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NEW VERSIONS OF REVERSE YOUNG AND HEINZ MEAN INEQUALITIES WITH THE KANTOROVICH CONSTANT

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Abstract. We show new versions of reverse Young inequalities by virtue of the Kantorovich constant, and utilizing the new reverse Young inequalities we give the reverses of the weighted arithmetic-geometric and geometric-harmonic mean inequalities for two positive operators. Also, new versions of reverse Young and Heinz mean inequalities for unitarily invariant norms are established.

1. INTRODUCTION

In what follows, let $M_n(\mathrm{C})$ be the space of all $n \times n$ complex matrices. For Hermitian matrices $A, B \in M_n(\mathrm{C})$, we write that $A \geq 0$ if A is positive semidefinite, $A > 0$ if A is positive definite, and $A \geq B$ if $A - B \geq 0$. The Hilbert-Schmidt (or Frobenius) norm of $A = [a_{ij}] \in M_n(\mathrm{C})$ is denoted by $\|A\|_F = \left(\sum\limits_{j=1}^{n} s_j^2(A)\right)^{\frac{1}{2}}$, where $s_1(A) \geq s_2(A) \geq \cdots \geq s_n(A)$ are the singular values of A. It is known that the Hilbert-Schmidt norm is unitarily invariant. For convenience, we often use the following notations:

$$\begin{aligned}
a\nabla_\mu b \; &= (1-\mu)a + \mu b,\ a!_\mu b = \left((1-\mu)a^{-1} + \mu b^{-1}\right)^{-1},\\
A\nabla_\mu B &= (1-\mu)A + \mu B,\ A\#_\mu B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\mu A^{\frac{1}{2}},\\
A!_\mu B \; &= \left((1-\mu)A^{-1} + \mu B^{-1}\right)^{-1},\ H_\mu(A,B) = \frac{A\#_\mu B + A\#_{1-\mu}B}{2},
\end{aligned}$$

where A, B are positive operators on a Hilbert space. When $\mu = \frac{1}{2}$, we write $A\nabla B$, $A\#B$, $A!B$, $a\nabla b$, $a!b$ and $H(A,B)$ for brevity, respectively.

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As is known to all, the famous Young inequality for scalars says that if $a, b > 0$ and $\mu \in [0,1]$, then

$$a^{1-\mu}b^{\mu} \leq a\nabla_{\mu}b \tag{1.1}$$

with equality if and only if $a = b$. The inequality (1.1) is also called μ-weighted arithmetic-geometric mean inequality. Replacing a, b by a^{-1}, b^{-1} in the Young inequality, respectively, we get the μ-weighted geometric-harmonic mean inequality

$$a^{1-\mu}b^{\mu} \geq a!_{\mu}b.$$

Tominaga [8] had proved a reverse Young inequality with the Specht's ratio

$$a\nabla_{\mu}b \leq S(h)a^{1-\mu}b^{\mu}, \tag{1.2}$$

where $a, b > 0$, $\mu \in [0,1]$, $h = \dfrac{b}{a}$, and the Specht's ratio [6] is denoted by

$$S(t) = \frac{t^{\frac{1}{t-1}}}{e\ln t^{\frac{1}{t-1}}}, for\, t > 0, t \neq 1, and\, S(1) = \lim_{t\to 1} S(t) = 1,$$

which has properties $S(t) = S\left(\frac{1}{t}\right) > 1$, and $S(t)$ is monotone increasing on $(1,\infty)$ and monotone decreasing on $(0,1)$.

Zou et. al.[4] refined Young inequality with the Kantorovich constant, and obtained the following results:

$$a\nabla_{\mu}b \geq \mathrm{K}(h,2)^{r}a^{1-\mu}b^{\mu}, \tag{1.3}$$

$$a^{1-\mu}b^{\mu} \geq \mathrm{K}(h,2)^{r}a!_{\mu}b \tag{1.4}$$

for all $\mu \in [0,1]$, where $r = \min\{\mu, 1-\mu\}$ and $h = \dfrac{b}{a}$. It admits two operator extensions

$$A\nabla_{\mu}B \geq \mathrm{K}(h,2)^{r}A\#_{\mu}B, \tag{1.5}$$

$$A\#_{\mu}B \geq \mathrm{K}(h,2)^{r}A!_{\mu}B \tag{1.6}$$

for positive operators A, B on a Hilbert space and the Kantorovich constant is denoted by

$$\mathrm{K}(t,2) = \frac{(t+1)^2}{4t}, for\, t > 0, and\, \mathrm{K}(1,2) = 1,$$

which has properties $\mathrm{K}(t, 2) = \mathrm{K}\left(\frac{1}{t}, 2\right) \geq 1 (t > 0)$, and $\mathrm{K}(t, 2)$ is monotone increasing on $[1, \infty)$ and monotone decreasing on $(0, 1]$.

Kittaneh and Manasrah [1, 2] improved the Young inequality, and obtained the following inequalities:

$$a\nabla_\mu b - r(\sqrt{a} - \sqrt{b})^2 \geq a^{1-\mu} b^\mu, \tag{1.7}$$

$$a\nabla_\mu b - R(\sqrt{a} - \sqrt{b})^2 \leq a^{1-\mu} b^\mu. \tag{1.8}$$

where $a, b > 0$, $\mu \in [0, 1]$, $R = \max\{1 - \mu, \mu\}$ and $r = \min\{\mu, 1 - \mu\}$.

Hirzallah and Kittaneh [9] obtained another refinement of the inequality (1.1):

$$(a\nabla_\mu b)^2 - r^2(a - b)^2 \geq (a^{1-\mu} b^\mu)^2. \tag{1.9}$$

The Heinz mean is defined as

$$H_\mu(a, b) = \frac{a^{1-\mu} b^\mu + a^\mu b^{1-\mu}}{2}$$

for $a, b \geq 0$ and $\mu \in [0, 1]$. It's easy to see that

$$\sqrt{ab} \leq H_\mu(a, b) \leq \frac{a + b}{2}.$$

The research of the Young and Heinz mean inequalities is interesting. For more results on the Young and Heinz mean inequalities, see[7], [11]and [12].

In this paper, we will present reverse of the improved Young inequalities (1.3) and (1.4) and obtain new versions of reverse ratio Young inequality with the Kantorovich constant. By virtue of these reverse inequalities, new versions of reverse Young and Heinz mean inequalities for operators and unitarily invariant norms are established.

2. New Versions of the Reverse Young Inequalities

In this section, we present reverses of the improved Young inequalities (1.3) and (1.4) and establish new versions of reverse ratio Young inequalities with the Kantorovich constant which are different from the improved inequalities (1.7) and (1.8). We also obtain some refinements of the Heinz mean inequalities.

Firstly, we will need the following lemma due to Mitroi [3, Corollary 3.1] to obtain our results. For more related work see [5].

Lemma 2.1. *For $i = 1, 2, \cdots, n$, we consider x_i belong to a fixed closed interval I, $p_i \geq 0$ with $\sum_{i=1}^{n} p_i = 1$ and $\bar{p} = \max\{p_1, p_2, \cdots, p_n\}$. If f is a convex function on I, then*

$$\sum_{i=1}^{n} p_i f(x_i) - f(\sum_{i=1}^{n} p_i x_i) \leq n\bar{p} \left[\sum_{i=1}^{n} \frac{1}{n} f(x_i) - f(\sum_{i=1}^{n} \frac{1}{n} x_i) \right].$$

If we take $f(x) = -\log x$ in Lemma 2.1, then we have the following:

Corollary 2.1. *If* $x_i \in I \subseteq (0, \infty)$, $p_i \geq 0\,(i = 1, 2, \cdots, n)$ *with* $\sum\limits_{i=1}^{n} p_i = 1$ *and* $\bar{p} = \max\{p_1, p_2, \cdots, p_n\}$, *then*

$$\frac{\sum\limits_{i=1}^{n} p_i x_i}{\prod\limits_{i=1}^{n} x_i^{p_i}} \leq \left(\frac{\frac{1}{n}\sum\limits_{i=1}^{n} x_i}{\prod\limits_{i=1}^{n} x_i^{\frac{1}{n}}} \right)^{n\bar{p}}.$$

We can get a special form when $n = 2$ in the above inequality, which is an extension of (1.2).

Corollary 2.2. *If* $a, b > 0,\ \mu \in [0, 1]$, *then*

$$a \nabla_\mu b \leq \mathrm{K}(h, 2)^R a^{1-\mu} b^\mu, \tag{2.1}$$

where $R = \max\{1 - \mu, \mu\}$ *and* $h = \dfrac{b}{a}$.

Remark 2.1. It is easy to see that the right-hand side of the inequality (2.1) and the corresponding side of the inequality (1.2) can not be compared, because the value of $\mathrm{K}(\sqrt{h}, 2)^R$ will change with R, neither the inequality (2.1) nor (1.2) is uniformly better than the other.

Replacing $a,\ b$ by $a^{-1},\ b^{-1}$ in the inequality (2.1), respectively, we have the counterpart of the inequality (2.1).

Corollary 2.3. *If* $a, b > 0,\ \mu \in [0, 1]$, *then*

$$a^{1-\mu} b^\mu \leq \mathrm{K}(h, 2)^R a !_\mu b. \tag{2.2}$$

We now show the reverse ratio inequality of the refined Young inequality (1.7).

Theorem 2.1. *If* $a, b > 0$, *then for any* $\mu \in [0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$, *the inequality*

$$(1 - \mu)\, a + \mu b - r(\sqrt{a} - \sqrt{b})^2 \leq \mathrm{K}(\sqrt{h}, 2)^{R'} a^{1-\mu} b^\mu \tag{2.3}$$

holds, where $h = \dfrac{b}{a}, r = \min\{\mu, 1 - \mu\}$ *and* $R' = \max\{2r, 1 - 2r\}$.

Proof. Firstly, we consider the case $\mu \in \left[0, \frac{1}{2}\right)$, by the inequality (2.1), we have

$$\begin{aligned}(1 - \mu)\, a + \mu b - \mu(\sqrt{a} - \sqrt{b})^2 &= (1 - 2\mu)\, a + 2\mu\sqrt{ab} \\ &\leq \mathrm{K}(\sqrt{h}, 2)^{R'} a^{1-\mu} b^\mu.\end{aligned}$$

Conversely, if $\mu \in \left(\frac{1}{2}, 1\right]$, then we have

$$\begin{aligned}(1-\mu)\,a+\mu b-(1-\mu)\,(\sqrt{a}-\sqrt{b})^2 &= (2\mu-1)\,b+2\,(1-\mu)\,\sqrt{ab}\\ &\le \mathrm{K}(\sqrt{h},2)^{R'}a^{1-\mu}b^{\mu}.\end{aligned}$$

From what has been discussed above, for any $\mu \in [0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$, the inequality

$$(1-\mu)\,a+\mu b-r(\sqrt{a}-\sqrt{b})^2 \le \mathrm{K}(\sqrt{h},2)^{R'}a^{1-\mu}b^{\mu}$$

always holdes. ∎

Similar to Theorem 2.1, by the inequality (1.2), it's easy to get the following

Theorem 2.2. *If $a, b > 0$, then for any $\mu \in [0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$, then*

$$(1-\mu)\,a+\mu b-r(\sqrt{a}-\sqrt{b})^2 \le S(\sqrt{h})a^{1-\mu}b^{\mu}. \tag{2.4}$$

The inequality (2.4) was obtained by S. Furuichi with a different technique, but our method is more transparent and simpler than the one given in [10].

Remark 2.2. It is easy to see that the right-hand side of the inequality (2.3) and the corresponding side of the inequality (2.4) can not be compared, because the value of $\mathrm{K}(\sqrt{h},2)^{R'}$ will change with R', so neither the inequality (2.3) nor (2.4) is uniformly better than the other (But the inequality (2.3) is indeed a new version of reverse ratio Young inequality).

The reverse ratio inequality (2.3) can be presented as

$$(1-\mu)\,a+\mu b \le \mathrm{K}(\sqrt{h},2)^{R'}a^{1-\mu}b^{\mu}+r(\sqrt{a}-\sqrt{b})^2.$$

So $\mathrm{K}(\sqrt{h},2)^{R'}a^{1-\mu}b^{\mu}+r(\sqrt{a}-\sqrt{b})^2$ can be considered as the upper bound of μ-weighted arithmetic mean. The reverse Young inequality (1.8) can also be considered as the upper bound of μ-weighted arithmetic mean. But the two upper bounds cannot be compared.

Next, we will prove reverse ratio inequality of the refined Young inequality (1.9).

Theorem 2.3. *If $a, b > 0$, then for any $\mu \in [0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$, the inequality*

$$((1-\mu)\,a+\mu b)^2-r^2(a-b)^2 \le \mathrm{K}(h,2)^{R'}\left(a^{1-\mu}b^{\mu}\right)^2 \tag{2.5}$$

holds, where $h=\dfrac{b}{a}$, $r=\min\{\mu,1-\mu\}$ and $R'=\max\{2r,1-2r\}$.

Proof. The proof is similar to Theorem 2.1, so we omit it. ■

If we replace a by a^2 and b by b^2, the inequality (2.3) can be rewritten as the following form

$$(1-\mu)\,a^2+\mu b^2-r(a-b)^2\le \mathrm{K}(h,2)^{R'}\left(a^{1-\mu}b^{\mu}\right)^2. \tag{2.6}$$

A reverse of the Heinz mean inequality can be sated as follows:

Theorem 2.4. *If* $a,b>0$ *and* $\mu\in[0,\frac{1}{2})\cup(\frac{1}{2},1]$, *then*

$$\mathrm{K}(h,2)^{R'}\left(a^{1-\mu}b^{\mu}+a^{\mu}b^{1-\mu}\right)^2+2r(a-b)^2\ge(a+b)^2. \tag{2.7}$$

where $h=\dfrac{b}{a}$, $r=\min\{\mu,1-\mu\}$ *and* $R'=\max\{2r,1-2r\}$.

Proof. By the inequality (2.6) and $\mathrm{K}(h,2)^{R'}\ge 1$, we have

$$\begin{aligned}
&(a+b)^2-\mathrm{K}(h,2)^{R'}\left(a^{1-\mu}b^{\mu}+a^{\mu}b^{1-\mu}\right)^2\\
=\;&a^2+b^2+2ab-\mathrm{K}(h,2)^{R'}\left((a^{1-\mu}b^{\mu})^2+(a^{\mu}b^{1-\mu})^2+2ab\right)\\
=\;&(1-\mu)\,a^2+\mu b^2-\mathrm{K}(h,2)^{R'}(a^{1-\mu}b^{\mu})^2+\mu a^2+(1-\mu)\,b^2\\
&-\mathrm{K}(h,2)^{R'}(a^{\mu}b^{1-\mu})^2+\left(1-\mathrm{K}(h,2)^{R'}\right)2ab\\
\le\;&r(a-b)^2+r(a-b)^2+0\\
=\;&2r(a-b)^2.
\end{aligned}$$

Hence

$$\mathrm{K}(h,2)^{R'}\left(a^{1-\mu}b^{\mu}+a^{\mu}b^{1-\mu}\right)^2+2r(a-b)^2\ge(a+b)^2.$$ ■

3. Reverse Young and Heinz Mean Inequalities for Operators

In this section, the operator versions of these inequalities proved in section 2 are established.

Theorem 3.1. *Suppose two invertible positive operators* A *and* B, I *represents an identity operator and positive real number* m,m',M,M' *satisfy either of the following conditions:*

(i) $0<mI\le A\le m'I<M'I\le B\le MI.$
(ii) $0<mI\le B\le m'I<M'I\le A\le MI.$

then

$$A\nabla_{\mu}B\le \mathrm{K}(h,2)^{R}A\#_{\mu}B \tag{3.1}$$

for all $\mu \in [0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$, *where* $R = \max\{1-\mu, \mu\}$, $h = \frac{M}{m}$ *and* $h' = \frac{M'}{m'}$.

Proof. From the inequality (2.1), we have

$$(1-\mu) + \mu x \leq \mathrm{K}(x, 2)^R x^\mu$$

for any $x > 0$, and hence

$$(1-\mu)\, I + \mu X \leq \max_{h' \leq x \leq h} \mathrm{K}(x, 2)^R X^\mu$$

for the positive operator X such that $0 < h'I \leq X \leq hI$.

Substituting $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ for X in the above inequality:

In the case of i), $I < h'I = \frac{M'}{m'}I \leq A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \leq \frac{M}{m}I = hI$, we have

$$(1-\mu)\, I + \mu A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \leq \max_{h' \leq x \leq h} \mathrm{K}(x, 2)^R (A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\mu.$$

Since the Kantorovich constant $\mathrm{K}(t, 2) = \dfrac{(t+1)^2}{4t}$ is an increasing function for $t > 1$, then

$$(1-\mu)\, I + \mu A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \leq \mathrm{K}(h, 2)^R (A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\mu. \tag{3.2}$$

In the case of ii), $0 < \frac{1}{h}I \leq A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \leq \frac{1}{h'}I < I$, we have

$$(1-\mu)\, I + \mu A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \leq \max_{\frac{1}{h} \leq x \leq \frac{1}{h'}} \mathrm{K}(x, 2)^R (A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\mu.$$

Since the Kantorovich constant $\mathrm{K}(t, 2) = \dfrac{(t+1)^2}{4t}$ is an decreasing function for $0 < t < 1$, then

$$(1-\mu)\, I + \mu A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \leq \mathrm{K}(\frac{1}{h}, 2)^R (A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\mu, \tag{3.3}$$

Multiplying both sides by $A^{\frac{1}{2}}$ to the inequalities (3.2) and (3.3) and using $\mathrm{K}(\frac{1}{h}, 2) = \mathrm{K}(h, 2)$ for $h > 0$, we can deduce

$$A \nabla_\mu B \leq \mathrm{K}(h, 2)^R A \#_\mu B. \qquad \blacksquare$$

By replacing A, B by A^{-1}, B^{-1} in the inequality (3.1), respectively, then the reverse weighted geometric-harmonic operator mean inequality can be obtained.

Corollary 3.1. *Assume the conditions as in Theorem 3.1, then*

$$A \#_\mu B \leq \mathrm{K}(h, 2)^R A !_\mu B.$$

By virtue of Theorem 2.1, we have the reverse ratio inequality of the refined Young inequality (1.7) for positive operators.

Theorem 3.2. *Suppose two invertible positive operators A and B, I represents an identity operator and positive real number m, m', M, M' satisfy either of the following conditions:*

(i) $0 < mI \leq A \leq m'I < M'I \leq B \leq MI.$

(ii) $0 < mI \leq B \leq m'I < M'I \leq A \leq MI.$

then

$$(1-\mu)\,A + \mu B - 2r(A\nabla B - A\#B) \leq \mathrm{K}(\sqrt{h}, 2)^{R'} A\#_{\mu}B \tag{3.4}$$

for all $\mu \in [0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$, *where* $r = \min\{\mu, 1-\mu\}$, $R' = \max\{2r, 1-2r\}$, $h = \frac{M}{m}$ *and* $h' = \frac{M'}{m'}$.

Proof. From Theorem 2.1, we have

$$(1-\mu) + \mu x - r(1-\sqrt{x})^2 \leq \mathrm{K}(\sqrt{h}, 2)^{R'} x^{\mu}$$

for any $x > 0$, hence

$$(1-\mu)\,I + \mu X - r(I - X^{\frac{1}{2}})^2 \leq \mathrm{K}(\sqrt{h}, 2)^{R'} X^{\mu}$$

and

$$(1-\mu)\,I + \mu X - r(I - 2X^{\frac{1}{2}} + X) \leq \mathrm{K}(\sqrt{h}, 2)^{R'} X^{\mu}$$

for the positive operator X such that $0 < h'I \leq X \leq hI$.

Substituting $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ for X in the above inequality, by the similar process of Theorem 3.1, we have

$$\begin{aligned} &(1-\mu)\,I + \mu A^{-\frac{1}{2}}BA^{-\frac{1}{2}} - r[I - 2(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}} + A^{-\frac{1}{2}}BA^{-\frac{1}{2}}] \\ &\leq \mathrm{K}(\sqrt{h}, 2)^{R'}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\mu}. \end{aligned} \tag{3.5}$$

Multiplying both sides by $A^{\frac{1}{2}}$ to the inequality (3.5), we can deduce

$$(1-\mu)\,A + \mu B - 2r(A\nabla B - A\#B) \leq \mathrm{K}(\sqrt{h}, 2)^{R'} A\#_{\mu}B. \qquad \blacksquare$$

Based on the inequality (2.4), we have

Corollary 3.2. *Assume the conditions as in Theorem 3.2, then*

$$(1-\mu)\,A + \mu B - 2r(A\nabla B - A\#B) \leq S(\sqrt{h}) A\#_{\mu}B.$$

As direct consequences of Corollary 2.2 and Corollary 2.3, we have several inequalities with respect to the Heinz mean

$$\mathrm{K}(h,2)^{r}\frac{a!_\mu b+b!_\mu a}{2}\le H_\mu(a,b)\le \mathrm{K}(h,2)^{-r}\frac{a+b}{2}, \tag{3.6}$$

$$\mathrm{K}(h,2)^{-R}\frac{a+b}{2}\le H_\mu(a,b)\le \mathrm{K}(h,2)^{R}\frac{a!_\mu b+b!_\mu a}{2}, \tag{3.7}$$

where $R=\max\{1-\mu,\mu\}, r=\min\{1-\mu,\mu\}$ and $h=\dfrac{b}{a}$.

Theorem 3.3. *Suppose two invertible positive operators A and B, I represents an identity operator and positive real number m, m', M, M' satisfy either of the following conditions:*
(i) $0<mI\le A\le m'I<M'I\le B\le MI$.
(ii) $0<mI\le B\le m'I<M'I\le A\le MI$.
then

$$\mathrm{K}(h,2)^{r}\frac{A!_\mu B+B!_\mu A}{2}\le H_\mu(A,B)\le \mathrm{K}(h,2)^{-r}\frac{A+B}{2}, \tag{3.8}$$

$$\mathrm{K}(h,2)^{-R}\frac{A+B}{2}\le H_\mu(A,B)\le \mathrm{K}(h,2)^{R}\frac{A!_\mu B+B!_\mu A}{2} \tag{3.9}$$

for all $\mu\in[0,\frac{1}{2})\cup(\frac{1}{2},1]$, *where* $R=\max\{1-\mu,\mu\}, r=\min\{1-\mu,\mu\}, h=\frac{M}{m}$ *and* $h'=\frac{M'}{m'}$.

Proof. We consider the second one of the inequalities (3.9), from the corresponding one of (3.7), we have

$$H_\mu(1,x)\le \mathrm{K}(h,2)^{R}\frac{\big((1-\mu)+\mu x^{-1}\big)^{-1}+\big((1-\mu)x^{-1}+\mu\big)^{-1}}{2}$$

for any $x>0$, and hence

$$H_\mu(I,X)\le \mathrm{K}(h,2)^{R}\frac{\big((1-\mu)I+\mu X^{-1}\big)^{-1}+\big((1-\mu)X^{-1}+\mu I\big)^{-1}}{2}$$

for the positive operator X such that $0<h'I\le X\le hI$.

Substituting $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ for X in the above inequality, through the similar process of Theorem 3.1, we have

$$\begin{aligned}&H_\mu(I,A^{-\frac{1}{2}}BA^{-\frac{1}{2}})\\ &\le \mathrm{K}(h,2)^{R}\frac{\Big((1-\mu)I+\mu(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{-1}\Big)^{-1}+\Big((1-\mu)(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{-1}+\mu I\Big)^{-1}}{2}.\end{aligned} \tag{3.10}$$

Multiplying both sides by $A^{\frac{1}{2}}$ to the inequality (3.10), we can deduce the second inequality of (3.9).

The rest of the inequalities (3.8) and (3.9) can be proven through the similar method, so we omit it. ∎

As a direct consequence of Theorem 2.1, we have

$$\mathrm{K}(\sqrt{h},2)^{R'}\left(a^{1-\mu}b^{\mu}+a^{\mu}b^{1-\mu}\right)+2r(\sqrt{a}-\sqrt{b})^{2}\geq a+b,$$

and so

$$\mathrm{K}(\sqrt{h},2)^{R'}H_u(a,b)+r(\sqrt{a}-\sqrt{b})^{2}\geq \frac{a+b}{2}. \tag{3.11}$$

Theorem 3.4. *Suppose two invertible positive operators A, B, I represents an identity operator and positive real number m, m', M, M' satisfy either of the following conditions:*

(i) $0 < mI \leq A \leq m'I < M'I \leq B \leq MI.$

(ii) $0 < mI \leq B \leq m'I < M'I \leq A \leq MI.$

then

$$\mathrm{K}(\sqrt{h},2)^{R'}H_u(A,B)+2r(A\nabla B-A\#B)\geq \frac{A+B}{2}$$

for all $\mu\in[0,\frac{1}{2})\cup(\frac{1}{2},1]$, *where* $R'=\max\{1-2r,2r\}, r=\min\{1-\mu,\mu\}, h=\frac{M}{m}$ *and* $h'=\frac{M'}{m'}$.

Proof. By the inequality (3.11) or Theorem 3.2, we can obtain the consequence directly by the similar method. ∎

4. Reverse Ratio Young and Heinz Mean Inequalities for Unitarily Invariant Norms

In the last section, we will discuss the reverse ratio Young inequality (2.5) and Heinz mean inequality (2.7) for unitarily invariant norms.

Based on the refined Young inequality (2.2) and Heinz mean inequality (2.4) in [2], Kittaneh and Manasrah have showed that if $A, B, X \in M_n(\mathrm{C})$ with A and B positive semidefinite matrices and $\mu\in[0,1]$, then

$$\|(1-\mu)AX+\mu XB\|_F^2\leq \left\|A^{1-\mu}XB^{\mu}\right\|_F^2+R^2\left\|AX-XB\right\|_F^2, \tag{4.1}$$

$$\|AX+XB\|_F^2\leq \left\|A^{1-\mu}XB^{\mu}+A^{\mu}XB^{1-\mu}\right\|_F^2+2R\left\|AX-XB\right\|_F^2. \tag{4.2}$$

Two new refined forms of inequalities (4.1) and (4.2) for the Hilbert-Schmidt norm are presented.

Theorem 4.1. *Suppose* $A, B, X \in M_n(\mathrm{C})$ *such that* A *and* B *are two positive definite matrices and satisfy* $0 < mI \leq A, B \leq MI$, *where* I *represents an identity matrix and* $m, M \in \mathrm{R}$. *For any* $\mu \in [0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$, *then we have*

$$\|(1-\mu)AX + \mu XB\|_F^2 \leq \mathrm{K}(h,2)^{R'} \left\|A^{1-\mu}XB^{\mu}\right\|_F^2 + r^2 \|AX - XB\|_F^2 ,$$

where $h = \frac{M}{m}$, $r = \min\{\mu, 1-\mu\}$ *and* $R' = \max\{2r, 1-2r\}$.

Proof. Since A and B are positive definite, it follows by the spectral theorem that there exist unitary matrices $U, V \in M_n(\mathrm{C})$ such that

$$A = U\Lambda_1 U^*, B = V\Lambda_2 V^*,$$

where $\Lambda_1 = diag(\lambda_1, \lambda_2, \cdots, \lambda_n)$, $\Lambda_2 = diag(\nu_1, \nu_2, \cdots, \nu_n)$, $\lambda_i, \nu_i \geq 0$, $i = 1, 2, \cdots, n$.

Let $Y = U^*XV = [y_{ij}]$, $i, j = 1, 2, ..., n$. Then

$$\begin{aligned} (1-\mu)AX + \mu XB &= U((1-\mu)\Lambda_1 Y + \mu Y\Lambda_2)V^* \\ &= U[((1-\mu)\lambda_i + \mu\nu_j)y_{ij}]V^*, \end{aligned}$$

$$AX - XB = U[(\lambda_i - \nu_j)y_{ij}]V^*,$$

and

$$A^{1-\mu}XB^{\mu} = U(\lambda_i^{1-\mu}\nu_j^{\mu}y_{ij})V^*.$$

Now by the inequality (2.5) and the unitarily invariant of the Hilbert-Schmidt norm, we have

$$\begin{aligned} \|(1-\mu)AX + \mu XB\|_F^2 &= \sum_{i,j=1}^{n} ((1-\mu)\lambda_i + \mu\nu_j)^2 |y_{ij}|^2 \\ &\leq \sum_{i,j=1}^{n} \left(\max \mathrm{K}(t_{ij},2)^{R'} (\lambda_i^{1-\mu}\nu_j^{\mu})^2 + r^2(\lambda_i - \nu_j)^2\right) |y_{ij}|^2, \end{aligned}$$

where $t_{ij} = \lambda_i / \nu_j$.

According to the conditions $0 < mI \leq A, B \leq MI$, $\dfrac{m}{M} = \dfrac{1}{h} \leq t_{ij} = \dfrac{\lambda_i}{\nu_j} \leq h = \dfrac{M}{m}$ and the properties of the Kantorovich constant, we can get

$$\begin{aligned} \|(1-\mu)AX + \mu XB\|_F^2 &\leq \sum_{i,j=1}^{n} \left(\mathrm{K}(h,2)^{R'} (\lambda_i^{1-\mu}\nu_j^{\mu})^2 + r^2(\lambda_i - \nu_j)^2\right) |y_{ij}|^2 \\ &= \mathrm{K}(h,2)^{R'} \sum_{i,j=1}^{n} (\lambda_i^{1-\mu}\nu_j^{\mu})^2 |y_{ij}|^2 + r^2 \sum_{i,j=1}^{n} (\lambda_i - \nu_j)^2 |y_{ij}|^2 \\ &= \mathrm{K}(h,2)^{R'} \left\|A^{1-\mu}XB^{\mu}\right\|_F^2 + r^2 \|AX - XB\|_F^2 . \end{aligned}$$

This completes the proof. ■

Theorem 4.2. *Suppose* $A, B, X \in M_n(\mathrm{C})$ *such that* A *and* B *are two positive definite matrices and satisfy* $0 < mI \leq A, B \leq MI$, *where* I *represents an identity matrix and* $m, M \in \mathrm{R}$. *For any* $\mu \in [0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$, *then we have*

$$\|AX + XB\|_F^2 \leq \mathrm{K}(h,2)^{R'} \left\|A^{1-\mu}XB^{\mu} + A^{\mu}XB^{1-\mu}\right\|_F^2 + 2r\,\|AX - XB\|_F^2\,,$$

where $h = \frac{M}{m}$, $r = \min\{\mu, 1-\mu\}$ *and* $R' = \max\{2r, 1-2r\}$.

Proof. Since A and B are positive definite, it follows by the spectral theorem that there exist unitary matrices $U, V \in M_n(\mathrm{C})$ such that

$$A = U\Lambda_1 U^*, B = V\Lambda_2 V^*,$$

where $\Lambda_1 = diag(\lambda_1, \lambda_2, \cdots, \lambda_n)$, $\Lambda_2 = diag(\nu_1, \nu_2, \cdots, \nu_n)$, $\lambda_i, \nu_i \geq 0$, $i = 1, 2, \cdots, n$.

Let $Y = U^*XV = [y_{ij}]$, $i, j = 1, 2, ..., n$. Then

$$A^{1-\mu}XB^{\mu} + A^{\mu}XB^{1-\mu} = U(\Lambda_1^{1-\mu}Y\Lambda_2^{\mu} + \Lambda_1^{\mu}Y\Lambda_2^{1-\mu})V^*.$$

Therefore,

$$\left\|A^{1-\mu}XB^{\mu} + A^{\mu}XB^{1-\mu}\right\|_F^2 = \sum_{i,j=1}^{n} (\lambda_i^{1-\mu}\nu_j^{\mu} + \lambda_i^{\mu}\nu_j^{1-\mu})^2 |y_{ij}|^2.$$

Now by the inequality (2.7) and the unitarily invariant of the Hilbert- Schmidt norm, we have

$$\begin{aligned}\|AX + XB\|_F^2 &= \sum_{i,j=1}^{n} (\lambda_i + \nu_j)^2 |y_{ij}|^2 \\ &\leq \sum_{i,j=1}^{n} \Big(\max \mathrm{K}(t_{ij},2)^{R'}(\lambda_i^{1-\mu}\nu_j^{\mu} + \lambda_i^{\mu}\nu_j^{1-\mu})^2 + 2r(\lambda_i - \nu_j)^2\Big)|y_{ij}|^2,\end{aligned}$$

where $t_{ij} = \lambda_i/\nu_j$.

According to the conditions $0 < mI \leq A, B \leq MI$, $\dfrac{m}{M} = \dfrac{1}{h} \leq t_{ij} = \dfrac{\lambda_i}{\nu_j} \leq h = \dfrac{M}{m}$ and the properties of the Kantorovich constant, we can get

$$\begin{aligned}&\|(1-\mu)AX + \mu XB\|_F^2 \\ \leq\ & \sum_{i,j=1}^{n} \Big(\mathrm{K}(h,2)^{R'}(\lambda_i^{1-\mu}\nu_j^{\mu} + \lambda_i^{\mu}\nu_j^{1-\mu})^2 + 2r(\lambda_i - \nu_j)^2\Big)|y_{ij}|^2 \\ =\ & \mathrm{K}(h,2)^{R'} \sum_{i,j=1}^{n} (\lambda_i^{1-\mu}\nu_j^{\mu} + \lambda_i^{\mu}\nu_j^{1-\mu})^2 |y_{ij}|^2 + 2r \sum_{i,j=1}^{n} (\lambda_i - \nu_j)^2 |y_{ij}|^2 \\ =\ & \mathrm{K}(h,2)^{R'} \left\|A^{1-\mu}XB^{\mu} + A^{\mu}XB^{1-\mu}\right\|_F^2 + 2r\,\|AX - XB\|_F^2\,.\end{aligned}$$

This completes the proof. ■

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