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THE PROX-TIKHONOV-LIKE FORWARD-BACKWARD METHOD AND APPLICATIONS

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Abstract. It is known, by Rockafellar [*SIAM J. Control Optim.*, 14 (1976), 877-898], that the proximal point algorithm (PPA) converges weakly to a zero of a maximal monotone operator in a Hilbert space, but it fails to converge strongly. Lehdili and Moudafi [*Optimization*, 37(1996), 239-252] introduced the new prox-Tikhonov regularization method for PPA to generate a strongly convergent sequence and established a convergence property for it by using the technique of variational distance in the same space setting. In this paper, the prox-Tikhonov regularization method for the proximal point algorithm of finding a zero for an accretive operator in the framework of Banach space is proposed. Conditions which guarantee the strong convergence of this algorithm to a particular element of the solution set is provided. An inexact variant of this method with non-summable error sequence is also discussed.

1. INTRODUCTION

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let C be a nonempty closed convex subset of H. The class of all proper, lower semicontinuous, convex functions from C to $(-\infty, \infty]$ is denoted by $\Gamma_0(C)$. The normal cone for C at a point $u \in C$ is

$$N_C(u) = \{z \in H : \langle u - v, z \rangle \geq 0 \text{ for all } v \in C\}.$$

Let $A : C \to 2^H$ and $B : C \to H$ be monotone operators. The inclusion problem is to find $z \in C$ such that

$$0 \in (A + B)z. \tag{1.1}$$

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Many nonlinear problems arising in applied areas such as image recovery, signal processing, and machine learning can be mathematically modeled in form of inclusion problem (1.1). For instance, a stationary solution to the initial value problem of the evolution equation

$$0 \in \frac{\partial u}{\partial t} + Fu, u_0 = u(0)$$

can be recast as (1.1) when the governing maximal monotone F is of the form $F = A + B$, see, for example, [10].

Consider $\psi \in \Gamma_0(H)$, and set $A = \partial\psi$. Then, the inclusion problem (1.1) is equivalent to the mixed variational inequality problem (in short, MVI) of finding $x^* \in C$ such that

$$\langle Bx^*, v - x^* \rangle + \psi(v) - \psi(x^*) \geq 0, \quad \text{for all } v \in C. \tag{1.2}$$

The central problem is to iteratively find the solution of the inclusion problem (1.1) when A and B are two monotone operators in a Hilbert space H. One method for finding solutions of problem (1.1) is *splitting method*. A splitting method for (1.1) means an iterative method for which each iteration involves only with the individual operators A and B, but not the sum $A+B$. Splitting methods for linear equations were introduced by Peaceman and Rachford [11] and Douglas and Rachford [12]. Extensions to nonlinear equations in Hilbert spaces were carried out by Kellogg [8] and Lions and Mercier [10] (see also [15]).

In this paper, we are interested in the following variational inclusion problem:

$$\text{Find } z \in C \text{ such that } 0 \in Az + Bz, \tag{$\mathscr{P}$}$$

in the framework of a Banach space X, where C is a nonempty closed convex subset of X, $B : C \to X$ is a monotone operator and $A \subseteq X \times X$ an accretive operator such that $\overline{D(A)} \subseteq C \subseteq \bigcap_{t>0} R(I + tA)$. In the sequel, we assume that Zer$(A+B)$, the set of solutions of problem ($\mathscr{P}$) is nonempty. The inclusion problem ($\mathscr{P}$) is more general in nature. For instance, if B is the operator constantly zero, the problem ($\mathscr{P}$) reduces

$$\text{to find } z \in C \text{ such that } 0 \in Az. \tag{1.3}$$

One popular method for solving inclusion problem (1.3) is the proximal point algorithm of Rockafellar [17]. The proximal point like methods for finding solutions of problem (1.3) have been studied by Lehdili and Moudafi [9] and Tossings [20] in Hilbert spaces and by Sahu and Yao [16] in Banach spaces.

The purpose of this paper is to introduce a novel prox-Tikhonov-like forward-backward method to solve the accretive inclusion problem ($\mathscr{P}$) in general Banach spaces like the spaces L_p ($1 < p < \infty$) without using the technique of variational distance [9]. We also discuss inexact version of our prox-Tikhonov-like forward-backward method. We prove strong convergence of iterative sequences generated by our algorithms. To the best of our knowledge, it is among the first algorithm to tackle the case where A is not necessarily m-accretive operator. In section 2 we give geometry of Banach spaces, nonexpansive type mappings and their properties and accretive operators and their properties. We introduce the property ($\mathscr{N}$) for nonexpansivity of operators in Banach spaces. The property ($\mathscr{N}$) of certain classes of nonlinear operators shall be central tool for our splitting methods for solving inclusion problem ($\mathscr{P}$). Section 3 introduces a new prox-Tikhonov-like forward-backward method and its inexact version and states main theoretical results of the paper. We derive several known and unknown results in the context of the property ($\mathscr{N}$). Section 4 deals algorithms in general Banach spaces. In Section 5, we discuss applications of our algorithms to mixed variational inequalities and nonsmooth convex minimization problems. Our iterative methods unify, improve and generalize the corresponding results of fixed point problems, solutions of problems (1.2)-(1.3) and inclusion problem ($\mathscr{P}$).

2. Preliminaries and Notations

Throughout this paper, all vector spaces are real and we denote by $\mathbb{N}$ the set of natural numbers. Let X be a Banach space and $\mathcal{M} \subseteq X$. We denote Fix(T) the set of fixed points of a mapping $T : \mathcal{M} \to \mathcal{M}$. In the sequel, we always use $\Pi_{\mathcal{M}}$ to denote the collection of all contractions on $\mathcal{M}$ and S_X to denote the unit sphere $S_X = \{x \in X : \|x\| = 1\}$.

2.1. Geometry of Banach spaces

A Banach space X is said to be *strictly convex* if

$$x, y \in S_X \text{ with } x \neq y \Rightarrow \|(1-\lambda)x + \lambda y\| < 1 \text{ for all } \lambda \in (0,1).$$

The *modulus of convexity* of X is defined by

$$\delta_X(\epsilon) = \inf \left\{1 - \|x+y\|/2 : \|x\|, \|y\| \leq 1, \|x-y\| \geq \epsilon\right\},$$

for all $\epsilon \in [0,2]$. X is said to be uniformly convex if $\delta_X(0) = 0$, and $\delta_X(\epsilon) > 0$ for all $0 < \epsilon \leq 2$. The space X is said to be p-*uniformly convex* if there a constant $c_p > 0$ such that $\delta_X(\epsilon) \geq c_p \epsilon^p$. Every Hilbert space is 2-uniformly convex, while L^p is $\max\{p,2\}$-uniformly convex for every $p > 1$.

A Banach space X is said to be *smooth* provided the limit

$$\lim_{t \to 0} \frac{\|x+ty\| - \|x\|}{t} \text{ exists}$$

for each x and y in S_X. In this case, the norm of X is said to be *Gâteaux differentiable*. It is said to be *uniformly Gâteaux differentiable* if for each $y \in S_X$, this limit is attained uniformly for $x \in S_X$. Let $\rho_X : [0, \infty) \to [0, \infty)$ be the modulus of smoothness of X defined by

$$\rho_X(t) = \sup\left\{\frac{1}{2}(||x+y\| + \|x-y\|) - 1 : \, x \in S_X, \, \|y\| \le t\right\}.$$

A Banach space X is said to be uniformly smooth if $\dfrac{\rho_X(t)}{t} \to 0$ as $t \to 0$. A Banach space X is said to be *q-uniformly smooth* if there exists a fixed constant $c > 0$ such that $\rho_X(t) \le ct^q$. It is well-known that X is uniformly smooth if and only if the norm of X is uniformly Fréchet differentiable. If X is q-uniformly smooth, then $q \le 2$ and X is uniformly smooth, and hence the norm of X is uniformly Fréchet differentiable, in particular, the norm of X is Fréchet differentiable. Typical example of uniformly smooth Banach spaces is L^p, where $p > 1$. More precisely, L^p is $\min\{p, 2\}$-uniformly smooth for every $p > 1$. It is well known that every uniformly smooth space (e.g., L_p space, $1 < p < \infty$) has uniformly Gâteaux differentiable norm (see, e.g., [1]).

Lemma 2.1. [24]. *Let $p > 1$ be a given real number and X be a Banach space. Then, X is p-uniformly convex if and only if there exists a constant $c_p > 0$ such that*

$$\|tx + (1-t)y\|^p \quad \le \quad t\|x\|^p + (1-t)\|y\|^p - c_p W_p(t)\|x-y\|^p,$$

for all $x, y \in X$ and $t \in [0, 1]$, where $W_p(t) = (1-t)t^p + t(1-t)^p$.

2.2. Accretive operators

Let X be a real Banach space with dual space X^*. We denote by J the normalized duality mapping from X into 2^{X^*} defined by

$$J(x) := \left\{f^* \in X^* : \langle x, f^*\rangle = ||x||^2 = ||f^*||^2\right\}, \quad \text{for all } x \in X,$$

where $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing. For an operator $A : X \to 2^X$, we define its domain, range and graph as follows:

$$D(A) = \{x \in X : Ax \neq \emptyset\},$$

$$R(A) = \bigcup\{Az : z \in D(A)\},$$

and

$$G(A) = \{(x, y) \in X \times X : x \in D(A), y \in Ax\},$$

respectively. Thus, we write $A : X \to 2^X$ as follows: $A \subseteq X \times X$. The inverse A^{-1} of A is defined by

$$x \in A^{-1}y \text{ if and only if } y \in Ax.$$

The operator A is said to be *accretive* if for each $x_i \in D(A)$ and $y_i \in Ax_i$ $(i = 1, 2)$, there exists $j \in J(x_1 - x_2)$ such that $\langle y_1 - y_2, j\rangle \ge 0$. An accretive operator A is said to be *maximal accretive* if there is no proper accretive extension of A and *m-accretive* if $R(I+A)=X$, where I stands for the identity operator on X (It follows that $R(I+rA)=X$ for all $r>0$). If A is m-accretive, then it is maximal accretive, but the converse is not true in general. If A is accretive, then we can define, for each $\lambda>0$, a nonexpansive single-valued mapping $J_\lambda^A : R(1+\lambda A)\to D(A)$ by

$$J_\lambda^A=(I+\lambda A)^{-1}.$$

It is called the *resolvent* of A. It is well known that if A is an m-accretive operator on a Banach space X, then for each $\lambda>0$, the resolvent $J_\lambda^A=(I+\lambda A)^{-1}$ is a single-valued nonexpansive mapping whose domain is entire space X. An accretive operator A defined on a Banach space X is said to satisfy the *range condition* if $\overline{D(A)}\subset R(1+\lambda A)$ for all $\lambda>0$, where $\overline{D(A)}$ denotes the closure of the domain of A. It is well known that for an accretive operator A which satisfies the range condition, $A^{-1}(0)=\mathrm{Fix}(J_\lambda^A)$ for all $\lambda>0$. We also define the *Yosida approximation* A_r by $A_r=(I-J_r^A)/r$. We know that $A_r x\in AJ_r^A x$ for all $x\in R(I+rA)$ and $\|A_r x\|\le |Ax|=\inf\{\|y\|: y\in Ax\}$ for all $x\in D(A)\cap R(I+rA)$. We also know the following [19]: For each $\lambda,\mu>0$ and $x\in R(I+\lambda A)\cap R(I+\mu A)$, it holds that

$$\left\|J_\lambda^A x-J_\mu^A x\right\|\le\frac{|\lambda-\mu|}{\lambda}\|x-J_\lambda^A x\|. \tag{2.1}$$

Let C be a nonempty subset of a smooth Banach space X. An operator $T: C\to X$ is said to be *strongly accretive* if there exists $\eta>0$ such that

$$\langle Tx-Ty, J(x-y)\rangle\ge\nu\|x-y\|^2,\quad \text{for all } x,y\in C. \tag{2.2}$$

For $\eta>0$, an operator $T: C\to X$ is said to be ν-inverse strongly accretive [2] if

$$\langle Tx-Ty, J(x-y)\rangle\ge\eta\|Tx-Ty\|^2,\quad \text{for all } x,y\in C. \tag{2.3}$$

Remark 2.1. An inspection of (2.2) and (2.3) shows that every Lipschitzian strongly accretive operator is inverse strongly accretive.

The following results will be the key in the proof of our results.

Proposition 2.1. [5]. *Let X be a Banach space and $A: X\to 2^X$ be an m-accretive operator. Then, A is maximal accretive. If H is a Hilbert space, then $A: H\to 2^H$ is maximal accretive if and only if it is m-accretive.*

Proposition 2.2. *Let C be a nonempty closed convex subset of a Banach space X and $A\subseteq X\times X$ an accretive operator such that $\overline{D(A)}\subseteq C\subseteq\bigcap_{t>0}R(I+tA)$. Let $c>0$, $\lambda>0$, $x\in C$ and $B: C\to X$ an operator such that $(I-cB)x\in C$. Then,*

$$\begin{aligned}&\left\|J_c^A(I-cB)x-J_\lambda^A(I-\lambda B)x\right\|\\ &\le|c-\lambda|\|Bx\|+\frac{|c-\lambda|}{c}\left\|J_c^A(I-\lambda B)x-(I-\lambda B)x\right\|.\end{aligned}$$

Proof. From (2.1), we have

$$\begin{aligned}
&\left\| J_c^A(I-cB)x - J_\lambda^A(I-\lambda B)x \right\| \\
&\leq \left\| J_c^A(I-cB)x - J_c^A(I-\lambda B)x \right\| + \left\| J_c^A(I-\lambda B)x - J_\lambda^A(I-\lambda B)x \right\| \\
&\leq \|(I-cB)x-(I-\lambda B)x\| + \frac{|c-\lambda|}{c} \left\| J_c^A(I-\lambda B)x - (I-\lambda B)x \right\| \\
&= |c-\lambda|\|Bx\| + \frac{|c-\lambda|}{c} \left\| J_c^A(I-\lambda B)x - (I-\lambda B)x \right\|.
\end{aligned}$$

■

2.3. Nonexpansive type mappings

The notion of κ-strict pseudocontractive mapping was introduced by Browder and Petryshyn [4] as follows: Let C be a nonempty subset of a Hilbert space H. A mapping $T : C \to C$ is called *κ-strict pseudocontractive* if there exists a constant $\kappa \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa\|x - Tx - (y - Ty)\|^2, \quad \text{for all } x, y \in C.$$

Let C be a nonempty subset of a Banach space X. A mapping $T : C \to C$ is called *κ-strict pseudocontractive with respect to* $p \in (1, \infty)$ if there exists a constant $\kappa \in [0, 1)$ such that

$$\|Tx - Ty\|^p \leq \|x - y\|^p + \kappa\|x - Tx - (y - Ty)\|^p, \quad \text{for all } x, y \in C.$$

Thus, T is nonexpansive if and only if T is 0-strict pseudocontractive. The class of κ-strict pseudocontractive mappings is essentially wider than that of nonexpansive mappings.

A closed convex subset C of a Banach space X is said to have the *fixed-point property* for nonexpansive mappings if every nonexpansive mapping of a nonempty closed convex bounded subset M of C into itself has a fixed point in M.

A subset C of a Banach space X is said to be a *retract* of X if there exists a continuous mapping P from X onto C such that $Px = x$ for all x in C. We call such P a *retraction* of X onto C. It follows that if a mapping P is a retraction, then $Py = y$ for all y in the range of P. A retraction P is said to be *sunny* if $P(Px + t(x - Px)) = Px$ for each x in X and $t \geq 0$. If a sunny retraction P is also nonexpansive, then C is said to be a *sunny nonexpansive retract* of X.

Let C be a nonempty subset of a Banach space X and $x \in X$. An element $y_0 \in C$ is said to be a *best approximation* to x if $\|x - y_0\| = d(x, C)$, where $d(x, C) = \inf\limits_{y \in C} \|x - y\|$. The set of all best approximations from x to C is denoted by

$$\text{Proj}_C(x) = \{y \in C : \|x - y\| = d(x, C)\}.$$

This defines a mapping Proj_C from X into 2^C and is called the *nearest point projection mapping* (*metric projection mapping*) onto C. It is well known that if C is a nonempty closed convex subset of a Hilbert space H, then the nearest point projection Proj_C from H onto C is the unique sunny nonexpansive retraction of H onto C. It is also known that $\text{Proj}_C x \in C$ and

$$\langle x - \text{Proj}_C(x), \text{Proj}_C(x) - y \rangle \geq 0, \quad \text{for all } x \in H, y \in C.$$

We need the following facts for proving our main results.

Lemma 2.2. [7, Lemma 13.1]. *Let C be a convex subset of a smooth Banach space X, D a nonempty subset of C and P a retraction from C onto D. Then, the following statements are equivalent:*

(a) *P is a sunny and nonexpansive.*

(b) *$\langle x - Px, J(z - Px) \rangle \leq 0$ for all $x \in C, z \in D$.*

(c) *$\langle x - y, J(Px - Py) \rangle \geq \|Px - Py\|^2$ for all $x, y \in C$.*

Lemma 2.3. [21, Corollary 3.4]. *Let X be a reflexive Banach space with a uniformly Gâteaux differentiable norm and C a nonempty closed convex subset of X with fixed point property for nonexpansive self-mappings. Let $T : C \to C$ be a nonexpansive mapping such that Fix$(T) \neq \emptyset$. Then, the following statements hold:*

(a) *Fix(T) is a sunny nonexpansive retract of C.*

(b) *For each fixed $f \in \Pi_C$ and every $t \in (0, 1)$, there exists a unique fixed point $v_t \in C$ of the contraction $C \ni v \mapsto tfv + (1 - t)Tv$ defined by*

$$v_t = tfv_t + (1 - t)Tv_t, \tag{2.4}$$

converges strongly as $t \to 0$ to $x^* \in$ Fix(T), where $x^* = Qfx^*$ and Q is a sunny nonexpansive retraction from C onto Fix(T).

Lemma 2.4. [2, Lemma 2.8]. *Let C be a nonempty closed convex subset of a 2-uniformly smooth Banach space X. Let $\nu > 0$ and $A : C \to X$ be ν-inverse strongly accretive operator. If $0 < \lambda \leq \nu/K^2$, then $I - \lambda A$ is a nonexpansive mapping of C into X, where K is the 2-uniformly smoothness constant of X.*

Lemma 2.5. [13]. *Let C be a nonempty closed convex subset of a Hilbert space H. Let $\nu > 0$ and $A : C \to H$ be ν-inverse strongly monotone operator. If $0 < \lambda \leq 2\nu$, then $I - \lambda A$ is a nonexpansive mapping of C into X.*

Proposition 2.3. *Let $p > 1$ be a given real number. Let C be a nonempty closed convex subset of a p-uniformly convex Banach space X, $T : C \to C$ be a λ-strictly pseudocontractive with $\lambda < \min\left\{1, \frac{c_p}{2^{(p-2)}}\right\}$ and $B = I - T$. Let $\gamma_{X,B} = 1 - \frac{c_p\lambda}{2^{(p-2)}}$. Then, $T_w = (1 - w)I + wT$ is a nonexpansive mapping from C into itself for each $w \in (0, \gamma_{X,B})$.*

Proof. Let $w \in (0, \gamma_{X,B})$. Note $W_p(w) = (1-w)w^p + w(1-w)^p \geq w(1-w)/2^{(p-2)}$. Let $x, y \in C$. From Lemma 2.1, we have

$$\begin{aligned}
&\|T_w x - T_w y\|^p \\
&= \|(1-w)(x-y) + w(Tx - Ty)\|^p \\
&\leq (1-w)\|x-y\|^p + w\|Tx - Ty\|^p - c_p W_p(w)\|x - y - (Tx - Ty)\|^p \\
&\leq (1-w)\|x-y\|^p + w[\|x-y\|^p + \lambda\|x - y - (Tx - Ty)\|^p] \\
&\quad - c_p W_p(w)\|x - y - (Tx - Ty)\|^p \\
&\leq \|x-y\|^p + w\lambda\|x - y - (Tx - Ty)\|^p - \frac{c_p w(1-w)}{2^{(p-2)}}\|x - y - (Tx - Ty)\|^p \\
&= \|x-y\|^p - w\left[\frac{c_p}{2^{(p-2)}}(1-w) - \lambda\right]\|x - y - (Tx - Ty)\|^p.
\end{aligned}$$

Since $w \in (0, \gamma_{X,B})$ implies that $\frac{c_p}{2^{(p-2)}}(1-w) - \lambda > 0$. Therefore, T_w is nonexpansive. ∎

2.4. The property $(\mathscr{N})$

We introduce the property $(\mathscr{N})$ for nonexpansivity of operators.

Let C be a nonempty closed convex subset of a Banach space X. An operator $B : C \to X$ is said to satisfy the property $(\mathscr{N})$ on $(0, \gamma_{X,B})$ if there exists $\gamma_{X,B} \in (0, \infty]$, depends on X and B, such that $I - \xi B : C \to C$ is nonexpansive for each $\xi \in (0, \gamma_{X,B})$.

Remark 2.2. For a nonexpansive mapping $T : C \to C$ with $B = I - T$, the average mapping $T_w = I - wB$ is always nonexpansive for each $w \in (0, \gamma_{X,B})$, where $\gamma_{X,B} = 1$.

It turns out that the constant $\gamma_{X,B}$ closely depends upon geometric properties of the Banach spaces and operators B under consideration. We collect some examples of operators satisfying the property $(\mathscr{N})$ on $(0, \gamma_{X,B})$ in a variety of Banach spaces.

Example 2.1. Let C be a nonempty closed convex subset of a 2-uniformly smooth Banach space X. Let $\nu > 0$ and $B : C \to X$ be ν-inverse strongly accretive operator. If $R(I - \lambda B) \subseteq C$ for each $\lambda \in (0, \gamma_{X,B})$, where $\gamma_{X,B} = \nu/K^2$ and K is the 2-uniformly smoothness constant of X, then B has the property $(\mathscr{N})$ on $(0, \gamma_{X,B})$.

Proof. Let K be the 2-uniformly smoothness constant of X and $\lambda \in (0, \gamma_{X,B})$. Here $\gamma_{X,B} = \nu/K^2$. Suppose that $R(I - \lambda B) \subseteq C$. It follows from Lemma 2.4 that $I - \lambda B$ is a nonexpansive mapping from C into C. ∎

Example 2.2. Let $p > 1$ be a given real number. Let C be a nonempty closed convex subset of a p-uniformly convex Banach space X, $T : C \to C$ be a λ-strictly pseudocontractive with $\lambda < \min\left\{1, \frac{c_p}{2^{(p-2)}}\right\}$, $B = I - T$ and $\gamma_{X,B} = 1 - \frac{c_p \lambda}{2^{(p-2)}}$. Then, B has the property $(\mathscr{N})$ on $(0, \gamma_{X,B})$.

Proof. Lemma 2.3 shows that $T_w = I - wB$ is a nonexpansive mapping from C into itself for each $w \in (0, \gamma_{X,B})$. It follows that B has the property $(\mathscr{N})$ on $(0, \gamma_{X,B})$. ■

The property $(\mathscr{N})$ alludes to the fact that in order to solve the inclusion problem $(\mathscr{P})$, it suffices to find a fixed point of a nonexpansive mapping $J_r^{A,B}$ defined by (2.5).

Proposition 2.4. *Let C be a nonempty closed convex subset of a Banach space X, $A \subseteq X \times X$ an accretive operator such that $\overline{D(A)} \subseteq C \subseteq \bigcap_{t>0} R(I + tA)$ and $B : C \to X$ an operator such that Zer$(A + B) \neq \emptyset$ and B has the property $(\mathscr{N})$ on $(0, \gamma_{X,B})$, where $\gamma_{X,B}$ is a constant depends on X and B. For $r \in (0, \gamma_{X,B})$, define an operator $J_r^{A,B} : C \to C$ by*

$$J_r^{A,B} x = J_r^A (I - rB)x, x \in C. \tag{2.5}$$

Then, the following statements hold.

(a) *$J_r^{A,B}$ is nonexpansive.*

(b) *Fix$(J_r^{A,B})$ = Zer$(A + B)$.*

Proof. Let $r \in (0, \gamma_{X,B})$. By the property $(\mathscr{N})$, $I - rB$ is a nonexpansive mapping from C into itself.

(a) Since J_r^A is nonexpansive, one can see that $J_r^{A,B}$ is nonexpansive.

(b) From the definition of $J_r^{A,B}$, we have

$$\begin{aligned} z = J_r^{A,B} z &\Leftrightarrow z = J_r^A (I - rB)z \\ &\Leftrightarrow z = (I + rA)^{-1}(I - rB)z \\ &\Leftrightarrow z - rBz \in (I + rA)z \\ &\Leftrightarrow 0 \in Az + Bz. \end{aligned}$$

■

2.5. Approximating fixed point sequence

Let (X, d) be a metric space and $T : X \to X$ a mapping. A bounded sequence $\{x_n\}$ in X is said to an *approximating fixed point sequence* of T if $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

The following auxiliary results will be needed in the sequel for the proof of our main results:

Proposition 2.5. *Let (X, d) be a metric space, $T : X \to X$ a uniformly continuous mapping and $\{x_n\} \subset X$ be an approximating fixed point sequence of T. Then, $\{y_n\}$ is an approximating fixed point sequence of T whenever $\{y_n\}$ is in X such that $\lim_{n\to\infty} d(x_n, y_n) = 0$.*

Proof. Let $\{z_n\}$ be a sequence in X such that $\lim_{n\to\infty} d(x_n, z_n) = 0$. Since $\{x_n\}$ is an approximating fixed point sequence of T and T is uniformly continuous, we have

$$d(z_n, Tz_n) \le d(z_n, x_n) + d(x_n, Tx_n) + d(Tx_n, Tz_n) \to 0 \text{ as } n \to \infty. \qquad \blacksquare$$

Lemma 2.6. [21, Lemma 2.12]. *Let X be a Banach space with a uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of X, $f : C \to C$ a continuous mapping, $T : C \to C$ a nonexpansive mapping and $\{x_n\}$ a bounded sequence in C such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Suppose that $\{z_t\}$ is a path in C defined by*

$$z_t = tfz_t + (1-t)Tz_t,\ t \in (0, 1),$$

such that $z_t \to z$ as $t \to 0$. Then, $\limsup_{n\to\infty}\langle fz - z, J(x_n - z)\rangle \le 0$.

3. Algorithms on Banach Spaces with Uniformly Gâteaux Differentiable Norms

Let C be a nonempty closed convex subset of a Banach space X. Let $A \subseteq X \times X$ be an accretive operator such that $\overline{D(A)} \subseteq C \subseteq \bigcap_{t>0} R(I + tA)$, $B : C \to X$ an operator such that $\mathrm{Zer}(A + B) \ne \emptyset$ and B has the property $(\mathscr{N})$ on $(0, \gamma_{X,B})$, i.e., $I - \xi B$ is nonexpansive from C into itself for each $\xi \in (0, \gamma_{X,B})$, where $\gamma_{X,B}$ is a constant depends on X and B.

Noticing that $J_r^{A,B}$ defined by (2.5) is already split. Therefore, a fixed point iterative algorithm for $J_r^{A,B}$ on C corresponds to a splitting algorithm for inclusion problem $(\mathscr{P})$. Motivated by above fact and prox-Tikhonov method [9, 16], our prox-Tikhonov-like forward-backward splitting method is then defined to generate a sequence $\{x_n\}$ in C according to the recursive formula: Starting with $x_1 \in C$ and after $x_n \in C$ is defined, we define the next iterate x_{n+1} as follows:

$$x_{n+1} = J_{c_n}^A(I - c_nB)((1-\alpha_n)x_n + \alpha_n fx_n), \quad \text{for all } n \in \mathbb{N}, \tag{3.1}$$

where $f \in \Pi_C$, $\{\alpha_n\}$ is a sequence in $(0, 1]$ and $\{c_n\}$ is a regularization sequence in $(0, \gamma_{X,B})$.

We shall study our prox-Tikhonov-like forward-backward splitting method under the following conditions:

(C1) $\lim\limits_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and either $\sum_{n=1}^{\infty} \left|\alpha_n - \alpha_{n+1}\right| < \infty$ or $\lim\limits_{n\to\infty} \left|1 - \dfrac{\alpha_n}{\alpha_{n+1}}\right| = 0$, (C2) $0 < \varepsilon \le c_n$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} \left| c_n - c_{n+1} \right| < \infty$.

Now we are ready to prove a main result of this section for solving problem $(\mathscr{P})$ in the framework of Banach space with a uniformly Gâteaux differentiable norm.

Theorem 3.1. *Let X be a reflexive Banach space with a uniformly Gâteaux differentiable norm. Let C be a nonempty closed convex subset of X such that C has the fixed-point property for nonexpansive mappings. Let $A \subseteq X \times X$ be an accretive operator such that $\overline{D(A)} \subseteq C \subseteq \bigcap_{t>0} R(I + tA)$, $B : C \to X$ an operator such that Zer$(A + B) \neq \emptyset$ and B has the property $(\mathscr{N})$ on $(0, \gamma_{X,B})$. For given $f \in \Pi_C$ and $x_1 \in C$, let $\{x_n\}$ be a sequence in C generated by (3.1), where $\{\alpha_n\}$ is a sequence in $(0, 1]$ and $\{c_n\}$ is a regularization sequence in $(0, \gamma_{X,B})$ satisfying conditions (C1)–(C2). Then, $\{x_n\}$ converges strongly to $x^* \in$ Zer$(A + B)$, where $x^* = Qfx^*$ and Q is a sunny nonexpansive retraction of C onto Zer$(A + B)$.*

Proof. By assumptions for regularization sequence $\{c_n\}$, we have $\lim_{n\to\infty} c_n = c$ for some $c \ge \varepsilon$. Define $T_n := J_{c_n}^A(I - c_nB)$ and $T := J_c^A(I - cB)$. From Proposition 2.4, we obtain that T_n and T are nonexpansive with Fix(T_n) = Fix(T) = Zer$(A + B)$. For $t \in (0, 1)$, invoking Lemma 2.3, there exists a path $\{v_t\}$ in C defined by (2.4) which is strongly convergent as $t \to 0$ to $x^* \in$ Fix(T) = Zer$(A + B)$, where $x^* = Qfx^*$ and Q is a sunny nonexpansive retraction of C onto Zer$(A + B)$. Set $y_n := (1-\alpha_n)x_n + \alpha_n fx_n$. Let κ_f denote the Lipschitz constant of f. We now proceed with the following steps:

Step 1. *$\{x_n\}$ and $\{y_n\}$ are bounded.*

Let κ_f be the contraction constant of f. Note that

$$\|y_n - x^*\| \le (1 - \alpha_n)\|x_n - x^*\| + \alpha_n\|fx_n - x^*\|. \tag{3.2}$$

Invoking (3.1), we have

$$\begin{aligned} &\|x_{n+1} - x^*\| = \|T_n y_n - x^*\| \\ &\le (1 - \alpha_n)\|x_n - x^*\| + \alpha_n\|fx_n - x^*\| \\ &\le (1 - \alpha_n)\|x_n - x^*\| + \alpha_n(\|fx_n - fx^*\| + \|fx^* - x^*\|) \\ &\le (1 - (1 - \kappa_f)\alpha_n)\|x_n - x^*\| + \alpha_n\|fx^* - x^*\| \\ &\le \max\{\|x_n - x^*\|, \|fx^* - x^*\|/(1 - \kappa_f)\} \\ &\vdots \\ &\le \max\{\|x_1 - x^*\|, \|fx^* - x^*\|/(1 - \kappa_f)\}. \end{aligned}$$

Thus, $\{x_n\}$ is bounded and hence, from (3.2), $\{y_n\}$ is bounded.

Step 2. *$\{x_n\}$ is asymptotically regular, i.e., $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$.*

Let K_1 be a constant such that $K_1 = \max\{\sup_{n\in\mathbb{N}} \|x_n\|, \sup_{n\in\mathbb{N}} \|fx_n\|\}$. Observe that

$$\begin{aligned}
&||y_n - y_{n-1}|| \\
&= ||(1-\alpha_n)x_n + \alpha_n f x_n - (1-\alpha_{n-1})x_{n-1} - \alpha_{n-1} f x_{n-1}\| \\
&= ||(1-\alpha_n)x_n - (1-\alpha_n)x_{n-1} + (1-\alpha_n)x_{n-1} + \alpha_n(fx_n - fx_{n-1}) \\
&\quad +\alpha_n f x_{n-1} - (1-\alpha_{n-1})x_{n-1} - \alpha_{n-1} f x_{n-1}\| \\
&\le (1-\alpha_n)||x_n - x_{n-1}\| + \alpha_n\kappa_f\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|(\|x_{n-1}\| + \|fx_{n-1}\|) \\
&\le (1-(1-\kappa_f)\alpha_n)||x_n - x_{n-1}\| + 2|\alpha_n - \alpha_{n-1}|K_1.
\end{aligned}$$

It follows from (3.1) that

$$\begin{aligned}
&||x_{n+1} - x_n|| \\
&= ||T_n y_n - T_{n-1}y_{n-1}|| \\
&\le ||T_n y_n - T_n y_{n-1}\| + \|T_n y_{n-1} - T_{n-1}y_{n-1}|| \\
&\le ||y_n - y_{n-1}\| + \|T_n y_{n-1} - T_{n-1}y_{n-1}|| \\
&\le [1-(1-\kappa_f)\alpha_n]||x_n - x_{n-1}\| + 2|\alpha_n - \alpha_{n-1}|K_1 + \|T_n y_{n-1} - T_{n-1}y_{n-1}||.
\end{aligned}$$

From the definition of T_n and Proposition 2.2, we have

$$\begin{aligned}
&\|T_{n+1}y_n - T_n y_n\| \\
&= \|J^A_{c_{n+1}}(I - c_{n+1}B)y_n - J^A_{c_n}(I - c_n B)y_n\| \\
&\le |c_{n+1} - c_n|\|By_n\| + \frac{|c_{n+1} - c_n|}{\varepsilon}\|J^A_{c_{n+1}}(I - c_n B)y_n - (I - c_n B)y_n\| \\
&\le 2|c_{n+1} - c_n|K_2,
\end{aligned}$$

where $K_2 = \sup_{n\in\mathbb{N}}\left\{\|By_n\| + \frac{1}{\varepsilon}\|J^A_{c_{n+1}}(I - c_n B)y_n - (I - c_n B)y_n\|\right\}$. Hence,

$$||x_{n+1} - x_n|| \le [1-(1-\kappa_f)\alpha_n]||x_n - x_{n-1}\| + 2|\alpha_n - \alpha_{n-1}|K_1 + 2|c_n - c_{n-1}|K_2.$$

By conditions (C2)–(C2) and [23, Lemma 2.5], we obtain that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$.

Step 3. $||y_n - Ty_n|| \to 0$ *as* $n \to \infty$.

Noticing that $\|y_n - x_n\| = \alpha_n\|x_n - fx_n\| \to 0$ as $n \to \infty$. From (3.1), we have

$$\|x_{n+1} - T_n x_n\| = \|T_n y_n - T_n x_n\| \le \|y_n - x_n\| \to 0 \text{ as } n \to \infty.$$

and

$$\|x_n - T_n x_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - T_n x_n\| \to 0 \text{ as } n \to \infty. \tag{3.3}$$

From Proposition 2.2, we have

$$\begin{aligned}
&\|T_n x_n - T x_n\| \\
&= \|J^A_{c_n}(I - c_n B)x_n - J^A_c(I - cB)x_n\| \\
&\le |c_n - c|\|Bx_n\| + \frac{|c_n - c|}{c}\|(I - cB)x_n - J^A_c(I - cB)x_n\| \\
&\le 2|c_n - c|K_3,
\end{aligned}$$

where $K_3 = \sup_{n\in\mathbb{N}}\left\{\|Bx_n\| + \frac{1}{\varepsilon}\|(I - cB)x_n - J^A_c(I - cB)x_n\|\right\}$. Hence,

$$\begin{aligned}
\|x_{n+1} - Tx_{n+1}\| &\le \|x_{n+1} - T_n x_n\| + \|T_n x_n - T x_n\| + \|Tx_n - Tx_{n+1}\| \\
&\le \|x_{n+1} - T_n x_n\| + 2|c_n - c|K_3 + \|x_n - x_{n+1}\| \to 0 \quad \text{as } n \to \infty,
\end{aligned}$$

i.e., $\{x_n\}$ is an approximating fixed point sequence of T. Since $||y_n - x_n|| \to 0$, it follows from Proposition 2.5 that $\{y_n\}$ is an approximating fixed point sequence of T.

Step 4. $\{x_n\}$ *converges strongly to* x^*.

Noticing that T is nonexpansive with Fix(T) = Zer$(A + B)$. Set $\sigma_n := \langle fx^* - x^*, J(y_n - x^*)\rangle$. Since $||y_n - Ty_n|| \to 0$ as $n \to \infty$ and path $\{v_t\}$ in C defined by (2.4) is strongly convergent, as $t \to 0$, to $x^* \in$ Fix(T), it follows from Lemma 2.6 that $\limsup_{n\to\infty} \sigma_n \le 0$. From (3.1), we have

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&= \|J^A_{c_n}(I - c_n B)y_n - x^*\|^2 \\
&\le \|y_n - x^*\|^2 = \|(1 - \alpha_n)(x_n - x^*) + \alpha_n(fx_n - fx^* + fx^* - x^*)\|^2 \\
&\le \|(1 - \alpha_n)(x_n - x^*) + \alpha_n(fx_n - fx^*)\|^2 + 2\alpha_n\langle f(x^*) - x^*, J(y_n - x^*)\rangle \\
&\le (1 - (1 - \kappa_f)\alpha_n)^2\|x_n - x^*\|^2 + 2\alpha_n\langle f(x^*) - x^*, J(y_n - x^*)\rangle \\
&\le (1 - (1 - \kappa_f)\alpha_n)\|x_n - x^*\|^2 + 2\alpha_n\langle f(x^*) - x^*, J(y_n - x^*)\rangle.
\end{aligned}$$

Noticing that $\limsup_{n\to\infty}\langle f(x^*) - x^*, J(y_n - x^*)\rangle \le 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Therefore, we conclude from [23, Lemma 2.5] that $\{x_n\}$ converges strongly to x^*. ■

We now consider inexact variant of algorithm (3.1) for solution of problem $(\mathscr{P})$.

Let X be a Banach space, $A \subseteq X \times X$ an m-accretive operator and $B : X \to X$ an operator such that Zer$(A + B) \neq \emptyset$ and B has the property $(\mathscr{N})$ on $(0, \gamma_{X,B})$. Our inexact prox-Tikhonov regularized generalized forward-backward splitting method is then defined to generate a sequence $\{z_n\}$ in X according to the recursive formula: Starting with $z_1 \in X$ and after $z_n \in X$ is defined, we define the next iterate z_{n+1} as follows:

$$\text{(3.4)} \qquad \begin{cases} u_n &= (1 - \alpha_n)z_n + \alpha_n f z_n; \\ z_{n+1} &= J^A_{c_n}(u_n - c_n(Bu_n + b_n)) + e_n \text{ for all } n \in \mathbb{N}, \end{cases}$$

where $f \in \Pi_X$, α_n is a relaxation parameter in $(0, 1]$, $\{c_n\}$ is a regularization sequence in $(0, \gamma_{X,B})$ and $\{b_n\}$ and $\{e_n\}$ are sequences of errors in X.

Next, we apply Theorem 3.1 to establish a strong convergence theorem for algorithm (3.4) with error sequence which may not be sumable.

Theorem 3.2. *Let X be a reflexive Banach space with a uniformly Gâteaux differentiable norm such that X has the fixed point property for nonexpansive mappings. Let $A \subseteq X \times X$ be an m-accretive operator and $B : X \to X$ be an operator such that B has the property $(\mathcal{N})$ on $(0, \gamma_{X,B})$. For given $f \in \Pi_X$ and $z_1 \in X$, let $\{z_n\}$ be a sequence in X generated by (3.4), where $\{\alpha_n\}$ is a sequence in $(0, 1]$, $\{c_n\}$ is a regularization sequence in $(0, \gamma_{X,B})$ and $\{b_n\}$ and $\{e_n\}$ are sequences of errors in X satisfying conditions (C1)–(C4):*

$(C3)$ $\sum_{n=1}^{\infty} \|e_n\| < \infty$ *or* $\lim\limits_{n\to\infty} \|e_n\|/\alpha_n = 0$,

$(C4)$ $\sum_{n=1}^{\infty} \|b_n\|c_n < \infty$ *or* $\lim\limits_{n\to\infty} (\|b_n\|c_n)/\alpha_n = 0$.

Then, $\{z_n\}$ converges strongly to $x^* \in$ Zer$(A + B)$, where $x^* = Qfx^*$ and Q is a sunny nonexpansive retraction of X onto Zer$(A + B)$.

Proof. For $x_1 = z_1 \in X$, let $\{x_n\}$ be the iterative sequence in X defined by (3.1). It follows from Theorem 3.1 that $\{x_n\}$ converges strongly to $x^* \in$ Zer$(A+B)$, where $x^* = Qfx^*$ and Q is a sunny nonexpansive retraction of X onto Zer$(A + B)$. Set $y_n := (1 - \alpha_n)x_n + \alpha_n f x_n$. From (3.1) and (3.4), we have

$$\begin{aligned}
\|z_{n+1} - x_{n+1}\| &= \|J^A_{c_n}(u_n - c_n(Bu_n + b_n)) + e_n - J^A_{c_n}(y_n - c_nBy_n)\| \\
&\leq \|(I - c_nB)u_n - b_nc_n - (I - c_nB)y_n\| + \|e_n\| \\
&\leq \|(I - c_nB)u_n - (I - c_nB)y_n\| + c_n\|b_n\| + \|e_n\| \\
&\leq \|u_n - y_n\| + c_n\|b_n\| + \|e_n\| \\
&= \|(1 - \alpha_n)(z_n - x_n) + \alpha_n(fz_n - fz_n)\| + c_n\|b_n\| + \|e_n\| \\
&\leq (1 - (1 - \kappa_f)\alpha_n)\|z_n - x_n\| + c_n\|b_n\| + \|e_n\| \text{ for all } n \in \mathbb{N}.
\end{aligned}$$

By [23, Lemma 2.5], we have $\|z_n - x_n\| \to 0$. Therefore, $\{z_n\}$ converges strongly to x^*. ∎

As we have discussed in section 2.4 that there are some classes of nonlinear operators which enjoy the property $(\mathcal{N})$ in suitable Banach spaces. Therefore, we are able to derive the some new and known results from Theorems 3.1 and 3.2.

Corollary 3.1. *Let X be a reflexive Banach space with a uniformly Gâteaux differentiable norm. Let C be a closed convex subset of X, $A \subseteq X \times X$ an accretive operator such that $\overline{D(A)} \subseteq C \subseteq \bigcap_{t>0} R(I + tA)$, and $T : C \to C$ be a nonexpansive mapping with $B = I - T$ such that Zer$(A + B) \neq \emptyset$. Suppose that C has the fixed* *point property for nonexpansive mappings. For given $f \in \Pi_C$ and $x_1 \in C$, let $\{x_n\}$ be a sequence in C defined by*

$$x_{n+1} = J_{c_n}^A((1-c_n)I + c_nT)[(1-\alpha_n)x_n + \alpha_n f x_n], \quad \textit{for all } n \in \mathbb{N}, \tag{3.5}$$

where $\{\alpha_n\}$ is a sequence in $(0,1]$ and $\{c_n\}$ is a sequence in $(0,1)$ satisfying conditions (C1)–(C2), where $\gamma_{X,B} = 1$. Then, $\{x_n\}$ converges strongly to $x^* \in$ Zer$(A+B)$, where $x^* = Qfx^*$ and Q is a sunny nonexpansive retraction of C onto Zer$(A+B)$.

Proof. Note T is nonexpansive with $B = I - T$. It follows from Remark 2.2 that the average mapping $T_w = I - wB$ is always nonexpansive for each $w \in (0, \gamma_{X,B})$, where $\gamma_{X,B} = 1$. ∎

Remark 3.3. If $T = I$, then algorithm (3.5) reduces proximal point algorithm studied by Sahu and Yao [16] in the framework of a reflexive Banach space. In case of Hilbert space H, if $fx = u$ and $T = I$, then (3.5) reduces to the proximal point algorithm studied in Song and Yang [18] and Xu [22]. Therefore, Corollary 3.1 extends results of [9, 16, 18, 22] in the context of the inclusion problem ($\mathscr{P}$) in the Banach space setting.

The following result is a generalization of those results concerning with approximation of fixed points of inverse strongly accretive operators.

Corollary 3.2. *Let C be a nonempty closed convex subset of a 2-uniformly smooth Banach space X. Let $\nu > 0$ and $B : C \to X$ be ν-inverse strongly accretive operator such that $R(I - \lambda B) \subseteq C$ for each $\lambda \in (0, \gamma_{X,B})$, where $\gamma_{X,B} = \nu/K^2$ and K is the 2-uniformly smoothness constant of X. Let $A \subseteq X \times X$ an accretive operator such that $\overline{D(A)} \subseteq C \subseteq \bigcap_{t>0} R(I+tA)$ such that Zer$(A+B) \neq \emptyset$. Suppose that C has the fixed-point property for nonexpansive mappings. For given $f \in \Pi_C$ and $x_1 \in C$, let $\{x_n\}$ be a sequence in C defined by (3.1), where $\{\alpha_n\}$ is a sequence in $(0,1]$ and $\{c_n\}$ is a sequence in $(0,1)$ satisfying conditions (C1)-(C2). Then, $\{x_n\}$ converges strongly to $x^* \in$ Zer$(A+B)$, where $x^* = Qfx^*$ and Q is a sunny nonexpansive retraction of C onto Zer$(A+B)$.*

Proof. It follows from Example 2.1 that B has the property ($\mathscr{N}$) on $(0, \gamma_{X,B})$, where $\gamma_{X,B} = \nu/K^2$. ∎

Noticing that, for a λ-strictly pseudocontractive mapping $T : C \to C$ with $B = I - T$, the average mapping $T_w = I - wB$ is nonexpansive under some geometric conditions. From Example 2.2, we are able to derive the following result.

Corollary 3.3. *Let $p > 1$ be a given real number and X be a p-uniformly convex Banach space with a uniformly Gâteaux differentiable norm. Let C be a nonempty closed convex subset of X and $A \subseteq X \times X$ an accretive operator such that $\overline{D(A)} \subseteq$*

$C \subseteq \bigcap_{t>0} R(I + tA)$. *Let* $T : C \to C$ *be* λ*-strictly pseudocontractive with* $\lambda < \min\left\{1, \frac{c_p}{2^{(p-2)}}\right\}$ *and* $B = I - T$ *such that Zer*$(A + B) \neq \emptyset$. *Let* $\gamma_{X,B} = 1 - \frac{c_p \lambda}{2^{(p-2)}}$. *For given* $f \in \Pi_C$ *and* $x_1 \in C$, *let* $\{x_n\}$ *be a sequence in* C *defined by (3.5), where* $\{\alpha_n\}$ *is a sequence in* $(0, 1]$ *and* $\{c_n\}$ *is a sequence in* $(0, \gamma_{X,B})$ *satisfying conditions (C1)–(C2). Then* $\{x_n\}$ *converges strongly to* $x^* \in$ *Zer*$(A+B)$, *where* $x^* = Qfx^*$ *and* Q *is a sunny nonexpansive retraction of* C *onto Zer*$(A + B)$.

If A is the operator constantly zero, then Corollary 3.3 yields

Corollary 3.4. *Let* $p > 1$ *be a given real number and* X *be a* p*-uniformly convex Banach space with a uniformly Gâteaux differentiable norm. Let* C *be a nonempty closed convex subset of* X *and let* $T : C \to C$ *be* λ*-strictly pseudocontractive with* $\lambda < \min\left\{1, \frac{c_p}{2^{(p-2)}}\right\}$ *and* $B = I - T$ *such that Fix*$(T) \neq \emptyset$. *Let* $\gamma_{X,B} = 1 - \frac{c_p \lambda}{2^{(p-2)}}$. *For given* $f \in \Pi_C$ *and* $x_1 \in C$, *let* $\{x_n\}$ *be a sequence in* C *defined by*

$$x_{n+1} = (I - c_n B)((1 - \alpha_n)x_n + \alpha_n f x_n), \quad \textit{for all } n \in \mathbb{N},$$

where $\{\alpha_n\}$ *is a sequence in* $(0, 1]$ *and* $\{c_n\}$ *is a sequence in* $(0, \gamma_{X,B})$ *satisfying conditions (C1)–(C2). Then,* $\{x_n\}$ *converges strongly to* $x^* \in$ *Fix*(T), *where* $x^* = Qfx^*$ *and* Q *is a sunny nonexpansive retraction of* C *onto Fix*(T).

Corollary 3.5. *Let* X *be a reflexive Banach space with a uniformly Gâteaux differentiable norm such that* X *has the fixed point property for nonexpansive mappings and* $A \subseteq X \times X$ *be an* m*-accretive operator such that* $A^{-1}0 \neq \emptyset$. *For given* $f \in \Pi_X$ *and* $x_1 \in X$, *let* $\{x_n\}$ *be a sequence in* X *generated by*

$$x_{n+1} = J^A_{c_n}((1 - \alpha_n)x_n + \alpha_n f x_n), \quad \textit{for all } n \in \mathbb{N},$$

where $\{\alpha_n\}$ *is a sequence in* $(0, 1]$ *and* $\{c_n\}$ *is a regularization sequence in* $(0, \gamma_X)$ *satisfying conditions (C1)–(C2), where* $\gamma_X = \infty$. *Then,* $\{x_n\}$ *converges strongly to* $x^* \in A^{-1}0$, *where* $x^* = Qfx^*$ *and* Q *is a sunny nonexpansive retraction of* X *onto* $A^{-1}0$.

4. Algorithms on General Banach Spaces

Let X be a Banach space. Recall that a mapping $T : D(T) \to X$ is said to be ϕ*-expansive* if there exists a continuous or nondecreasing function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(0) = 0$ and $\phi(t) > 0$ for $t > 0$ such that

$$\|Tx - Ty\| \geq \phi(\|x - y\|), \quad \text{for all } x, y \in D(T).$$

Here we shall use the following result, which can be found in [6].

Theorem 4.1. [6, Theorem 3.2]. *Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ be a mapping such that*

(i) *T is a Φ-1-set contraction,*

(ii) *there exist $R > 0$ and $x_0 \in C$ such that $Tx - x_0 \neq \lambda(x - x_0)$ for all $x \cap S_R(x_0)$ and for all $\lambda > 1$,*

where $S_R(x_0)$ is the closed ball with radius R and center $x_0 \in X$. Then, there exists an approximating fixed point sequence $\{x_n\}$ of T. Furthermore, if

(iii) *$I - T : C \to R(I - T)$ is ϕ-expansive,*

then, T has a unique fixed point $x^* \in C$ and $x_n \to x^*$ as $n \to \infty$.

Now our purpose is to establish strong convergence theorems for the unique solution of inclusion problem ($\mathscr{P}$) in general Banach spaces.

Theorem 4.2. *Let C be a nonempty closed convex subset of a Banach space X. Let $A \subseteq X \times X$ be an accretive operator such that $\overline{D(A)} \subseteq C \subseteq \bigcap_{t>0} R(I + tA)$ and $B : C \to X$ an operator such that Zer$(A + B) \neq \emptyset$ and B has the property ($\mathscr{N}$) on $(0, \gamma_{X,B})$. Let $r \in (0, \gamma_{X,B})$ and $J_r^{A,B} : C \to C$ be an operator defined by (2.5) such that $J_r^{A,B}$ satisfies the following conditions:*

$(R1)$ *$I - J_r^{A,B} : C \to R(I - J_r^{A,B})$ is ϕ-expansive,*

$(R2)$ *there exist $R > 0$ and $x_0 \in C$ such that $J_r^{A,B}x - x_0 \neq \lambda(x - x_0)$ for all $x \cap S_R(x_0)$ and for all $\lambda > 1$.*

For given $u, x_1 \in C$, let $\{x_n\}$ be a sequence in C generated by

$$x_{n+1} = J_r^{A,B}((1 - \alpha_n)x_n + \alpha_n u), \quad \textit{for all } n \in \mathbb{N}, \tag{4.1}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1]$ satisfying condition (C1). Then, $\{x_n\}$ converges strongly to the unique solution of inclusion problem ($\mathscr{P}$).

Proof. Since $J_r^{A,B}$ is nonexpansive, then $J_r^{A,B}$ is 1-set contractive for the Kuratowskii measure of noncompactness. Thus, $J_r^{A,B}$ satisfies the assumptions of Theorem 4.1. Therefore, there exists a unique fixed point $x^* \in C$ for $J_r^{A,B}$. It is easy to see from (3.3) that $||x_n - J_r^{A,B}x_n|| \to 0$, i.e., $\{x_n\}$ is an approximating fixed point sequence of $J_r^{A,B}$. Therefore, from Theorem 4.1 the sequence $\{x_n\}$ converges to the unique fixed point x^* of $J_r^{A,B}$. ■

Theorem 4.3. *Let X be a Banach space, $A \subseteq X \times X$ be an accretive operator and $B : X \to X$ an operator such that Zer$(A + B) \neq \emptyset$ and B has the property ($\mathscr{N}$) on $(0, \gamma_{X,B})$. Let $r \in (0, \gamma_{X,B})$ and $J_r^{A,B} : X \to X$ be an operator defined by $J_r^{A,B} = J_r^A(I - rB)$ such that $J_r^{A,B}$ satisfies the following conditions:*

$(R1)$ *$I - J_r^{A,B} : X \to R(I - J_r^{A,B})$ is ϕ-expansive,*

$(R2)$ *there exist* $R > 0$ *and* $x_0 \in X$ *such that* $J_r^{A,B}x - x_0 \neq \lambda(x - x_0)$ *for all* $x \cap S_R(x_0)$ *and for all* $\lambda > 1$.

For given $u, z_1 \in X$, *let* $\{z_n\}$ *be a sequence in* X *generated by*

$$\begin{cases} u_n = & (1-\alpha_n)z_n + \alpha_n u; \\ z_{n+1} = & J_r^A(u_n - r(Bu_n + b_n)) + e_n \text{ for all } n \in \mathbb{N}, \end{cases} \tag{4.2}$$

where $\{\alpha_n\}$ *is a sequence in* $(0, 1]$ *satisfying conditions (C1), (C3) and* $(C4)'$:

$(C4)'$ $\sum_{n=1}^{\infty} \|b_n\| < \infty$ *or* $\lim\limits_{n\to\infty} \|b_n\|/\alpha_n = 0$.

Then, $\{z_n\}$ *converges strongly to the unique solution of inclusion problem* ($\mathscr{P}$).

Proof. For $x_1 = z_1 \in X$, let $\{x_n\}$ be the iterative sequence in X defined by (4.1). It follows from Theorem 4.2 that $\{x_n\}$ converges strongly to $x^* \in X$, which is a unique solution of inclusion problem ($\mathscr{P}$). Set $y_n := (1-\alpha_n)x_n + \alpha_n u$. From (4.1) and (4.2), we have

$$\begin{aligned} \|z_{n+1} - x_{n+1}\| &= \|J_r^A(u_n - r(Bu_n + b_n)) + e_n - J_r^A(y_n - rBy_n)\| \\ &\leq \|(I - rB)u_n - b_n r - (I - rB)y_n\| + \|e_n\| \\ &\leq \|(I - rB)u_n - (I - rB)y_n\| + r\|b_n\| + \|e_n\| \\ &\leq \|u_n - y_n\| + r\|b_n\| + \|e_n\| \\ &= (1-\alpha_n)\|z_n - x_n\| + r\|b_n\| + \|e_n\| \text{ for all } n \in \mathbb{N}. \end{aligned}$$

By [23, Lemma 2.5], we have $\|z_n - x_n\| \to 0$. Therefore, $\{z_n\}$ converges strongly to x^*. ∎

Note that if T is a nonexpansive mapping from a nonempty closed convex subset C of a Banach space X into itself and if $B = I - T$, then, from Remark 2.2, we conclude B has the property ($\mathscr{N}$) on $(0, \gamma_{X,B})$, where $\gamma_{X,B} = 1$.

Corollary 4.6. *Let* C *be a nonempty closed convex subset of Banach space* X *and* $A \subseteq X \times X$ *an accretive operator such that* $\overline{D(A)} \subseteq C \subseteq \bigcap_{t>0} R(I + tA)$. *Let* $T : C \to C$ *be a nonexpansive mapping with* $B = I - T$ *such that* Zer$(A + B) \neq \emptyset$. *Let* $r \in (0, 1)$ *and* $J_r^{A,B} : C \to C$ *be an operator defined by (2.5) such that* $J_r^{A,B}$ *satisfies the following conditions:*

$(R1)$ $I - J_r^{A,B} : C \to R(I - J_r^{A,B})$ *is* ϕ*-expansive,*

$(R2)$ *there exist* $R > 0$ *and* $x_0 \in C$ *such that* $J_r^{A,B}x - x_0 \neq \lambda(x - x_0)$ *for all* $x \cap S_R(x_0)$ *and for all* $\lambda > 1$.

For given $u, x_1 \in C$, *let* $\{x_n\}$ *be a sequence in* C *generated by (4.1), where* $\{\alpha_n\}$ *is a sequence in* $(0, 1]$ *satisfying condition (C1). Then,* $\{x_n\}$ *converges strongly to the unique solution of inclusion problem* ($\mathscr{P}$).

Recently, Falset and Prez [6] proved that the sequence $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n x_1 + (1-\alpha_n)Tx_n \text{ for all } n \in \mathbb{N},$$

converges strongly to the unique fixed point of a nonexpansive mapping T in a general Banach space under suitable mapping conditions. In order to find the unique solution of the inclusion problem ($\mathscr{P}$) when $A \subseteq X \times X$ is an accretive operator and T is nonexpansive with $B = I - T$, we infer that Corollary 4.6 is new and a more general result in an arbitrary Banach space.

5. Applications

5.1. Application to mixed variational inequalities

The following basic result of subdifferentials can be found in [25].

Lemma 5.7. [25, Theorem 3.1.11]. *Let $\psi \in \Gamma_0(H)$. Then, $\partial\psi$ is maximal monotone.*

Let $\psi \in \Gamma_0(H)$ with subdifferential $\partial\psi$. It is well known that

$$\psi(z) = \min_{x \in H} \psi(x) \Leftrightarrow 0 \in \partial\psi(z). \tag{5.1}$$

Noticing that $\partial\psi$ is maximal monotone and $prox_\psi = \partial\psi$ is Moreau's proximity operator [14]. Thus, $(I + c\partial\psi_1)^{-1} = prox_{c\psi}$ for some $c > 0$.

As a special case of problem (1.1), the mixed variational inequality problem (1.2), can be solved via Lemma 5.7 and algorithm (3.5) as follows.

Theorem 5.1. *Let C be a nonempty closed convex subset of Hilbert space H and $\psi \in \Gamma_0(H)$ such that $\overline{D(\partial\psi)} \subseteq C \subseteq \bigcap_{h>0} R(I+h\partial\psi)$. Let $\nu > 0$ and $B : C \to H$ be ν-inverse strongly monotone operator such that $(\partial\psi+B)^{-1}0 \neq \emptyset$ and $R(I-\xi B) \subseteq C$ for each $\xi \in (0, 2\nu)$. Let $\{x_n\}$ be a sequence in C generated by*

$$x_{n+1} = prox_{c_n\psi}(I - c_n B)((1-\alpha_n)x_n + \alpha_n f x_n), \quad \textit{for all } n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in $(0, 1]$ and $\{c_n\}$ is a regularization sequence in $(0, \gamma_{X,B})$ satisfying conditions (C1)–(C2), where $\gamma_{X,B} = 2\nu$. Then, $\{x_n\}$ converges strongly to x^*, where $x^* = Proj_{(prox_\psi+B)^{-1}0} f x^*$.

Proof. Note B is ν-inverse-strongly monotone. It follows from Lemma 2.5 that B has the property ($\mathscr{N}$) on $(0, \gamma_{X,B})$, where $\gamma_{X,B} = 2\nu$. Therefore, result follows from Theorem 3.1. ∎

5.2. Application to nonsmooth convex optimization

Let C be a nonempty closed convex subset of a real Hilbert space H. It is well known that if $\psi \in \Gamma_0(H)$ is Gâteaux differentiable at $x \in H$ with gradient $\nabla\psi(x)$, then $\partial\psi(x) = \{\nabla\psi(x)\}$.

Consider the convex optimization problem:

$$\min_{x\in C}(\psi_1(x)+\psi_2(x)), \tag{5.2}$$

where $\psi_1 \in \Gamma_0(H)$ such that ψ_1 is not essentially smooth function and $\psi_2 : H \to \mathbb{R}$ is a convex and differentiable with a L-Lipschitz continuous gradient $\nabla\psi_2$. Denote by Ω the solution set of problem (5.2); that is,

$$\Omega = \{z \in C : \psi_1(z) + \psi_2(z) = \min_{x\in C}(\psi_1(x) + \psi_2(x))\}.$$

Assume that $\Omega \neq \emptyset$.

It is known that if $\psi : H \to (-\infty, \infty]$ is proper, lower semicontinuous and convex and $\varphi : H \to \mathbb{R}$ is continuous and convex, then

$$\partial(\psi + \varphi) = \partial\psi + \partial\varphi. \tag{5.3}$$

It follows from (5.1) and (5.3) that

$$z \in \Omega \Leftrightarrow 0 \in (\partial\psi_1 + \partial\psi_2)z$$

Noticing, from [3], that L-Lipschitz condition of the gradient $\nabla\psi_2$ implies that $\nabla\psi_2$ is $(1/L)$-inverse-strongly monotone. The operator $J_r^{A,B}$ defined by (2.5) is a composition of two nonexpansive self-mappings. Hence, for any $c \in (0, 2/L)$, solutions of problem (5.7) are characterized by the fixed point equation

$$x = \underbrace{prox_{c\psi_1}}\underbrace{(I - c\nabla\psi_2)}x.$$

We now apply Theorem 5.1 for finding numerical solutions of nonsmooth convex optimization problem (5.2).

Theorem 5.2. *Let C be a nonempty closed convex subset of Hilbert space H. Let $\psi_1 \in \Gamma_0(H)$ and $\psi_2 : H \to \mathbb{R}$ a convex and differentiable with a L-Lipschitz continuous gradient $B = \nabla\psi_2$ such that $R(I - \xi\nabla\psi_2) \subseteq C$ for each $\xi \in (0, 2/L)$. Assume that $\overline{D(\partial\psi_1)} \subseteq C \subseteq \bigcap_{h>0} R(I + h\partial\psi_1)$ and $(\partial\psi_1 + \nabla\psi_2)^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be a sequence in C generated by*

$$x_{n+1} = prox_{c_n\psi_1}(I - c_n\nabla\psi_2)((1-\alpha_n)x_n + \alpha_n f x_n) \text{ for all } n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in $(0, 1]$ and $\{c_n\}$ is a regularization sequence in $(0, \gamma_{X,B})$ satisfying conditions (C1)–(C2), where $\gamma_{X,B} = 2/L$. Then, $\{x_n\}$ converges strongly to x^*, where $x^* = Proj_{(prox_{\psi_1} + \nabla\psi_2)^{-1}0} f x^*$

Proof. Note $\nabla\psi_2$ is $(1/L)$-inverse-strongly monotone. It follows from Lemma 2.5 that $B = \nabla\psi_2$ has the property ($\mathscr{N}$) on $(0, \gamma_{X,B})$, where $\gamma_{X,B} = 2/L$. Therefore, result follows from Theorem 5.1. ∎

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