

## DIRECTIONAL DERIVATIVES IN SET OPTIMIZATION WITH THE SET LESS ORDER RELATION

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**Abstract.** Based on a special concept of the difference of sets, a new notion of the directional derivative of a set-valued map is presented. This theory is applied to set optimization problems with the known set less order relation, and it results in necessary and sufficient optimality conditions.

### 1. INTRODUCTION

Set optimization has developed as an extension of continuous optimization to problems with set-valued maps. Early investigations have already been carried out in the 1970s. Although most of the papers on set optimization work with the notion of a minimizer and variants of it, nowadays one works with a more realistic order relation for the comparison of sets which has been introduced to optimization by Kuroiwa (e.g., see [14]; a first publication has been given by Kuroiwa, Tanaka and Ha [15]). Outside the optimization community this notion has been used by Young [22] in algebra, by Nishnianidze [18] in fixed point theory and by Chiriaev and Walster [2] in computer science and interval analysis. Since Chiriaev and Walster introduced the name “set less” for the comparison of sets, which is also implemented in the FORTRAN compiler f95 of SUN Microsystems [21], we also use this name in the present paper. In [11] even more realistic order relations have been proposed. Set optimization has important real-world applications in socio-economics which have been discussed by Neukel [17].

Several authors have already investigated directional derivatives of set-valued maps. A first approach to directional derivatives has been given by Kuroiwa [16] in 2009. Using a special embedding technique directional derivatives are introduced for set-valued maps and optimality conditions for the lower set less order relation are presented. In a paper by Hoheisel, Kanzow, Mordukhovich and Phan [7, 8] so-called restrictive graphical derivatives are introduced which are special directional derivatives, where a

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Received June 12, 2014, accepted August 7, 2014.

Communicated by Jen-Chih Yao.

2010 *Mathematics Subject Classification*: 49J53, 06A99, 90C46.

*Key words and phrases*: Directional derivatives, Set optimization, Optimality conditions.

translation (the difference of a set and a point) is considered instead of the difference of sets consisting of more than one point. The approach by Hamel and Schrage [5, 6] is based on a residuation operation and on the solution concept of an infimizer. In a recent paper Pilecka [19] presents directional derivatives using the geometrical difference of sets in combination with the lower set less order relation.

The present paper runs in another quite different direction: directional derivatives of set-valued maps are developed from a computational point of view with respect to the set less order relation which is generally more useful in real-world applications and more difficult to treat. Here we interpret a directional derivative as a limit of difference quotients. In the real- and single-valued case these difference quotients are of the form “ $\frac{0}{0}$ ” and in order to have comparable results in the set-valued case, set differences have to be carefully defined. Baier and Farkhi [1] give a good survey on possible set differences. The approach of the present paper adapts the Demyanov difference [3, 20] to the structure of set optimization problems with the set less order relation. This new set difference is then qualified for the appropriate definition of difference quotients and directional derivatives. This approach can be used numerically and it can be applied in order to formulate optimality conditions in set optimization.

This paper is organized as follows. In Section 2, we present the new set difference and we define Lipschitz continuity with respect to this set difference in Section 3. Directional derivatives are introduced in Section 4. Necessary and sufficient optimality conditions for constrained set optimization problems with the set less order relation are given in Section 5.

## 2. SET DIFFERENCE

In this section we investigate sets in the following setting.

**Assumption 2.1.** Let  $(Y, \|\cdot\|_Y)$  be a real normed space partially ordered by a convex cone  $C \neq Y$ .

Let  $T := C^* \cap \{\ell \in Y^* \mid \|\ell\|_{Y^*} = 1\}$  denote the subset of the unit sphere of the topological dual space  $Y^*$  belonging to the dual cone  $C^*$ . For an arbitrary nonempty set  $A \subset Y$  and every  $\ell \in T$  we consider the minimization problem

$$(1) \quad \min_{y \in A} \ell(y)$$

and the maximization problem

$$(2) \quad \max_{y \in A} \ell(y).$$

At least one solution of these problems exists whenever the constrained set  $A$  is weakly compact. If the set  $A$  is strictly convex, then such a solution is even unique.

**Definition 2.1.** A subset  $A$  with nonempty interior in a real topological linear space is called *strictly convex* iff for arbitrary  $a_1, a_2 \in A$  with  $a_1 \neq a_2$

$$\lambda a_1 + (1 - \lambda)a_2 \in \text{int}(A) \text{ for all } \lambda \in (0, 1)$$

(here  $\text{int}(\cdot)$  denotes the interior of a set).

Strictly convex sets are “round” sets. Polyhedral sets are not strictly convex. With the following proposition we recall a well-known result in convex optimization.

**Proposition 2.1.** *Let Assumption 2.1 be satisfied. For every  $\ell \in T$  and every strictly convex and weakly compact set  $A \subset Y$  the optimization problems (1) and (2) are uniquely solvable.*

*Proof.* For simplicity, we prove this result only for the minimization problem (1) for an arbitrary  $\ell \in T$ . Since this problem has at least one solution, assume that  $a_1$  and  $a_2$  are two different minimal solutions. Because of  $\ell \neq 0_{Y^*}$  there is some  $y \in Y$  with  $\ell(y) = -1$ . For an arbitrary  $\lambda \in (0, 1)$  and a sufficiently small  $\mu > 0$  we conclude

$$\lambda a_1 + (1 - \lambda)a_2 + \mu y \in A$$

and then we get

$$\ell(\lambda a_1 + (1 - \lambda)a_2 + \mu y) = \ell(\lambda a_1 + (1 - \lambda)a_2) + \underbrace{\mu \ell(y)}_{=-1} < \ell(\lambda a_1 + (1 - \lambda)a_2).$$

This is a contradiction to the fact that  $\lambda a_1 + (1 - \lambda)a_2$  is a minimal solution (e.g., compare [9, Thm. 2.14]). ■

For convenience, a solution of the minization problem (1) is denoted by  $y_{\min}(\ell, A)$  and  $y_{\max}(\ell, A)$  denotes a solution of the maximization problem (2).

Now we are able to present the basic definition of the difference of sets.

**Definition 2.2.** Let Assumption 2.1 be satisfied and let two sets  $A, B \in Y$  be given so that for every  $\ell \in T$  the solutions  $y_{\min}(\ell, A)$ ,  $y_{\min}(\ell, B)$ ,  $y_{\max}(\ell, A)$  and  $y_{\max}(\ell, B)$  are unique. Then the set

$$A \ominus B := \bigcup_{\ell \in T} \{y_{\min}(\ell, A) - y_{\min}(\ell, B), y_{\max}(\ell, A) - y_{\max}(\ell, B)\}$$

is called the *set difference* of the sets  $A$  and  $B$ .

The set difference  $A \ominus B$  consists of all differences of supporting points of the sets  $A$  and  $B$  given by supporting hyperplanes defined by continuous linear functionals  $\ell \in T$ . Figure 1 illustrates elements of  $A \ominus B$ .

Definition 2.2 follows the lines of Demyanov's definition of a difference of sets (see [3] and also [20]). In both definitions differences of supporting points are considered. This is an essential tool in set optimization. But there are also two important differences between the two definitions. In Definition 2.2 the continuous linear functionals are restricted to the set  $T$  (which fits to the vectorization approach in set optimization) and one does not consider the closure of the convex hull of these difference vectors.

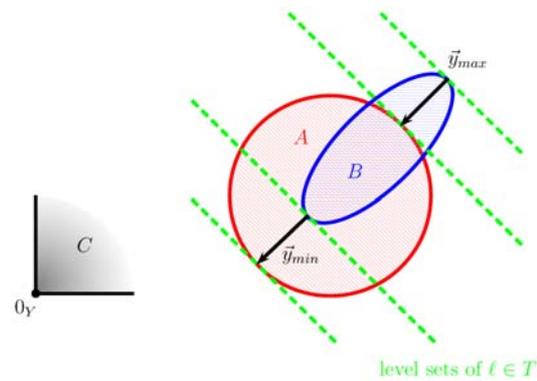


Fig. 1. Illustration of the vectors  $y_{\min}(\ell, A) - y_{\min}(\ell, B) =: \vec{y}_{\min}$  and  $y_{\max}(\ell, A) - y_{\max}(\ell, B) =: \vec{y}_{\max}$  in Definition 2.2.

**Example 2.1.** We consider the simple case  $Y := \mathbb{R}$  and  $C := \mathbb{R}_+$ . Then we have  $C^* = \mathbb{R}_+$  and  $T = \{1\}$ . For arbitrary closed intervals  $A := [a_1, a_2]$  and  $B := [b_1, b_2]$  we obtain for  $\ell = 1$  the optimal solutions  $y_{\min}(\ell, A) = a_1$ ,  $y_{\min}(\ell, B) = b_1$ ,  $y_{\max}(\ell, A) = a_2$  and  $y_{\max}(\ell, B) = b_2$ . Then it follows  $A \ominus B = \{a_1 - b_1, a_2 - b_2\}$ . This shows that this set difference has to do with the change of the lower and upper interval bounds (in contrast to the usual difference in interval arithmetic given by  $[a_1, a_2] - [b_1, b_2] = [a_1 - b_2, a_2 - b_1]$ ).

**Remark 2.1.** In Definition 2.2 it is assumed that the optimization problems (1) and (2) are uniquely solvable for two sets  $A$  and  $B$ . Without loss of generality we consider only the minimization problem (1). If for some  $\ell \in T$  the minimization problem  $\min_{y \in A} \ell(y)$  or  $\min_{y \in B} \ell(y)$  is solvable but not uniquely solvable, then we define the sets of minimal solutions  $Y_{\min}(\ell, A)$  and  $Y_{\min}(\ell, B)$ . In this case we assume that  $(Y, \|\cdot\|_Y)$  is a real reflexive Banach space and we consider the sets

$$(3) \quad \begin{aligned} & \{y_{\min}(\ell, A) - y_{\min}(\ell, B) \mid \\ & \|y_{\min}(\ell, A) - y_{\min}(\ell, B)\|_Y = \min_{y \in Y_{\min}(\ell, B)} \|y_{\min}(\ell, A) - y\|_Y \\ & \text{or } \|y_{\min}(\ell, A) - y_{\min}(\ell, B)\|_Y = \min_{y \in Y_{\min}(\ell, A)} \|y_{\min}(\ell, B) - y\|_Y \}. \end{aligned}$$

The afore-mentioned approximation problems are solvable, if the constraint sets  $Y_{min}(\ell, A)$  and  $Y_{min}(\ell, B)$  are closed and convex (e.g., compare [9, Thm. 2.18]). Notice that the sets  $Y_{min}(\ell, A)$  and  $Y_{min}(\ell, B)$  are convex, if the sets  $A$  and  $B$  are convex (e.g., compare [9, Thm. 2.14]).

The set (3) is the so-called metric difference of the sets  $Y_{min}(\ell, A)$  and  $Y_{min}(\ell, B)$  (see [4] and [1]). This metric difference can be used for an extension of Definition 2.2. Then polyhedral sets can also be investigated in reflexive Banach spaces.

Next we discuss some properties of the difference of sets. Recall that the Minkowski sum of two nonempty sets  $A$  and  $B$  is defined by  $A+B := \{a+b \mid a \in A \text{ and } b \in B\}$  and the multiplication of the set  $A$  by some  $\lambda \in \mathbb{R}$  is denoted by  $\lambda A := \{\lambda a \mid a \in A\}$ .

**Lemma 2.1.** *Let Assumption 2.1 be satisfied. For nonempty sets  $A, B, A_1, A_2, B_1, B_2 \subset Y$  it follows (if the following differences exist):*

- (a)  $A \ominus B = -(B \ominus A)$
- (b)  $A \ominus B \subset (A \ominus C) + (C \ominus B)$  for all nonempty sets  $C \subset Y$
- (c)  $(\alpha A) \ominus (\alpha B) = \alpha(A \ominus B)$  for all  $\alpha \geq 0$
- (d)  $(\alpha A) \ominus (\alpha B) = |\alpha|(B \ominus A)$  for all  $\alpha < 0$
- (e)  $(\alpha A) \ominus (\beta A) \subset (\alpha - \beta)A$  for all  $\alpha \geq \beta \geq 0$
- (f)  $(A_1 + A_2) \ominus (B_1 + B_2) \subset (A_1 \ominus B_1) + (A_2 \ominus B_2)$
- (g)  $(\{c\} + A) \ominus (\{d\} + B) = \{c - d\} + (A \ominus B)$  for all  $c, d \in Y$
- (h)  $A \ominus \{0_Y\} \subset A$
- (i)  $A \ominus A = \{0_Y\}$

*Proof.* We generally assume in this proof that  $\ell \in T$  is arbitrarily chosen.

- (a) With the equality

$$y_{min}(\ell, A) - y_{min}(\ell, B) = -(y_{min}(\ell, B) - y_{min}(\ell, A))$$

and a corresponding equality for maximal solutions we get the assertion.

- (b) The assertion follows from the equality

$$y_{min}(\ell, A) - y_{min}(\ell, B) = (y_{min}(\ell, A) - y_{min}(\ell, C)) + (y_{min}(\ell, C) - y_{min}(\ell, B))$$

and the associated equality for maximal solutions.

- (c) Since for every  $\alpha \geq 0$

$$y_{min}(\ell, \alpha A) - y_{min}(\ell, \alpha B) = \alpha(y_{min}(\ell, A) - y_{min}(\ell, B)),$$

we get the desired result.

- (d) We follow the proof in (c) and apply the result in (a).

(e) Here we note the equality

$$y_{\min}(\ell, \alpha A) - y_{\min}(\ell, \beta A) = (\alpha - \beta)y_{\min}(\ell, A).$$

(f) Notice that

$$\begin{aligned} & y_{\min}(\ell, A_1 + A_2) - y_{\min}(\ell, B_1 + B_2) \\ &= y_{\min}(\ell, A_1) + y_{\min}(\ell, A_2) - y_{\min}(\ell, B_1) - y_{\min}(\ell, B_2) \\ &= (y_{\min}(\ell, A_1) - y_{\min}(\ell, B_1)) + (y_{\min}(\ell, A_2) - y_{\min}(\ell, B_2)). \end{aligned}$$

(g) The assertion follows from the equality

$$y_{\min}(\ell, \{c\} + A) - y_{\min}(\ell, \{d\} + B) = (c - d) + y_{\min}(\ell, A) - y_{\min}(\ell, B).$$

(h) We observe that

$$A \ominus \{0_Y\} = \bigcup_{\ell \in T} \{y_{\min}(\ell, A), y_{\max}(\ell, A)\} \subset A.$$

(i) The equality  $A \ominus A = \{0_Y\}$  follows directly from the definition of the set difference. ■

Baier and Farkhi [1] have formulated axioms which should be fulfilled by set differences in order to be able to prove basic results. Lemma 2.1 shows that most of these axioms are fulfilled for the set difference in Definition 2.2.

**Example 2.2.** Let Assumption 2.1 be satisfied, let  $\hat{S}$  be a nonempty set, and let  $A \subset Y$  be a weakly compact and strictly convex set.

(a) We investigate the set-valued map  $F : \hat{S} \rightrightarrows Y$  with

$$F(x) = \varphi(x) \cdot A \text{ for all } x \in \hat{S}$$

where  $\varphi : \hat{S} \rightarrow \mathbb{R}$  is any real-valued functional. For arbitrary  $x_1, x_2 \in \hat{S}$  we then obtain

$$\begin{aligned} & F(x_1) \ominus F(x_2) \\ &= (\varphi(x_1)A) \ominus (\varphi(x_2)A) \\ &= \bigcup_{\ell \in T} \{y_{\min}(\ell, \varphi(x_1)A) - y_{\min}(\ell, \varphi(x_2)A), y_{\max}(\ell, \varphi(x_1)A) - y_{\max}(\ell, \varphi(x_2)A)\} \\ &= \bigcup_{\ell \in T} \{(\varphi(x_1) - \varphi(x_2))y_{\min}(\ell, A), (\varphi(x_1) - \varphi(x_2))y_{\max}(\ell, A)\} \\ &= (\varphi(x_1) - \varphi(x_2))(A \ominus \{0_Y\}). \end{aligned}$$

(b) Now we investigate the set-valued map  $F : \hat{S} \rightrightarrows Y$  with

$$F(x) = \{f(x)\} + A \text{ for all } x \in \hat{S}$$

where  $f : \hat{S} \rightarrow Y$  is any vector function. For arbitrary  $x_1, x_2 \in \hat{S}$  we conclude with Lemma 2.1, (g) and (i)

$$\begin{aligned} F(x_1) \ominus F(x_2) &= (\{f(x_1)\} + A) \ominus (\{f(x_2)\} + A) \\ &= \{f(x_1) - f(x_2)\} + \underbrace{A \ominus A}_{=\{0_Y\}} \\ &= \{f(x_1) - f(x_2)\}. \end{aligned}$$

### 3. LIPSCHITZ CONTINUITY

In this short section we follow the lines of Baier and Farkhi [1] and define Lipschitz continuity with the set difference  $\ominus$  as in [1, Def. 3.1].

**Definition 3.1.** Let  $(X, \|\cdot\|_X)$  and  $(Y, \|\cdot\|_Y)$  be real normed spaces, let  $\hat{S}$  be a nonempty subset of  $X$  and let  $F : \hat{S} \rightrightarrows Y$  be a set-valued map. The set-valued map  $F$  is called *Lipschitz continuous* with *Lipschitz constant*  $L > 0$  iff

$$\|F(x_1) \ominus F(x_2)\| \leq L\|x_1 - x_2\|_X \text{ for all } x_1, x_2 \in \hat{S}$$

where  $\|A\| := \sup_{y \in A} \|y\|_Y$  for some nonempty set  $A$  in  $Y$ .

Next we investigate the set-valued maps discussed in Example 2.2.

**Example 3.1.** Let  $(X, \|\cdot\|_X)$  and  $(Y, \|\cdot\|_Y)$  be real normed spaces, let  $\hat{S}$  be a nonempty subset of  $X$  and let  $A \subset Y$  be a weakly compact and strictly convex set.

(a) For the set-valued map  $F : \hat{S} \rightrightarrows Y$  with

$$F(x) = \varphi(x) \cdot A \text{ for all } x \in \hat{S}$$

where  $\varphi : \hat{S} \rightarrow \mathbb{R}$  is an arbitrary real-valued Lipschitz continuous functional with Lipschitz constant  $L > 0$ , we have with Example 2.2, (a)

$$F(x_1) \ominus F(x_2) = (\varphi(x_1) - \varphi(x_2))(A \ominus \{0_Y\})$$

and with Lemma 2.1, (b) it follows for every  $y \in (\varphi(x_1) - \varphi(x_2))(A \ominus \{0_Y\})$

$$\begin{aligned} \|y\|_Y &\leq |\varphi(x_1) - \varphi(x_2)| \cdot \underbrace{\|A \ominus \{0_Y\}\|}_{\subset A} \\ &\leq L\|x_1 - x_2\|_X \cdot \|A\|. \end{aligned}$$

This implies

$$\|F(x_1) \ominus F(x_2)\| = \sup_{y \in F(x_1) \ominus F(x_2)} \|y\|_Y \leq L \cdot \|A\| \cdot \|x_1 - x_2\|_X \text{ for all } x_1, x_2 \in \hat{S}.$$

Consequently, the set-valued map  $F$  is Lipschitz continuous (a related result can be found in [1, Lemma 3.5]).

(b) For the set-valued map  $F : \hat{S} \rightrightarrows Y$  with

$$F(x) = \{f(x)\} + A \text{ for all } x \in \hat{S}$$

where  $f : \hat{S} \rightarrow Y$  is an arbitrary vector function being Lipschitz continuous with Lipschitz constant  $L > 0$ , we get with Example 2.2, (b)

$$\begin{aligned} \|F(x_1) \ominus F(x_2)\| &= \|\{f(x_1) - f(x_2)\}\| \\ &= \|f(x_1) - f(x_2)\|_Y \\ &\leq L\|x_1 - x_2\|_X \text{ for all } x_1, x_2 \in \hat{S}. \end{aligned}$$

**Proposition 3.2.** *Let  $(X, \|\cdot\|_X)$  and  $(Y, \|\cdot\|_Y)$  be real normed spaces, let  $\hat{S}$  be a nonempty subset of  $X$  and let  $F_1, F_2 : \hat{S} \rightrightarrows Y$  be set-valued maps with weakly compact and strictly convex images.*

- (a) *If  $F_1$  is Lipschitz continuous with Lipschitz constant  $L_1$ , then for every  $\alpha \in \mathbb{R}$  the map  $\alpha F_1$  is Lipschitz continuous with Lipschitz constant  $|\alpha|L_1$ .*
- (b) *If  $F_1$  and  $F_2$  are Lipschitz continuous with Lipschitz constants  $L_1$  and  $L_2$ , respectively, then the map  $F_1 + F_2$  is Lipschitz continuous with Lipschitz constant  $L_1 + L_2$ .*

*Proof.* (a) With Lemma 2.1, (c) and (d) we obtain for all  $\alpha \in \mathbb{R}$

$$\|(\alpha F_1(x_1)) \ominus (\alpha F_1(x_2))\| = |\alpha| \|F_1(x_1) \ominus F_1(x_2)\| \leq |\alpha| L_1 \|x_1 - x_2\|_X \text{ for all } x_1, x_2 \in \hat{S}.$$

(b) With Lemma 2.1, (f) we conclude for all  $x_1, x_2 \in \hat{S}$

$$\begin{aligned} &\|(F_1(x_1) + F_2(x_1)) \ominus (F_1(x_2) + F_2(x_2))\| \\ &\leq \|(F_1(x_1) \ominus F_1(x_2)) + (F_2(x_1) \ominus F_2(x_2))\| \\ &= \sup_{\substack{y_1 \in F_1(x_1) \ominus F_1(x_2) \\ y_2 \in F_2(x_1) \ominus F_2(x_2)}} \|y_1 + y_2\|_Y \\ &\leq \sup_{y_1 \in F_1(x_1) \ominus F_1(x_2)} \|y_1\|_Y + \sup_{y_2 \in F_2(x_1) \ominus F_2(x_2)} \|y_2\|_Y \\ &= \|(F_1(x_1) \ominus F_1(x_2))\| + \|(F_2(x_1) \ominus F_2(x_2))\| \\ &\leq L_1 \|x_1 - x_2\|_X + L_2 \|x_1 - x_2\|_X \\ &\leq (L_1 + L_2) \|x_1 - x_2\|_X. \quad \blacksquare \end{aligned}$$

Baier and Farkhi have shown in [1, Prop. 3.7] that the result of Proposition 3.2 already holds for the Demyanov difference of sets which is closely related to the set difference  $\ominus$  in this paper.

#### 4. DIRECTIONAL DERIVATIVE

The aim of this paper is to formulate directional derivatives of set-valued maps which are suitable in set optimization. These derivatives are formulated in the following setting.

**Assumption 4.1.** Let  $(Y, \|\cdot\|_Y)$  be a real normed space partially ordered by a convex cone  $C \neq Y$ , let  $X$  be a real linear space, let  $\hat{S}$  be a subset of  $X$  with nonempty interior  $\text{int}(\hat{S})$ , and let  $F : \hat{S} \rightrightarrows Y$  be a set-valued map.

According to the real- and single-valued case we will introduce the directional derivative of  $F$  at some  $x \in \hat{S}$  in some direction  $d \in X$  as a certain limit of sets  $\frac{1}{\lambda}(F(x + \lambda d) \ominus F(x))$  for appropriate  $\lambda > 0$ . In the real- and single-valued case such a difference quotient is nearly of the form “ $\frac{0}{0}$ ” and, therefore, in the set-valued case it makes certainly sense to use a set difference which becomes small for nearly the same sets. The set difference of this paper has for appropriate sets  $A \subset Y$  the property  $A \ominus A = \{0_Y\}$  (see Lemma 2.1, (i)). Consequently, the set difference  $\ominus$  seems to be qualified for the definition of directional derivatives.

It is obvious that the algebraic difference “ $-_a$ ” (see [1, Def. 2.1,(i)]) with

$$A -_a A := \{y_1 - y_2 \mid y_1, y_2 \in A\}$$

generally is much larger than  $\{0_Y\}$ . And the geometric difference “ $-_g$ ” (see [1, Def. 2.1,(ii)]) with

$$A -_g A := \{y \in Y \mid \{y\} + A \subset A\}$$

does not equal  $\{0_Y\}$ , in general. The so-called  $\ell$ -difference “ $-_\ell$ ” introduced in [19, Def. 5] has even the property  $A -_\ell A = C$  where  $C$  is the ordering cone in  $Y$ .

From a numerical point of view the set difference  $\ominus$  seems to be one possible tool for the introduction of directional derivatives of set-valued maps.

For the following let some  $x \in \text{int}(\hat{S})$  and some  $d \in X$  be arbitrarily given. Then we obtain for the difference quotient

$$\begin{aligned} \frac{1}{\lambda}(F(x + \lambda d) \ominus F(x)) &= \frac{1}{\lambda} \bigcup_{\ell \in T} \{y_{\min}(\ell, F(x + \lambda d)) - y_{\min}(\ell, F(x)), \\ &\quad y_{\max}(\ell, F(x + \lambda d)) - y_{\max}(\ell, F(x))\} \\ &= \bigcup_{\ell \in T} \left\{ \frac{1}{\lambda} (y_{\min}(\ell, F(x + \lambda d)) - y_{\min}(\ell, F(x))), \right. \\ &\quad \left. \frac{1}{\lambda} (y_{\max}(\ell, F(x + \lambda d)) - y_{\max}(\ell, F(x))) \right\} \end{aligned}$$

for sufficiently small  $\lambda > 0$ .

So, we consider difference quotients of the minimal and maximal solution functions. If these vector functions are directionally differentiable, we use the notation

$$(4) \quad D_{\min}F(x, d, \ell) := \lim_{\lambda \rightarrow 0^+} \frac{1}{\lambda} (y_{\min}(\ell, F(x + \lambda d)) - y_{\min}(\ell, F(x))) \text{ for all } \ell \in T$$

and

$$(5) \quad D_{\max}F(x, d, \ell) := \lim_{\lambda \rightarrow 0^+} \frac{1}{\lambda} (y_{\max}(\ell, F(x + \lambda d)) - y_{\max}(\ell, F(x))) \text{ for all } \ell \in T.$$

This approach motivates the following definition.

**Definition 4.1.** Let Assumption 4.1 be satisfied and let some  $x \in \text{int}(\hat{S})$  and some  $d \in X$  be arbitrarily given. Let the directional derivatives  $D_{\min}F(x, d, \ell)$  and  $D_{\max}F(x, d, \ell)$  (defined in (4) and (5)) exist for all  $\ell \in T$ . Then the set

$$DF(x, d) := \bigcup_{\ell \in T} \{D_{\min}F(x, d, \ell), D_{\max}F(x, d, \ell)\}$$

is called the *directional derivative* of  $F$  at  $x$  in the direction  $d$ .

**Example 4.1.** We pick up on Example 3.1 in [12]. Here we have  $Y := \mathbb{R}^2$ ,  $C := \mathbb{R}_+^2$ ,  $X := \hat{S} := \mathbb{R}$  and  $F : \hat{S} \rightrightarrows Y$  with

$$F(x) := \{(y_1, y_2) \in \mathbb{R}^2 \mid (y_1 - 2x^2)^2 + (y_2 - 2x^2)^2 \leq (x^2 + 1)^2\} \text{ for all } x \in \mathbb{R}.$$

Following [12, Ex. 3.1] we have the minimal solutions

$$y_{\min}(\ell, F(x)) = (2x^2, 2x^2) - (x^2 + 1)\ell \text{ for all } \ell \in T \text{ and all } x \in \mathbb{R}$$

and the maximal solutions

$$y_{\max}(\ell, F(x)) = (2x^2, 2x^2) + (x^2 + 1)\ell \text{ for all } \ell \in T \text{ and all } x \in \mathbb{R}.$$

Then we get the difference quotients

$$\begin{aligned} & \frac{1}{\lambda} (y_{\min}(\ell, F(x + \lambda d)) - y_{\min}(\ell, F(x))) \\ &= \frac{1}{\lambda} ((4\lambda dx + 2\lambda^2 d^2, 4\lambda dx + 2\lambda^2 d^2) - (2\lambda dx + \lambda^2 d^2)\ell) \\ &= \frac{1}{\lambda} (2\lambda dx + \lambda^2 d^2)(2 - \ell_1, 2 - \ell_2) \\ &= (2dx + \lambda d^2)(2 - \ell_1, 2 - \ell_2) \\ & \text{for all } x \in \mathbb{R}, d \in \mathbb{R}, \ell \in T \text{ and sufficiently small } \lambda > 0 \end{aligned}$$

and the directional derivative

$$\begin{aligned} D_{\min}F(x, d, \ell) &= \lim_{\lambda \rightarrow 0^+} \frac{1}{\lambda} (y_{\min}(\ell, F(x + \lambda d)) - y_{\min}(\ell, F(x))) \\ &= 2dx(2 - \ell_1, 2 - \ell_2) \\ &\text{for all } x \in \mathbb{R}, d \in \mathbb{R} \text{ and } \ell \in T. \end{aligned}$$

Analogously, we conclude

$$D_{\max}F(x, d, \ell) = 2dx(2 + \ell_1, 2 + \ell_2) \text{ for all } x \in \mathbb{R}, d \in \mathbb{R} \text{ and } \ell \in T.$$

Then we obtain the directional derivative of  $F$  at some  $x \in \mathbb{R}$  in the direction  $d \in \mathbb{R}$

$$\begin{aligned} DF(x, d) &= 2dx \bigcup_{\ell \in T} \{(2 - \ell_1, 2 - \ell_2), (2 + \ell_1, 2 + \ell_2)\} \\ &= 2dx \bigcup_{\ell \in T} \{(2, 2) \pm \ell\}. \end{aligned}$$

Figure 2 illustrates this directional derivative for  $x := \frac{1}{2}$  and  $d := 1$ .

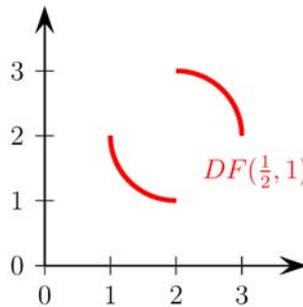


Fig. 2. Illustration of the directional derivative  $DF(\frac{1}{2}, 1)$  in Example 4.1.

**Example 4.2.** (a) We pick up on Example 2.2, (a) and investigate the set-valued map  $F : \hat{S} \rightrightarrows Y$  with

$$F(x) = \varphi(x) \cdot A \text{ for all } x \in \hat{S}.$$

In addition, we now assume that the set  $\hat{S}$  has a nonempty interior  $\text{int}(\hat{S})$  and the functional  $\varphi$  has a directional derivative  $D\varphi(x, d)$  at some  $x \in \text{int}(\hat{S})$  in some direction  $d \in X$ . With Example 2.2,(a) we obtain the difference quotient

$$\frac{1}{\lambda} (F(x + \lambda d) \ominus F(x)) = \frac{1}{\lambda} (\varphi(x + \lambda d) - \varphi(x))(A \ominus \{0_Y\})$$

for sufficiently small  $\lambda > 0$ .

So, we get the directional derivative

$$DF(x, d) = D\varphi(x, d)(A \ominus \{0_Y\}).$$

(b) Now we turn our attention to part (b) of Example 2.2. Under the additional assumption that the vector function  $f$  has a directional derivative  $Df(x, d)$ , we obtain the difference quotient

$$\frac{1}{\lambda}(F(x + \lambda d) \ominus F(x)) = \left\{ \frac{1}{\lambda}(f(x + \lambda d) - f(x)) \right\} \text{ for sufficiently small } \lambda > 0$$

and, therefore, we conclude

$$DF(x, d) = \{Df(x, d)\}.$$

The next proposition shows that directional derivatives of set-valued maps are positive homogeneous with respect to the direction.

**Proposition 4.1.** *Let Assumption 4.1 be satisfied and assume that the set-valued map  $F$  has a directional derivative  $DF(x, d)$  at some  $x \in \text{int}(\hat{S})$  in some direction  $d \in X$ . Then*

$$DF(x, \alpha d) = \alpha DF(x, d) \text{ for all } \alpha \geq 0.$$

*Proof.* The assertion is obvious for  $\alpha = 0$ . Therefore, we assume that some  $\alpha > 0$  is arbitrarily chosen. Then we have

$$\begin{aligned} DF(x, \alpha d) &= \bigcup_{\ell \in T} \{D_{\min} F(x, \alpha d, \ell), D_{\max} F(x, \alpha d, \ell)\} \\ &= \bigcup_{\ell \in T} \left\{ \alpha \lim_{\alpha \lambda \rightarrow 0_+} \frac{1}{\alpha \lambda} (y_{\min}(\ell, F(x + \alpha \lambda d)) - y_{\min}(\ell, F(x))), \right. \\ &\quad \left. \alpha \lim_{\alpha \lambda \rightarrow 0_+} \frac{1}{\alpha \lambda} (y_{\max}(\ell, F(x + \alpha \lambda d)) - y_{\max}(\ell, F(x))) \right\} \\ &= \alpha \bigcup_{\ell \in T} \{D_{\min} F(x, d, \ell), D_{\max} F(x, d, \ell)\} \\ &= \alpha DF(x, d). \quad \blacksquare \end{aligned}$$

In the case of Lipschitz continuity difference quotients of set-valued maps satisfy the following convergence result.

**Proposition 4.2.** *Let Assumption 4.1 be satisfied and let some  $x \in \text{int}(\hat{S})$  and some  $d \in X$  be arbitrarily given. For sufficiently small  $\lambda > 0$  we have*

$$\lim_{d \rightarrow 0_X} \left\| \frac{1}{\lambda}(F(x + \lambda d) \ominus F(x)) \right\| = 0.$$

*Proof.* Since  $F$  is Lipschitz continuous with Lipschitz constant  $L > 0$ , we obtain for sufficiently small  $\lambda > 0$

$$\left\| \frac{1}{\lambda}(F(x + \lambda d) \ominus F(x)) \right\| = \frac{1}{\lambda}L\|\lambda d\|_X = L\|d\|_X$$

which implies the assertion. ■

The directional derivative has a rich mathematical structure so that a numerical approximation of this set can be done on a computer.

**Example 4.3.** Consider the set-valued map  $F : (0, \infty) \rightrightarrows \mathbb{R}^2$  with

$$F(x) := \{(y_1, y_2) \in \mathbb{R}^2 \mid (y_1^2 + y_2^2)^2 - 2x^2(y_1^2 - y_2^2) \leq 0\} \text{ for all } x \in (0, \infty).$$

For every  $x \in (0, \infty)$  the boundary of the set  $F(x)$  is a Lemniscate with parameter  $x$ . Figure 3 illustrates  $F$  at  $x := 1$ . For the following computations let the ordering cone  $C := \mathbb{R}_+^2$  be chosen. An approximation of the directional derivative of  $F$  at  $x = 1$  in the direction  $d := 1$  is illustrated in Figure 4. This approximation is calculated for  $\lambda := 10^{-4}$ . The set  $T$  is discretized by 1,000 equidistant points which implies that 4,000 optimization problems have to be solved (this number can be reduced to 2,000, if one exploits the symmetry of the set). These problems are solved with the optimization tool *fmincon* in MATLAB.

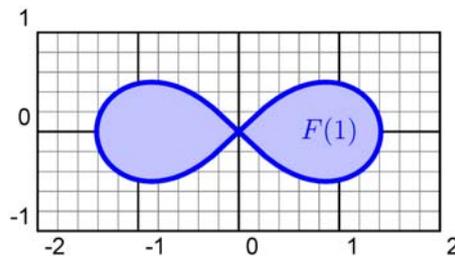


Fig. 3. Illustration of the set  $F(1)$  (with a Lemniscate as boundary) in Example 4.3.

### 5. OPTIMALITY CONDITIONS IN SET OPTIMIZATION

In this section we apply the developed theory to set optimization problems in the following setting.

**Assumption 5.1.** Let  $(Y, \|\cdot\|_Y)$  be a real normed space partially ordered by a closed convex cone  $C \neq Y$ , let  $X$  be a real linear space, let  $S$  be a nonempty subset of  $X$ , let  $\hat{S}$  be an open superset of  $S$ , and let  $F : \hat{S} \rightrightarrows Y$  be a set-valued map. Let the set  $F(x)$  be nonempty and weakly compact for all  $x \in S$ , and let the sets  $F(x) + C$  and  $F(x) - C$  be convex for all  $x \in S$  (here we set  $F(x) \pm C := \{y \pm c \mid y \in F(x) \text{ and } c \in C\}$ ).

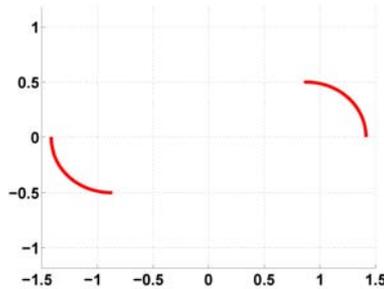


Fig. 4. Illustration of the directional derivative  $DF(1,1)$  of the set-valued map  $F$  in Example 4.3.

Notice that for a closed convex cone  $C$  and a weakly compact set  $F(x)$  (with  $x \in S$ ) the sets  $F(x) + C$  and  $F(x) - C$  are also closed. The set  $F(x)$  has not to be convex. Under Assumption 5.1 we now investigate the set optimization problem

$$(6) \quad \min_{x \in S} F(x).$$

Minimal solutions of this problem are understood using the set less order relation.

**Definition 5.1.** Let Assumption 5.1 be satisfied.

- (a) Let nonempty sets  $A, B \subset Y$  be given. Then the *set less* order relation  $\preceq_s$  is defined by

$$A \preceq_s B \quad :\iff \quad B \subset A + C \text{ and } A \subset B - C.$$

- (b)  $\bar{x} \in S$  is called a *minimal solution* of the set optimization problem (6) iff  $F(\bar{x})$  is a minimal element of the system of sets  $F(x)$  (with arbitrary  $x \in S$ ), i.e.

$$F(x) \preceq_s F(\bar{x}), x \in S \implies F(\bar{x}) \preceq_s F(x).$$

The original definition of the set less order relation is replaced by a characterization with the ordering cone  $C$  in Definition 5.1,(a) (e.g., see [11, Prop. 3.1,(a)]).

For the formulation of a vectorization result presented in [12, Thm. 3.1] we consider continuous linear functionals  $\ell$  in the set  $T := C^* \cap \{\ell \in Y^* \mid \|\ell\|_{Y^*} = 1\}$ . Let  $\mathcal{A}$  be a system of nonempty and weakly compact subsets of  $Y$  and let  $\mathcal{R}^2(T)$  denote the space of functions on  $T$  with values in  $\overline{\mathbb{R}}^2$ . Then we define the map  $v : \mathcal{A} \rightarrow \mathcal{R}^2(T)$  pointwise by

$$v(A)(\ell) := \begin{pmatrix} \min_{a \in A} \ell^T a \\ \max_{a \in A} \ell^T a \end{pmatrix} \text{ for all } A \in \mathcal{A} \text{ and all } \ell \in T.$$

With this map  $v$  we recall a vectorization result in set optimization in a modified form.

**Theorem 5.1.** Let Assumption 5.1 be satisfied.  $\bar{x} \in S$  is a minimal solution of the set optimization problem (6) with respect to the set less order relation  $\preceq_s$  if and only if  $\bar{x} \in S$  is a minimal solution of the vector optimization problem

$$(7) \quad \min_{x \in S} v(F(x))$$

with respect to the componentwise and pointwise order relation.

*Proof.* This result follows from [12, Thm. 3.1] with one modification. By Remark 2.2 in [12] the set  $C^* \setminus \{0_{Y^*}\}$ , which plays a central role in the key lemma [12, Lemma 2.1], can be replaced by the set  $T$  so that we consider functions on  $T$ . ■

Now we are able to formulate a necessary optimality condition for the set optimization problem (6) using the concept of the directional derivative.

**Theorem 5.2.** Let Assumption 5.1 be satisfied. If  $\bar{x} \in S$  is a minimal solution of the set optimization problem (6) and if for all  $x \in S$  the directional derivative  $DF(\bar{x}, x - \bar{x})$  exists and  $\bar{x} + \lambda(x - \bar{x}) \in S$  for sufficiently small  $\lambda > 0$ , then

$$(8) \quad \forall x \in S \quad \exists \ell \in T \quad \max \left\{ \ell(D_{min}F(\bar{x}, x - \bar{x}, \ell)), \ell(D_{max}F(\bar{x}, x - \bar{x}, \ell)) \right\} \geq 0.$$

If, in addition, the interior  $\text{int}(C)$  of the ordering cone is nonempty, then the necessary optimality condition (8) implies

$$(9) \quad DF(\bar{x}, x - \bar{x}) \not\subset -\text{int}(C) \text{ for all } x \in S.$$

*Proof.* (a) We proof the first part of this theorem by contraposition. Assume that the necessary condition (8) is not true, i.e. there is some  $x \in S$  so that for all  $\ell \in T$

$$\max \left\{ \ell(D_{min}F(\bar{x}, x - \bar{x}, \ell)), \ell(D_{max}F(\bar{x}, x - \bar{x}, \ell)) \right\} < 0.$$

Since functionals in  $T$  are continuous and linear, we conclude for all  $\ell \in T$

$$\begin{aligned} 0 &> \ell(D_{min}F(\bar{x}, x - \bar{x}, \ell)) \\ &= \ell \left( \lim_{\lambda \rightarrow 0^+} \frac{1}{\lambda} (y_{min}(\ell, F(\bar{x} + \lambda(x - \bar{x}))) - y_{min}(\ell, F(\bar{x}))) \right) \\ &= \lim_{\lambda \rightarrow 0^+} \frac{1}{\lambda} \left( \underbrace{\ell(y_{min}(\ell, F(\bar{x} + \lambda(x - \bar{x})))}_{=: \varphi_{min}(\ell, F(\bar{x} + \lambda(x - \bar{x})))} - \underbrace{\ell(y_{min}(\ell, F(\bar{x})))}_{=: \varphi_{min}(\ell, F(\bar{x}))} \right) \end{aligned}$$

(the last term denotes the directional derivative of the minimal value functional where  $\varphi_{min}(\ell, F(\bar{x} + \lambda(x - \bar{x})))$  and  $\varphi_{min}(\ell, F(\bar{x}))$  denote the minimal values of the minimization problems  $\min_{y \in F(\bar{x} + \lambda(x - \bar{x}))} \ell(y)$  and  $\min_{y \in F(\bar{x})} \ell(y)$ , respectively). So, there is some  $\bar{\lambda} > 0$  with  $\bar{x} + \bar{\lambda}(x - \bar{x}) \in S$  so that

$$0 > \frac{1}{\bar{\lambda}} (\varphi_{min}(\ell, F(\bar{x} + \bar{\lambda}(x - \bar{x}))) - \varphi_{min}(\ell, F(\bar{x})))$$

implying

$$(10) \quad \varphi_{min}(\ell, F(\bar{x} + \bar{\lambda}(x - \bar{x}))) < \varphi_{min}(\ell, F(\bar{x})).$$

In analogy we get for the maximal value functionals of the maximization problems

$$\max_{y \in F(\bar{x} + \bar{\lambda}(x - \bar{x}))} \ell(y) \text{ and } \max_{y \in F(\bar{x})} \ell(y)$$

$$(11) \quad \varphi_{max}(\ell, F(\bar{x} + \bar{\lambda}(x - \bar{x}))) < \varphi_{max}(\ell, F(\bar{x})).$$

The inequalities (10) and (11) imply that  $\bar{x}$  is not a minimal solution of the vector optimization problem (7) (with respect to the componentwise and pointwise order relation). Consequently, by Theorem 5.1  $\bar{x}$  is not a minimal solution of the set optimization problem (6) with respect to the set less order relation  $\preceq_s$ .

(b) We prove the implication “(8) $\Rightarrow$ (9)” by contraposition. Assume that the condition (9) is not satisfied, i.e. there is some  $x \in S$  with  $DF(\bar{x}, x - \bar{x}) \subset -\text{int}(C)$ . Then we get for all  $\ell \in T$

$$D_{min}F(\bar{x}, x - \bar{x}, \ell) \in -\text{int}(C)$$

and

$$D_{max}F(\bar{x}, x - \bar{x}, \ell) \in -\text{int}(C).$$

By a well-known characterization of the interior of the ordering cone (e.g. see [10, Lemma 3.21,(c)]) we conclude for all  $\ell \in T \subset C^* \setminus \{0_{Y^*}\}$

$$\ell(D_{min}F(\bar{x}, x - \bar{x}, \ell)) < 0$$

and

$$\ell(D_{max}F(\bar{x}, x - \bar{x}, \ell)) < 0$$

resulting in

$$\max \left\{ \ell(D_{min}F(\bar{x}, x - \bar{x}, \ell)), \ell(D_{max}F(\bar{x}, x - \bar{x}, \ell)) \right\} < 0 \text{ for all } \ell \in T.$$

So, the condition (8) is not satisfied. ■

Theorem 5.2 with the necessary optimality condition (9) can be interpreted as follows: If  $\bar{x}$  is a solution of the set optimization problem (6), then there is no feasible point  $x \in S$  so that the directional derivative  $DF(\bar{x}, x - \bar{x})$  is contained in  $-\text{int}(C)$ .

The necessary optimality condition (8) extends a similar condition which is given for unconstrained smooth problems in finite dimensional spaces in [13, Thm. 3.1].

It is well-known from nonlinear optimization that pseudoconvexity of the objective functional ensures that a necessary optimality condition is also sufficient. We go in line with this approach and present a concept of pseudoconvexity for set-valued maps which seems to be more complex than in the real- and single-valued case. A motivation of this notion is given by the following known result (see [12, Cor. 2.2]).

**Theorem 5.3.** Let Assumption 5.1 be satisfied. An element  $\bar{x}$  is a minimal solution of the set optimization problem (6) if and only if

$$\forall x \in S \left( \forall \ell \in T \min \left\{ \ell(y_{\min}(\ell, F(x))) - \ell(y_{\min}(\ell, F(\bar{x}))), \right. \right. \\ \left. \left. \ell(y_{\max}(\ell, F(x))) - \ell(y_{\max}(\ell, F(\bar{x}))) \right\} \geq 0 \right) \\ \text{or} \\ \left( \exists \ell \in T \max \left\{ \ell(y_{\min}(\ell, F(x))) - \ell(y_{\min}(\ell, F(\bar{x}))), \right. \right. \\ \left. \left. \ell(y_{\max}(\ell, F(x))) - \ell(y_{\max}(\ell, F(\bar{x}))) \right\} > 0 \right).$$

*Proof.* By [12, Remark 2.2] the set  $C^* \setminus \{0_{Y^*}\}$  can be replaced by the set  $T$ , and then Corollary 2.2 in [12] states that  $\bar{x}$  is a minimal solution of the set optimization problem (6) if and only if there is no  $x \in S$  so that

$$\nexists \ell \in T : \min_{y \in F(x)} \ell(y) - \min_{y \in F(\bar{x})} \ell(y) < 0 \text{ or } \max_{y \in F(x)} \ell(y) - \max_{y \in F(\bar{x})} \ell(y) < 0$$

and

$$\forall \ell \in T : \min_{y \in F(x)} \ell(y) - \min_{y \in F(\bar{x})} \ell(y) \leq 0 \text{ and } \max_{y \in F(x)} \ell(y) - \max_{y \in F(\bar{x})} \ell(y) \leq 0.$$

Hence, the assertion is evident. ■

**Definition 5.2.** Let Assumption 5.1 be satisfied and let the directional derivative  $DF(\bar{x}, x - \bar{x})$  exist at some  $\bar{x} \in S$  in every direction  $x - \bar{x}$  with  $x \in S$ . The set-valued map  $F$  is called *pseudoconvex* at  $\bar{x} \in S$  iff

$$\forall x \in S \exists \ell \in T \max \left\{ \ell(D_{\min}F(\bar{x}, x - \bar{x}, \ell)), \ell(D_{\max}F(\bar{x}, x - \bar{x}, \ell)) \right\} \geq 0 \\ \implies \forall x \in S \left( \forall \ell \in T \min \left\{ \ell(y_{\min}(\ell, F(x))) - \ell(y_{\min}(\ell, F(\bar{x}))), \right. \right. \\ \left. \left. \ell(y_{\max}(\ell, F(x))) - \ell(y_{\max}(\ell, F(\bar{x}))) \right\} \geq 0 \right) \\ \text{or} \\ \left( \exists \ell \in T \max \left\{ \ell(y_{\min}(\ell, F(x))) - \ell(y_{\min}(\ell, F(\bar{x}))), \right. \right. \\ \left. \left. \ell(y_{\max}(\ell, F(x))) - \ell(y_{\max}(\ell, F(\bar{x}))) \right\} > 0 \right).$$

**Theorem 5.4.** Let Assumption 5.1 be satisfied, let the directional derivative  $DF(\bar{x}, x - \bar{x})$  exist at some  $\bar{x} \in S$  in every direction  $x - \bar{x}$  with  $x \in S$ , and let  $F$  be pseudoconvex at  $\bar{x}$ . If the condition (8) is satisfied, then  $\bar{x}$  is a minimal solution of the set optimization problem (6).

*Proof.* Let the condition (8) be satisfied at some  $\bar{x} \in S$ , i.e.

$$\forall x \in S \quad \exists \ell \in T \quad \max \left\{ \ell(D_{\min}F(\bar{x}, x - \bar{x}, \ell)), \ell(D_{\max}F(\bar{x}, x - \bar{x}, \ell)) \right\} \geq 0.$$

Since  $F$  is pseudoconvex at  $\bar{x}$ , it follows

$$\forall x \in S \quad \left( \forall \ell \in T \quad \min \left\{ \ell(y_{\min}(\ell, F(x))) - \ell(y_{\min}(\ell, F(\bar{x}))), \right. \right. \\ \left. \left. \ell(y_{\max}(\ell, F(x))) - \ell(y_{\max}(\ell, F(\bar{x}))) \right\} \geq 0 \right)$$

or

$$\left( \exists \ell \in T \quad \max \left\{ \ell(y_{\min}(\ell, F(x))) - \ell(y_{\min}(\ell, F(\bar{x}))), \right. \right. \\ \left. \left. \ell(y_{\max}(\ell, F(x))) - \ell(y_{\max}(\ell, F(\bar{x}))) \right\} > 0 \right).$$

By Theorem 5.3 we then conclude that  $\bar{x}$  is a minimal solution of the set optimization problem (6). ■

**Example 5.1.** Again we consider Example 4.1 with the additional constrained set  $S := [\frac{1}{2}, 1]$  (see also [13, Ex. 4.1]). It is obvious from the geometrical construction of the sets  $F(x)$  with  $x \in [\frac{1}{2}, 1]$  that  $\bar{x} := \frac{1}{2}$  is a minimal solution of this set optimization problem (compare Figure 5).

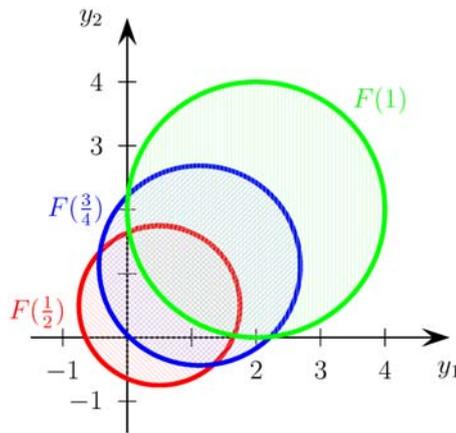


Fig. 5. Illustration of the sets  $F(\frac{1}{2})$ ,  $F(\frac{3}{4})$  and  $F(1)$  in Example 5.1.

In Example 4.1 the directional derivatives

$$D_{\min}F(\bar{x}, x - \bar{x}, \ell) = 2(x - \bar{x})\bar{x}(2 - \ell_1, 2 - \ell_2) \text{ for all } x \in S \text{ and all } \ell \in T$$

and

$$D_{max}F(\bar{x}, x - \bar{x}, \ell) = 2(x - \bar{x})\bar{x}(2 + \ell_1, 2 + \ell_2) \text{ for all } x \in S \text{ and all } \ell \in T$$

have already been determined. For every  $x \in S$  and every  $\ell \in T$  we then conclude

$$\ell^T D_{min}F(\bar{x}, x - \bar{x}, \ell) = 2(x - \bar{x})\bar{x}(2(\ell_1 + \ell_2) - 1)$$

and

$$\ell^T D_{max}F(\bar{x}, x - \bar{x}, \ell) = 2(x - \bar{x})\bar{x}(2(\ell_1 + \ell_2) + 1),$$

if we work with the Euclidean norm in  $\mathbb{R}^2$ . Consequently, we get for every  $x \in S$  and every  $\ell \in T$

$$\begin{aligned} & \max \left\{ \ell^T D_{min}F(\bar{x}, x - \bar{x}, \ell), \ell^T D_{max}F(\bar{x}, x - \bar{x}, \ell) \right\} \\ &= 2 \underbrace{(x - \bar{x})}_{\geq 0} \underbrace{\bar{x}}_{=\frac{1}{2}} \underbrace{(2(\ell_1 + \ell_2))}_{\geq 1} + 1 \\ &\geq 0. \end{aligned}$$

Hence, the necessary optimality condition (8) is fulfilled at  $\bar{x} = \frac{1}{2}$ .

Next, we show that the set-valued map  $F$  is pseudoconvex at  $\bar{x}$ . It is remarked in Example 4.1 that for an arbitrary  $x \in S$  and an arbitrary  $\ell \in T$

$$y_{min}(\ell, F(x)) = (2x^2, 2x^2) - (x^2 + 1)\ell$$

and

$$y_{max}(\ell, F(x)) = (2x^2, 2x^2) + (x^2 + 1)\ell$$

are minimal solutions of the subproblems  $\min_{y \in F(x)} \ell^T y$  and  $\max_{y \in F(x)} \ell^T y$ , respectively. For every  $x \in S$  and every  $\ell \in T$  we then get

$$\ell^T y_{min}(\ell, F(x)) = x^2(2(\ell_1 + \ell_2) - 1) - 1$$

and

$$\ell^T y_{max}(\ell, F(x)) = x^2(2(\ell_1 + \ell_2) + 1) + 1.$$

This implies for every  $x \in S$  and every  $\ell \in T$

$$\begin{aligned} & \min \left\{ \ell(y_{min}(\ell, F(x))) - \ell(y_{min}(\ell, F(\bar{x}))), \ell(y_{max}(\ell, F(x))) - \ell(y_{max}(\ell, F(\bar{x}))) \right\} \\ &= \underbrace{\left(x^2 - \frac{1}{4}\right)}_{\geq 0} \min \left\{ \underbrace{2(\ell_1 + \ell_2)}_{\geq 1} - 1, 2(\ell_1 + \ell_2) + 1 \right\} \\ &\geq 0. \end{aligned}$$

This shows that  $F$  is pseudoconvex at  $\bar{x}$ , and by Theorem 5.4 we conclude that  $\bar{x}$  is a minimal solution of the considered set optimization problem.

## 6. CONCLUSION

This paper develops directional derivatives of set-valued maps from a numerical point of view. Under appropriate assumptions an approximation of such a derivative can even be calculated on a computer. Essentially, a directional derivative of a set-valued map describes the directional derivative of certain supporting points and characterizes the movement of certain border points of the considered set. Since the approach of this paper depends on vectorization, where convexity plays an important role, it would be helpful to investigate other transformations for nonconvex sets.

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