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ENDOMORPHISM RINGS OF MODULES OVER PRIME RINGS

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Abstract. Endomorphism rings of modules appear as the center of a ring, as the fix ring of a ring with group action or as the subring of constants of a derivation. This note discusses the question whether certain $*$-prime modules have a prime endomorphism ring. Several conditions are presented that guarantee the primeness of the endomorphism ring. The contours of a possible example of a $*$-prime module whose endomorphism ring is not prime are traced.

1. INTRODUCTION

Endomorphism rings of modules appear in many ring theoretical situations. For example the center $C(R)$ of a (unital, associative) ring R is isomorphic to the endomorphism ring of R seen as a bimodule over itself, i.e. as a left $R \otimes R^{op}$-module. The subring R^G of elements that are left invariant under the action of a group G on R is isomorphic to the endomorphism ring of R seen as a left module over the skew group ring $R * G$. The subring R^∂ of constants of a derivation ∂ of R is isomorphic to the endomorphism ring of R seen as a left module over its differential operator ring $R[x, \partial]$. More generally the subring R^H of elements invariant under the action of a Hopf algebra H acting on R is isomorphic to the endomorphism ring of R seen as left module over the smash product $R\#H$. This identifications motivated the use of module theory in the study of Hopf algebra actions in [4, 10, 11, 12].

Prime numbers and prime ideals are basic concepts in algebra. While the idea of a prime ideal is well established, the idea of a prime submodule of a module is not. The purely essence of a prime ideal had been destilled already by Birkhoff in the concept

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of a prime element in a partially ordered groupoid. In [3], Bican et al. introduced an operation on the lattice of submodules of a module, turning it into a partially ordered groupoid. Let R be any (associative, unital) ring and M a left R-module. For any submodules N, L we denote

$$N * L = N\mathrm{Hom}(M, L) = \sum\{(N)f \mid f : M \to L\}.$$

Note that we will write homomorphisms opposite of scalars, i.e. on the right side of an element. A submodule P is a prime element in M if for any two submodules N, L of M

$$N * L \subseteq P \Rightarrow N \subseteq P \quad \text{or} \quad L \subseteq P.$$

Those modules whose zero submodule is a prime element had been termed $*$-prime modules, i.e. $N * L \neq 0$ for all non-zero $N, L \subseteq M$. Of course for $M = R$, the $*$-product equals the product of left ideals and R is a $*$-prime left R-module if and only if it is a prime ring. The meaning of the module theoretic prime concept for a ring R with Hopf algebra action H seen as left $R\#H$-module has been studied in [12] in connection with an open question in this area, due to Miriam Cohen, asking whether $R\#H$ is a semiprime algebra provided R is semiprime and H is semisimple (see [5]). The main purpose of this note is to shed new light into the following question which had been left open in [12]:

Question. Is the endomorphism ring of a $*$-prime module a prime ring ?

From [12, Proposition 4.2] it is known that the answer is yes, if the $*$-prime module M satisfies a light projectivity condition. Although we were unable to answer this question completely we will indicate various sufficient conditions for a $*$-prime module to have a prime endomorphism ring which narrows down the class of possible examples that could provide a negative answer.

Let M be a left R-module. $S = \mathrm{End}_R(M)$ shall always denote the endomorphism ring of M. Since any $*$-prime module M has a prime annihilator ideal $\mathrm{Ann}(M)$ and since $\mathrm{Hom}_R(M, N) = \mathrm{Hom}_{R/\mathrm{Ann}(M)}(M, N)$ holds for any submodule N of M, we will assume throughout this note that M is a faithful left module over a (unital, associative) prime ring R.

1.1. Retractable modules

A $*$-prime module M is *retractable*, i.e. $\mathrm{Hom}(M, K) \neq 0$ whenever $0 \neq K \subseteq M$. Note that it is always true that a retractable module with prime endomorphism ring is a $*$-prime module (see [12, Theorem 4.1]) and our question is whether this sufficient condition is also necessary. The retractability condition (called *quotient like* in [9] and *slightly compressible* in [14]) stems from the non-degeneration of the standard Morita context (R, M, M^*, S) between a ring R and the endomorphism ring S of a module M via $M^* = \mathrm{Hom}(M, R)$ (see [17]). In the case of a group G acting on a ring R, the retractability of R as $R*G$-module says that every non-zero G-stable left ideal contains a non-zero fixed element. The Bergman-Isaacs theorem [2] says that R is retractable as left $R*G$-module if G is a finite group acting on a semiprime ring R such that no non-zero element of R has additive $|G|$-torsion. This fact had been used by Fisher and Montgomery in [8] to prove that $R*G$ is semiprime provided R is semiprime and has no $|G|$-torsion, which originally with [6] motivated Cohen's question for Hopf algebra actions.

For a locally nilpotent derivation ∂ of a ring R it had been shown in [4, Lemma 3.8] that R is always retractable as $R[x, \partial]$-module. Rings R that are retractable as $R \otimes R^{op}$-module are those whose non-zero ideals contain non-zero elements like for example in the case of semiprime PI-rings ([13, Theorem 2]), central Azumaya rings ([15, 26.4]) or enveloping algebras of semisimple Lie algebras ([7, 4.2.2]). The retractability condition can be expressed by saying that the function from the lattice of left R-submodules of the module M to the lattice of left ideals of S defined as $N \mapsto \mathrm{Hom}(M, N)$ for submodules N of M has the property that the only submodule mapped to the zero left ideal of S is the zero submodule.

1.2. Endoprime modules

It is known by [12, 1.3] that the endomorphism ring of a right R-module M is prime if and only if $\mathrm{Hom}(M/N, M) = 0$ for all non-zero fully invariant, M-generated submodules N of M. With slightly different notation, Haghany and Vedadi defined a module M to be *endoprime* if $\mathrm{Hom}(M/K, M) = 0$ for all non-zero fully invariant submodules K of M (see [9]). Thus endoprime modules have a prime endomorphism ring. Note that $\mathrm{Hom}(M, K)\mathrm{Hom}(M/K, M) = 0$ holds for all submodules K of M. Hence a retractable module M with prime endomorphism ring S is endoprime. In other words a retractable module has a prime endomorphism ring if and only if it is endoprime. Since $*$-prime modules are retractable, our question can be equivalently reformulated to

Question: Are $*$-prime modules endoprime in the sense of Haghany and Vedadi ?

1.3. Semi-projective modules

As mentioned before under a light projectivity condition our question has an affirmative answer. Recall from [15] that a module M is called *semi-projective* if any diagram

$$\begin{array}{ccccc} & & M & & \\ & & \downarrow g & & \\ M & \xrightarrow{f} & K & \longrightarrow & 0 \end{array}$$

with $K \subseteq M$ can be extended by some endomorphism of M. In other words, M is semi-projective if and only if for any endomorphism f of M we have $\mathrm{Hom}(M, (M)f) = Sf$.

Lemma 1.1. ([12, Proposition 4.2]). *A semi-projective module is $\star$-prime if and only if it is a retractable module with prime endomorphism ring.*

Let R be a ring and $B \subseteq \mathrm{End}_{\mathbb{Z}}(R)$ be a subring of the ring of $\mathbb{Z}$-linear endomorphisms of R such that all left multiplications $L_a : R \to R$ defined by $L_a(x) = ax$ for $a, x \in R$ belong to B. R becomes naturally a left B-module by evaluating of functions. The subring $R^B = \{(1)f \mid f \in B\}$ can be seen to be a generalized subring of invariants of R with respect to B. It is not difficult to see, that R^B is isomorphic to $\mathrm{End}_B(R)$ (see [11, Lemma 1.8]). This general situation mimics the case of R considered as a bimodule or R considered having a Hopf algebra H acting on it. To ask that R is a semi-projective as B-module, is to say that for each $x \in R^B$ one has $R^B x = (Rx) \cap R^B$.

Considering R as a bimodule, we let B to be the subring of $\mathrm{End}_{\mathbb{Z}}(R)$ generated by all left and right multiplications of elements of R. The B-module structure of R is identical with the bimodule structure of R. Then R is semi-projective as $R \otimes R^{op}$-module if for example all non-zero central elements of R are non-zero divisors in R. Because if x is central and ax is central for some $a \in R$, then for any $b \in R$ one has $(ab - ba)x = abx - bax = axb - axb = 0$, i.e. $ab = ba$ and a is central. Thus $Rx \cap C(A) = C(A)x$. In case R is $*$-prime as $R \otimes R^{op}$-module, $0 \neq x \in C(R)$ and $I = \mathrm{Ann}(x) = \{a \in R \mid ax = 0\}$ is its annihilator, the $*$-product of I and Rx is given by:

$$I * (Rx) = I\mathrm{Hom}_{R\otimes R^{op}}(R, Rx) = I((Rx) \cap C(R)) \subseteq Ix = 0.$$

Since we supposed that R is $*$-prime and $x \neq 0$, we get $I = 0$. This shows that no non-zero central element of R is a zero-divisor in R. Consequently we can state the following

Corollary 1.2. *A ring R is a $*$-prime $R \otimes R^{op}$-module if and only if the center of R is an integral domain and large in R.*

Here we say that a subring R' of R is large in R if any non-zero ideal of R contains a non-zero element of R'.

Let G be a group acting on R. It is known that R is a projective $R * G$-module if and only if G is a finite group and $|G|1$ is invertible in R. Thus in this case R is a $*$-prime $R * G$-module if and only if R^G is a prime ring.

If R is an algebra over a field F and ∂ is a locally nilpotent derivation of R and either $\mathrm{char}(F) = 0$ or $\partial^{\mathrm{char}(F)} = 0$, then R is self-projective as left $R[x, \partial]$-module by [4, Proposition 3.10]. Hence in this situation (using also [4, Lemma 3.8]) R is a $*$-prime left $R[x, \partial]$-module if and only if R^∂ is a prime ring.

2. Prime Endomorphism Rings

The purpose of this section is to gather conditions for a $*$-prime module to have a prime endomorphism ring. Denote by $\mathrm{l.ann}_S(I)$ (resp. by $\mathrm{r.ann}_S(I)$) the left (resp. right) annihilator in S of an ideal I.

Theorem 2.1. *The following statements are equivalent for a $*$-prime module M with endomorphism ring S:*

(a) *S is prime.*

(b) *S is semiprime.*

(c) $\mathrm{l.ann}_S(I) \subseteq \mathrm{r.ann}_S(I)$ *holds for any ideal I of S.*

(d) $gSf = 0 \Rightarrow fSg = 0$ *for all $f, g \in S$.*

Proof. $(a) \Rightarrow (b) \Rightarrow (c)$ is trivial since the left and right annihilator of an ideal coincide in a semiprime ring. $(c) \Rightarrow (a)$ Suppose that $IJ = 0$ for two ideals I, J of S. Then $M\mathrm{Hom}(M, MI)J \subseteq MIJ = 0$ implies $\mathrm{Hom}(M, MI)J = 0$. By (c) $J\mathrm{Hom}(M, MI) = 0$. Hence $(MJ) * (MI) = MJ\mathrm{Hom}(M, MI) = 0$ and since M is $*$-prime, we have $MI = 0$ or $MJ = 0$, i.e. $I = 0$ or $J = 0$. Thus S is prime.

Condition (d) is equivalent to saying that

$$\mathrm{l.ann}_S(SfS) = \mathrm{l.ann}_S(Sf) \subseteq \mathrm{r.ann}_S(fS) = \mathrm{r.ann}_S(SfS)$$

for all $f \in S$, which is a consequence of (c). On the other hand, assuming (d) condition (c) follows since for any non-zero ideal I we have $\mathrm{l.ann}_S(I) = \bigcap_{f \in I} \mathrm{l.ann}_S(SfS)$ and the analogous statement for $\mathrm{r.ann}_S(I)$. ■

Note that $(c) \Rightarrow (a)$ needed only the primeness condition for fully invariant submodules. These modules had been investigated by R.Wisbauer and I. Wijayanti and termed *fully prime* modules. We deduce two corollaries from the last theorem:

Corollary 2.2. *Let M be a left R-module with endomorphism ring S. Then S is prime and M is retractable if and only if M is $*$-prime and $gSf = 0$ implies $fSg = 0$ for all $f, g \in S$.*

As a particular case we recover the characterization of R being $*$-prime as bimodule (see 1.2):

Corollary 2.3. *Let M be a left R-module with commutative endomorphism ring S. Then M is $*$-prime if and only if M is retractable and S is an integral domain.*

Since semiprime PI-rings or central Azumaya rings have large center, we see that any such ring is a $*$-prime bimodule if and only if its center is a domain. The next result generalizes the fact that semi-projective $*$-prime modules have prime endomorphism.

Proposition 2.4. *Assume that for any non-zero ideal J of S which is essential as left and right ideal there exists a non-zero submodule N of M such that* $\mathrm{Hom}(M,N) \subseteq J$. *Then S is prime if M is $*$-prime.*

Proof. Let $I^2 = 0$ for an ideal I of S. Then $J = \mathrm{r.ann}_S(I) \cap \mathrm{l.ann}_S(I)$ is a non-zero ideal of S which is essential on both sides. By assumption $\mathrm{Hom}(M,N) \subseteq J$ for some non-zero submodule N of M. Thus $MI * N = MI\mathrm{Hom}(M,N) \subseteq MIJ = 0$ and as M is $*$-prime and N non-zero we have $I = 0$, i.e. S is semiprime and by Theorem 2.1 S is prime. ∎

A left R-module M is called *torsionless* if it is cogenerated by R. A result by Amitsur says that any faithful torsionless module over a prime ring has a prime endomorphism ring (see [1, Corollary 2.8]). The following Proposition gives sufficient conditions for a $*$-prime module M to be torsionless.

Proposition 2.5. *Let M be a faithful left R-module over a prime ring R. In any of the following cases M is torsionless and hence has a prime endomorphism ring.*

(1) *M is a $*$-prime module and is not a singular left R-module.*

(2) *M is a $*$-prime module and R is a left duo ring, i.e. any left ideal is twosided.*

(3) *M is non-singular and is cogenerated by all of its essential submodules.*

Proof. Note that any non-zero submodule N of M that is not singular contains a submodule which is isomorphic to a non-zero left ideal of R. To see this let $0 \neq x \in N$ be an element whose annihilator $A = \mathrm{l.ann}_R(x)$ is not essential in R. Let B be a complement of A, i.e. a left ideal of R which is maximal with respect to $A \cap B = 0$. Then $I = A \oplus B$ is an essential left ideal of R and $Ix \neq 0$ since B is non-zero. As $B \simeq Ix$, we see that B is isomorphic to a submodule of M.

(1) As explained above, if M is not singular, then there exists a non-zero left ideal B of R which is isomorphic to a submodule of M. Since M is cogenerated by any of its non-zero submodules, it is cogenerated by B and hence by R as $B \subseteq R$. Thus M is torsionless.

(2) Since M is a (faithful) prime module, every submodule is also faithful. By hypothesis $I = \mathrm{l.ann}_R(m)$ is two sided for any element m of R and hence $0 = \mathrm{Ann}(Rm) = \mathrm{Ann}(R/I) = I$, i.e. M is not singular and the result follows from (1).

(3) Let M be any non-zero nonsingular module that cogenerated by every essential submodule of itself. By Zorn's Lemma there exists a maximal direct sum $\bigoplus_I C_i$ of cyclic modules $C_i = Rm_i$ non of which is singular. Let $A_i = \mathrm{l.ann}_R(m_i)$ for each $i \in I$. Since A_i is not essential in R, there exists a non-zero complement B_i of A_i in R such that $K_i = A_i \oplus B_i$ is an essential left ideal of R. Let

a be any element in R such that $a \notin A_i$. Then there exists an essential left ideal E of R such that $Ea = Ra \cap K_i$. Because M is nonsingular, we have $0 \neq Eam_i \subseteq K_i m_i \cap Ram_i$. Thus $K_i m_i$ is an essential submodule of C_i and moreover $K_i m_i \simeq B_i$. Hence $N = \bigoplus_{i \in I} K_i m_i$ is essential in M and by hypothesis cogenerates M. Since $N \simeq \bigoplus_{i \in I} B_i \subseteq R^{(I)}$, M is torsionless. ∎

The Wisbauer category of a module M is the full subcategory of R-Mod consisting of submodules of quotients of direct sums of copies of M. For $M = R$, we have $\sigma[R] = R$-Mod. A module $N \in \sigma[M]$ is called M-singular if there are modules $K, L \in \sigma[M]$ with K being an essential submodule of L and $N \simeq L/K$. For $M = R$, R-singular modules are called singular. A module M is called *polyform* or *non-M-singular* if it does not contain any M-singular submodule or equivalently if $\mathrm{Hom}(L/K, M) = 0$ for all essential submodules $K \subseteq L \subseteq M$ (see [15]).

Proposition 2.6. *The endomorphism ring of a $*$-prime polyform module is a prime ring.*

Proof. Recall our general hypothesis that M is a faithful left module over a prime ring R. Let $I^2 = 0$ for some ideal I of S. Then MI is fully invariant. Note that any fully invariant submodule N of M is essential as M is $*$-prime, because for any non-zero submodule L of M we have that $0 \neq N * L = N\mathrm{Hom}(M, L) \subseteq N \cap L$. Thus MI is essential in M. Denote by $\pi : M \to M/MI$ the canonical projection, then $\pi I \subseteq \mathrm{Hom}(M/MI, M) = 0$ as M is polyform. Thus $I = 0$ and S is semiprime. By Theorem 2.1 S is prime. ∎

3. Simple Submodules in Weakly Compressible Modules

The purpose of this section is to see what can be said about the endomorphism ring of a $*$-prime module with non-zero socle. It is also clear that if a $*$-prime module contains a simple submodule S, then any simple submodule of M must be isomorphic to S. Moreover since $\mathrm{Soc}(M)$, the socle of M, is fully invariant, we have for any submodule L of M: $\mathrm{Soc}(M) * L \subseteq \mathrm{Soc}(M) \cap L$. Thus a $*$-prime module M has either zero socle or has an essential and homogeneous semisimple socle, i.e. isomorphic to a direct sum of copies of a simple module.

A submodule N is called semiprime if for any $K \subseteq M : K * K \subseteq N \Rightarrow K \subseteq N$. A module whose zero submodule is semiprime is called *weakly compressible* by Zelmanowitz (see [16]). Obviously $*$-prime modules are weakly compressible.

Lemma 3.1. *Any simple submodule of a weakly compressible module M is a direct summand.*

Proof. Let K be a simple submodule of M, then $0 \neq K * K = K\mathrm{Hom}(M, K)$ implies the existence of $f : M \to K$ such that $f(K)$ is non-zero, i.e. $f(K) = K$ as

K is simple. By Schur's Lemma $\mathrm{End}(K)$ is a division ring and hence there exists an inverse $g \in \mathrm{End}(K)$ of f restricted to K, i.e. $gf = id_K$. Considering g as a map from K to M we showed that f splits, i.e. K is a direct summand of M. ■

Since by the last Lemma, simple modules of a $*$-prime module are direct summands, we have the following

Corollary 3.2. *Any weakly compressible module with DCC or ACC on direct summands and non-zero socle is homogeneous semisimple.*

Recall that a ring R is said to be left quotient finite dimensional (qfd) if every cyclic left R-module has finite Goldie dimension. Any left noetherian or more general any ring with Krull dimension is qfd.

Theorem 3.3. *Let R be a semilocal or a left qfd ring, then any $*$-prime module with non-zero socle has a prime endomorphism ring.*

Proof. If M is a $*$-prime module with a non-zero socle, then $\mathrm{Soc}(M)$ is essential and homogeneous semisimple. Any cyclic C submodule of M is also a $*$-prime module with non-zero essential socle and by assumption has finite Goldie dimension (in case R is qfd) or finite dual Goldie dimension (in case R is semilocal). In either case C has ACC on direct summands and by Corollary 3.2 C is homogeneous simple. Thus $M = \mathrm{Soc}(M) \simeq E^{(\Lambda)}$ is homogeneous semisimple and $\mathrm{End}(M) \simeq \mathrm{End}(E^{(\Lambda)})$ is a prime ring. ■

Recall that a ring R has left Krull dimension 0 if it is left artinian and left Krull dimension 1 if every proper cyclic left R-module $M \neq R$ is artinian.

Proposition 3.4. *Any $*$-prime left module over a ring with left Krull dimension less or equal to 1 has a prime endomorphism ring.*

Proof. Let R be a ring with left Krull dimension ≤ 1 and let M be a $*$-prime left R-module. If M is not singular, then it has a prime endomorphism ring by Proposition 2.5. Suppose that M is singular and let C be a non-zero cyclic submodule of M, then C is also singular and hence proper, i.e. $C \simeq R/I$ with $I \neq 0$. By hypothesis R has Krull dimension ≤ 1 and thus C is artinian. This shows that M has a non-zero socle. By 3.3 M has a prime endomorphism. ■

This implies that for instance any $*$-prime module over the first Weyl algebra A_1 has a prime endomorphism ring.

4. Conclusion

Let $C(R)$ denote the center of R. Faithful *-prime modules M that are not singular have prime endomorphism ring by Proposition 2.5. This applies in particular to the following case:

(1) if M has a non-zero submodule which is finitely generated over $C(R)$ or
(2) if R has a non-zero left ideal which is finitely generated over $C(R)$ or
(3) if R has a non-zero left socle.

In case (1), if $N = C(R)x_1 + \cdots + C(R)x_n$, then

$$0 = \mathrm{Ann}(M) = \mathrm{Ann}(N) = \mathrm{Ann}(x_1) \cap \cdots \cap \mathrm{Ann}(x_n).$$

Thus not all of the elements x_i can be singular and M is not a singular module. Case (2) reduces to the first case, because if I is a non-zero left ideal of R which is finitely generated over $C(R)$, then since M is faithful, there must exist a non-zero element $m \in M$ with $N = Im \neq 0$. But then N is a non-zero submodule of M which is finitely generated over $C(R)$ and (1) applies.

In case (3) we also see that due to $0 = \mathrm{Ann}(M) = \bigcap_{x \in M} \mathrm{Ann}(x)$ not all the annihilators $\mathrm{Ann}(x)$ can be essential left ideals, since otherwise the left socle would be contained in $\mathrm{Ann}(M)$ and would be zero. Hence M is not a singular module.

From the preceding we can conclude that if there exists a $*$-prime faithful left R-module M whose endomorphism ring is not prime, then

- R is not a left duo ring;
- R has zero left socle
- R does not contain any non-zero left ideal which is finitely generated over $C(R)$;
- the Krull dimension of R is greater than 1;
- $\mathrm{End}(M)$ is not commutative;
- M is a singular left R-module which is neither torsionless nor semi-projective;
- M is not polyform, i.e. M is cogenerated by some M-singular submodule;
- no non-zero submodule of M is finitely generated over the center $C(R)$ of R;
- if M has non-zero socle, then R cannot be semilocal nor can R have Krull dimension.

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