \sum_{m=1}^{\infty}\{(1+\beta\alpha)\ (2m+1)|a_{2m+1}|\right.$$
$$\left. +2(\beta-1)|a_{2m+1}|\} - (\beta(2+\alpha) - 1)\right] r$$
$$\leq 0 \text{ by}(3.1).$$

Hence it follows that in $|z| < 1$

$$\left|\left(\frac{zf'(z)}{f(z) - f(-z)} - 1\right) / \left(\frac{\alpha z f'(z)}{f(z) - f(-z)} + 1\right)\right| < \beta$$

so that $f(z) \in S_s^*(\alpha, \beta)$. We note that

$$f(z) = z - \frac{(\beta(2+\alpha) - 1)}{(1+\beta\alpha)n + (\beta(1 - (-1)^n) - (1 - (-1)^n))}\ z^n$$

is an extremal function with respect to the theorem
since

$$\left|\left(\frac{zf'(z)}{f(z) - f(-z)} - 1\right) / \left(\frac{\alpha z f'(z)}{f(z) - f(-z)} + 1\right)\right| = \beta$$

for $z = 1, 0 \leq \alpha \leq 1, 1/2 < \beta \leq 1, n = 2, 3 \ldots$

Remark 3.1. Theorem 3.1 can be used to show that $na_n \to 0$ as slowly as we desire, that is, given any sequence $\epsilon_n \to 0$ there exists a f such that $|na_n| > \epsilon_n$ for infinitely many n. In fact, it is clear that $\sum_{n=1}^{\infty} n|a_n| \leq k$. Given

$\epsilon_n \to 0$ such that $|na_n| > \epsilon_n$ we choose $k \geq \sum_{n=1}^{\infty} n|a_n| > \sum_{n=1}^{\infty} \epsilon_n$. If $\epsilon_n \to 0$ is so chosen that $\sum_{n=1}^{\infty} \epsilon_n = k/2$ and $|a_n| > \frac{2\epsilon_n}{n}$, then $\sum_{n=1}^{\infty} n|a_n| > k$.

Hence there exists a f such that $|na_n| > \epsilon_n$ for infinitely many n. In fact, the function

$$f(z) = 1/2(\alpha\beta^2 + 1)\frac{z}{1-z} \in S_s^*(\alpha, \beta), 0 \leq \alpha \leq 1, 1/2 < \beta \leq 1, \text{ but}$$

$$\sum_{n=2}^{\infty} \left[\frac{(1+\beta\alpha)n}{\beta(2+\alpha)-1} + \frac{\beta(1-(-1)^n) - (1-(-1)^n)}{\beta(2+\alpha)-1}\right] |a_n| > 1.$$

4. Coefficient Estimates For The Class $S_c^*(\alpha, \beta)$

Theorem 4.1. *Let $f \in S_c^*(\alpha, \beta)$ and be given by* (1.1). Then for $n \geq 2$

$$(n+1)^2|a_n|^2 \leq 2(\alpha\beta^2 + 1)\left(\sum_{k=1}^{n} k|a_k|^2\right).$$

Proof. The theorem follows immediately from Lemma 2.2. The inequality in the above Theorem (3.2) is sharp as can be seen from

$$f(z) = 1/2(\alpha\beta^2 + 1)\frac{z}{(1-z)^2}.$$

Corollary 4.1. *Let $f \in S_c^*(\alpha, \beta)$ and suppose $A(r, f) \leq A$, a constant. Then for $n \geq 2$*

$$(n+1)|a_n| \leq \left(2(\alpha\beta^2 + 1)\frac{A}{\pi}\right)^{1/2}.$$

Remark 4.1. When $\alpha = \beta = 1$, we get the corresponding results of EL - Ashwah and Thomas [2].

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