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ON THE NUMBER OF LAPLACIAN EIGENVALUES OF TREES SMALLER THAN TWO

Lingling Zhou, Bo Zhou* and Zhibin Du

Abstract. Let $m_T[0,2)$ be the number of Laplacian eigenvalues of a tree T in $[0,2)$, multiplicities included. We give best possible upper bounds for $m_T[0,2)$ using the parameters such as the number of pendant vertices, diameter, matching number, and domination number, and characterize the trees T of order n with $m_T[0,2) = n-1$, $n-2$, and $\lceil \frac{n}{2} \rceil$, respectively, and in particular, show that $m_T[0,2) = \lceil \frac{n}{2} \rceil$ if and only if the matching number of T is $\lfloor \frac{n}{2} \rfloor$.

1. Introduction

We consider simple graphs. Let G be a graph with vertex set $V(G)$. For $v \in V(G)$, let $d_G(v)$ be the degree of v in G. The Laplacian matrix of G is defined as $L(G) = D(G) - A(G)$, where $D(G)$ is the degree diagonal matrix of G, and $A(G)$ is the adjacency matrix of G. The Laplacian eigenvalues of G are the eigenvalues of $L(G)$. Since $L(G)$ is a positive semi-definite matrix, the Laplacian eigenvalues of G are nonnegative real numbers. Let $\mu_1(G) \le \mu_2(G) \le \cdots \le \mu_n(G)$ be the Laplacian eigenvalues of G, arranged in nondecreasing order, where $n = |V(G)|$. Since each row sum of $L(G)$ is zero, $\mu_1(G) = 0$. Recall that $\mu_n(G) \le n$ (see [1, 5]). Thus all Laplacian eigenvalues of G belong to $[0,n]$. For a survey on Laplacian eigenvalues, see [11].

For a graph G on n vertices and an interval $I \subseteq [0,n]$, let $m_G I$ be the number of Laplacian eigenvalues of G, multiplicities included, that belong to I.

Grone and Merris [5] showed that for a graph with at least one edge, its largest Laplacian eigenvalue is at least the maximum degree plus one. Thus for a tree T on $n \ge 2$ vertices, $m_T[0,2) \le n-1$.

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*Corresponding author.

A vertex of a graph G is a pendant vertex if $d_G(v) = 1$. A vertex of G is a quasi-pendant vertex if it is adjacent to a pendant vertex.

For a graph G on n vertices with p pendant vertices, q quasi-pendant vertices, and diameter d, Grone *et al.* [6] showed that

$$m_G[0,1], m_G[1,n] \geq p\,,$$
$$m_G[0,1), m_G(1,n] \geq q\,,$$
$$m_G(2,n] \geq \left\lfloor \frac{d}{2} \right\rfloor,$$

and Merris [10] showed that if $n > 2q$, then

$$m_G(2,n] \geq q\,.$$

Braga *et al.* [3] showed that for a tree T on $n \geq 2$ vertices,

$$m_T[0,2) \geq \left\lceil \frac{n}{2} \right\rceil.$$

More results along this line may be found in [3, 7, 8].

In this paper, we give best possible upper bounds for $m_T[0,2)$ using the parameters of a tree T such as the number of pendant vertices, diameter, matching number, and domination number, provide a simple different proof for the lower bound in [3] mentioned above, characterize the trees T of order n with $m_T[0,2) = n-1$, $n-2$, and $\lceil \frac{n}{2} \rceil$, respectively, and in particular, show that $m_T[0,2) = \lceil \frac{n}{2} \rceil$ if and only if the matching number of T is $\lfloor \frac{n}{2} \rfloor$ (in Theorem 4.2).

2. Preliminaries

An algorithm for computing the number of Laplacian eigenvalues of a tree in an interval was proposed in [3] based on the algorithm for computing the number of adjacency eigenvalues of a tree in an interval [9]. For a tree T on n vertices, choose any vertex as the root of T, and label the vertices of T as $v_1, v_2, \ldots, v_n$ such that if v_i is a child of v_k, then $k > i$. The algorithm for computing $m_T[0,2)$ of a tree T is given as follows:

Input: tree T
Output: diagonal matrix D congruent to $L(T)$

Algorithm Diagonalize $L(T)$
  initialize $a_T(v) := d_T(v) - 2$ for all vertices v
  order vertices bottom up
  for $k = 1$ to n
   if v_k is a leaf **then** continue

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else if  a_T(c) ≠ 0 for all children c of v_k then
a_T(v_k) := a_T(v_k) − Σ_{c is a child of v_k} 1/a_T(c)
else
    select one child v_j of v_k for which a_T(v_j) = 0
    a_T(v_k) := −1/2
    a_T(v_j) := 2
    if v_k has a parent v_l, then remove the edge v_k v_l
end loop
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For a tree T with vertices $v_1, v_2, \ldots, v_n$ labelled as above, the weight of v_i in T is the i-th diagonal entry $a_T(v_i)$ of the diagonal matrix D obtained under the above algorithm, where $1 \le i \le n$. If $a_T(v_i) < 0$, we say v_i has a negative weight in T.

Lemma 2.1. [3] . *Suppose that T is a tree. Then $m_T[0,2)$ is equal to the number of vertices with negative weights in T.*

A double broom is a tree obtained by attaching some pendant vertices to the two end vertices of a path on at least two vertices. In particular, a star is also regarded as a double broom.

Lemma 2.2. *Let T be an n-vertex double broom with diameter d, where $1 \le d \le n-1$. Then $m_T[0,2) = \lfloor \frac{2n-d}{2} \rfloor$.*

Proof. Choosing a quasi-pendant vertex of T as the root of T. Then the result follows from Lemma 2.1 easily. ■

Lemma 2.3. *Let T be a tree with $v \in V(T)$, and T' be the tree obtained from T by attaching a path on two vertices to v. Then $m_{T'}[0,2) = m_T[0,2) + 1$.*

Proof. In both T and T', we choose v as the root. Note that the two vertices in T' not in T have weights 1 and -1, and $a_T(x) = a_{T'}(x)$ for $x \in V(T)$. Then the result follows from Lemma 2.1 clearly. ■

Lemma 2.4. *Let T be a tree with $v \in V(T)$, and T^* be the tree obtained from T by attaching two pendant vertices to v. Then $m_{T^*}[0,2) \ge m_T[0,2) + 1$.*

Proof. Let us choose v as the root of both T and T^*. Clearly, $a_T(x) = a_{T^*}(x)$ for $x \in V(T) \setminus \{v\}$. Denote by s the number of vertices in T different from v with negative weights. Note that each pendant vertex in T^* has weight -1. By Lemma 2.1, $m_{T^*}[0,2) \ge s + 2 = (s+1) + 1 \ge m_T[0,2) + 1$. ■

Lemma 2.5. [6] . *Let G be an n-vertex graph and G' a graph obtained from G by deleting an edge. Then*

$$0 = \mu_1(G') = \mu_1(G) \le \mu_2(G') \le \mu_2(G) \le \cdots \le \mu_n(G') \le \mu_n(G).$$

For a vertex v of a graph G, $G-v$ denotes the graph resulting from G by deleting v (and its incident edges). For an edge uv of a graph G (the complement of G, respectively), $G-uv$ ($G+uv$, respectively) denotes the graph resulting from G by deleting (adding, respectively) uv.

3. Upper Bounds for $m_T[0,2)$

For a tree T, if v is a vertex of T with exactly $d_T(v)-1 \geq 1$ pendant neighbors, then the subgraph induced by v and its $d_T(v)-1$ pendant neighbors is said to be a pendant star of T at v. If T is not a star, then T has some pendant stars.

Lemma 3.1. *Suppose that T is a tree with a pendant star at v, say T_1. If we choose a vertex of T outside T_1 as the root of T, then $a_T(v) > 0$.*

Proof. Clearly, $a_T(u) = -1$ for any pendant neighbor u of v in T. Thus

$$a_T(v) = d_T(v) - 2 - (d_T(v)-1)\frac{1}{a_T(u)} = 2d_T(v) - 3 > 0\,,$$

as desired. ■

Lemma 3.2. *Let T be a tree, and T_1 be the tree obtained from T by deleting a pendant vertex. Then $m_T[0,2) = m_{T_1}[0,2)$ or $m_{T_1}[0,2)+1$.*

Proof. Let v be a pendant vertex of T, being adjacent to u. By Lemma 2.5, $\mu_i(T) \leq \mu_{i+1}(T-uv) \leq \mu_{i+1}(T)$ for $1 \leq i \leq n-1$. Obviously, $T-uv$ consists of T_1 and an isolated vertex v. Thus $\mu_{i+1}(T-uv) = \mu_i(T_1)$ for $1 \leq i \leq n-1$. It follows that $\mu_i(T) \leq \mu_i(T_1) \leq \mu_{i+1}(T)$ for $1 \leq i \leq n-1$. From $\mu_i(T) \leq \mu_i(T_1)$, we have $m_T[0,2) \geq m_{T_1}[0,2)$, and from $\mu_i(T_1) \leq \mu_{i+1}(T)$, we have $m_{T_1}[0,2) \geq m_T[0,2)-1$. Thus we have the desired result. ■

Theorem 3.1. *Let T be an n-vertex tree with p pendant vertices, where $2 \leq p \leq n-1$. Then $m_T[0,2) \leq \left\lfloor \frac{n+p-1}{2} \right\rfloor$.*

Proof. We prove the result by induction on n.

If $n=3$, then T is a star with $p=2$, and by Lemma 2.2, $m_T[0,2) = 2 \leq \left\lfloor \frac{n+p-1}{2} \right\rfloor$.

Suppose that the result holds for all trees on less than $n \geq 4$ vertices with any possible number of pendant vertices. Let T be an n-vertex tree with p pendant vertices. Let v be an end vertex of a diametrical path of T, and u be the (unique) neighbor of v (on that diametrical path).

Suppose first that u is of degree two. Note that $T-v-u$ has at most p pendant vertices. Applying the induction hypothesis to $T-v-u$, we have $m_{T-v-u}[0,2) \leq \left\lfloor \frac{(n-2)+p-1}{2} \right\rfloor$. Then by Lemma 2.3, we have

$$m_T[0,2) = m_{T-v-u}[0,2) + 1 \le \left\lfloor \frac{(n-2)+p-1}{2} \right\rfloor + 1 = \left\lfloor \frac{n+p-1}{2} \right\rfloor.$$

Now suppose that u is of degree at least three. Note that $T-v$ has $p-1$ pendant vertices. Applying the induction hypothesis to $T-v$, we have $m_{T-v}[0,2) \le \left\lfloor \frac{(n-1)+(p-1)-1}{2} \right\rfloor$. Then by Lemma 3.2, we have

$$m_T[0,2) \le m_{T-v}[0,2) + 1 \le \left\lfloor \frac{(n-1)+(p-1)-1}{2} \right\rfloor + 1 = \left\lfloor \frac{n+p-1}{2} \right\rfloor.$$

The result follows. ■

Corollary 3.1. *Let T be an n-vertex tree with diameter d, where $2 \le d \le n-1$. Then $m_T[0,2) \le \lfloor \frac{2n-d}{2} \rfloor$.*

Proof. Denote by p the number of pendant vertices in T. Clearly, $p \le n-d+1$. Then the result follows from Theorem 3.1 easily. ■

The upper bounds in Theorem 3.1 and Corollary 3.1 are both tight since they are attained when T is an n-vertex double broom.

A matching of a graph is an edge subset in which no pair shares a common vertex. The matching number $\beta(G)$ of a graph G is the maximum cardinality of a matching of G.

Theorem 3.2. *Let T be an n-vertex tree with matching number β, where $1 \le \beta \le \lfloor \frac{n}{2} \rfloor$. Then $m_T[0,2) \le n-\beta$.*

Proof. We prove the result by induction on n.

The case $n=3$ follows obviously from Lemma 2.2.

Suppose that the result holds for all trees on less than $n \ge 4$ vertices with any possible matching number. Let T be an n-vertex tree with matching number β. Let v be an end vertex of a diametrical path of T, and u be the (unique) neighbor of v (on that diametrical path).

Suppose first that u is of degree two. Note that $T-v-u$ has matching number $\beta-1$. Applying the induction hypothesis to $T-v-u$, we have $m_{T-v-u}[0,2) \le (n-2)-(\beta-1) = n-\beta-1$. Now it follows from Lemma 2.3 that

$$m_T[0,2) = m_{T-v-u}[0,2) + 1 \le (n-\beta-1)+1 = n-\beta.$$

Now suppose that u is of degree at least three. Note that $T-v$ has matching number β. Applying the induction hypothesis to $T-v$, we have $m_{T-v}[0,2) \le n-1-\beta$. Now it follows from Lemma 3.2 that

$$m_T[0,2) \le m_{T-v}[0,2) + 1 \le (n-1-\beta)+1 = n-\beta.$$

The result follows. ■

A dominating set of a graph is a vertex subset whose closed neighborhood contains all vertices of the graph. The domination number of a graph G is the minimum cardinality of a dominating set of G.

A covering of a graph G is a vertex subset K such that every edge of G has at least one end vertex in K.

Corollary 3.2. *Let T be an n-vertex tree with domination number γ, where $1 \le \gamma \le \lfloor n/2 \rfloor$. Then $m_T[0,2) \le n - \gamma$.*

Proof. Denote by β the matching number of T. By König's theorem [2], β is equal to the minimum cardinality of a covering of G. Note that a covering of T is also a dominating set of T. Thus $\beta \ge \gamma$. Then the result follows from Theorem 3.2 easily. ■

The upper bounds in Theorem 3.2 and Corollary 3.2 are both tight since they are attained when T is an n-vertex tree obtained by attaching some paths on two vertices to the central vertex of a star.

Recall that $m_T[0,2) \le n-1$ for any tree T on $n \ge 2$ vertices [5], (which also follows from Theorem 3.2). Let $\mathcal{T}_n^1$ be the set of n-vertex trees (double brooms) with diameter three, where $n \ge 4$. Let $\mathcal{T}_n^2$ be the set of n-vertex double brooms with diameter four, where $n \ge 5$.

Theorem 3.3. *Let T be a tree on n vertices.*

(*i*) $m_T[0,2) = n-1$ *for* $n \ge 2$ *if and only if* $T \cong S_n$.

(*ii*) $m_T[0,2) = n-2$ *for* $n \ge 4$ *if and only if* $T \in \mathcal{T}_n^1 \cup \mathcal{T}_n^2$.

Proof. By Lemma 2.2, $m_T[0,2) = n-1$ if $T \cong S_n$, and $m_T[0,2) = n-2$ if $T \in \mathcal{T}_n^1 \cup \mathcal{T}_n^2$.

Suppose in the following that $T \notin \{S_n\} \cup \mathcal{T}_n^1 \cup \mathcal{T}_n^2$. Then $n \ge 6$. Let $P = v_0 v_1 \dots v_d$ be a diametrical path of T. Obviously, $d \ge 4$. Let T_1 be the pendant star of T at v_1, and T_2 be the pendant star of T at v_{d-1}.

If T_1 and T_2 are the only two vertex-disjoint pendant stars in T, then T is a double broom with $d \ge 5$, and thus by Lemma 2.2, $m_T[0,2) \le n-3$.

Suppose that there are at least three vertex-disjoint pendant stars in T. Let T_3 be a pendant star in T different from T_1 and T_2.

If $V(T) = V(T_1) \cup V(T_2) \cup V(T_3)$, then T is the tree obtained by attaching at least one pendant vertex to each vertex of P_3, and by choosing v_2 as the root of T and applying Lemma 2.1, we have $m_T[0,2) = n-3$.

Suppose that $V(T) \supset V(T_1) \cup V(T_2) \cup V(T_3)$. Let u be a vertex in T outside T_1, T_2, T_3. Choosing u as the root of T, and by Lemma 3.1, each of T_1, T_2, T_3 has one vertex which is not of negative weight. Thus, by Lemma 2.1, we have $m_T[0,2) \le n-3$.

Now the result follows easily. ■

4. A Lower Bound for $m_T[0, 2)$

For a tree T, if u is a pendant vertex of T being adjacent to a vertex v of degree two, then the subgraph of T induced by u and v is said to be a pendant P_2 of T. For a tree on at least three vertices, if there is no pendant P_2, then there are two pendant vertices sharing a common neighbor.

Deleting a pendant P_2 of a tree T is said to be a deleting pendant P_2 operation, and deleting a pendant P_2 of T or two pendant vertices of T sharing a common neighbor is said to be a generalized deleting pendant P_2 operation.

For a tree on n vertices, we can finally obtain P_1 for odd n and P_2 for even n by a series of generalized deleting pendant P_2 operations.

The following result has been obtained by Braga *et al.* [3]. Here we present a simple different reasoning.

Theorem 4.1. *Let T be a tree on $n \geq 2$ vertices. Then $m_T[0, 2) \geq \lceil \frac{n}{2} \rceil$.*

Proof. By Lemmas 2.3 and 2.4, each generalized deleting pendant P_2 operation decreases the number of Laplacian eigenvalues in $[0, 2)$ by at least one. Thus, if n is odd, then $m_T[0, 2) \geq m_{P_1}[0, 2) + \frac{n-1}{2} = \frac{n+1}{2}$, and if n is even, then $m_T[0, 2) \geq m_{P_2}[0, 2) + \frac{n-2}{2} = \frac{n}{2}$. ■

Lemma 4.1. *Let T be a tree with a diametrical path $P = v_0 v_1 \ldots v_d$, where $d \geq 4$, and for some i with $2 \leq i \leq d-2$, v_i is of degree three. Let $T' = T - v_i v_{i+1} + v_i^* v_{i+1}$, where v_i^* is the pendant neighbor of v_i outside P. Then $m_T[0, 2) \geq m_{T'}[0, 2)$.*

Proof. Let us choose v_i as the root of both T and T'. It is easily checked that $a_T(x) = a_{T'}(x)$ for $x \in V(T) \setminus \{v_i, v_i^*\}$, $a_T(v_i^*) = -1$,

$$a_T(v_i) = 2 - \frac{1}{a_T(v_{i-1})} - \frac{1}{a_T(v_{i+1})} ,$$

$$a_{T'}(v_i^*) = -\frac{1}{a_{T'}(v_{i+1})} = -\frac{1}{a_T(v_{i+1})} ,$$

$$a_{T'}(v_i) = -\frac{1}{a_{T'}(v_{i-1})} - \frac{1}{a_{T'}(v_i^*)} = -\frac{1}{a_T(v_{i-1})} + a_T(v_{i+1}) .$$

Denote by s the number of vertices in T different from v_i, v_i^* with negative weights. By Lemma 2.1, $m_T[0, 2) \geq s + 1$ and $m_{T'}[0, 2) \leq s + 2$.

Suppose by contradiction that $m_T[0, 2) < m_{T'}[0, 2)$. Then

$$s + 1 \leq m_T[0, 2) \leq m_{T'}[0, 2) - 1 \leq s + 1 ,$$

and thus $m_T[0, 2) = s+1$ and $m_{T'}[0, 2) = s+2$, implying that $a_T(v_i) \geq 0$, $a_{T'}(v_i^*) < 0$, and $a_{T'}(v_i) < 0$. From $a_{T'}(v_i^*) < 0$, we have $a_T(v_{i+1}) > 0$, and then

$$a_{T'}(v_i) - a_T(v_i) = a_T(v_{i+1}) + \frac{1}{a_T(v_{i+1})} - 2 \geq 0\,.$$

Thus $a_{T'}(v_i) \geq a_T(v_i) \geq 0$, which is a contradiction. ■

Attaching the path P_2 to a vertex of a tree T is called adding a pendant P_2 to T. By Lemma 2.3, each operation of adding a pendant P_2 increases the number of Laplacian eigenvalues in $[0, 2)$ by one.

Theorem 4.2. *Let T be a tree on $n \geq 2$ vertices. Then $m_T[0,2) = \lceil \frac{n}{2} \rceil$ if and only if $\beta(T) = \lfloor \frac{n}{2} \rfloor$.*

Proof. If $\beta(T) = \lfloor \frac{n}{2} \rfloor$, then by Theorem 3.2, we have

$$\left\lceil \frac{n}{2} \right\rceil \leq m_T[0,2) \leq n - \left\lfloor \frac{n}{2} \right\rfloor = \left\lceil \frac{n}{2} \right\rceil,$$

and thus $m_T[0,2) = \lceil \frac{n}{2} \rceil$.

Suppose that $m_T[0,2) = \lceil \frac{n}{2} \rceil$. We will prove that $\beta(T) = \lfloor \frac{n}{2} \rfloor$.

Claim 1. T is a tree obtainable from P_2 if n is even and from P_1 if n is odd by sequentially adding pendant P_2's.

Applying a series of deleting pendant P_2 operations from T, we may finally obtain a tree $T^{(1)}$ without pendant P_2. Let $n^{(1)} = |V(T^{(1)})|$. By Lemma 2.3, we have $m_{T^{(1)}}[0,2) = \left\lceil \frac{n^{(1)}}{2} \right\rceil$.

If $n^{(1)} = 1$ or 2, i.e., $T^{(1)} \cong P_1$ or P_2, then Claim 1 follows obviously. In the following, we will prove that $n^{(1)} = 1$ or 2.

Since $T^{(1)}$ has no pendant P_2, we have $n^{(1)} \neq 3$, and if $n^{(1)} = 4, 5$, then $T^{(1)}$ is a star, and thus $m_{T^{(1)}}[0,2) = n^{(1)} - 1 \neq \left\lceil \frac{n^{(1)}}{2} \right\rceil$, which is a contradiction, implying that $n^{(1)} \neq 4, 5$.

Suppose that $n^{(1)} \geq 6$. Let d be the diameter of $T^{(1)}$, and let $P = v_0 v_1 \ldots v_d$ be a diametrical path of $T^{(1)}$. Note that both v_1 and v_{d-1} are of degree at least three (since $T^{(1)}$ has no pendant P_2). If $d = 2, 3$, then $T^{(1)}$ is a double broom, by Theorem 3.3, $m_{T^{(1)}}[0,2) \geq n^{(1)} - 2 > \left\lceil \frac{n^{(1)}}{2} \right\rceil$, which is a contradiction. Thus $d \geq 4$.

Note that the deletion of edges in P from $T^{(1)}$ results in a forest with $d+1$ components, each of which contains exactly one vertex of P. Among such $d+1$ components, denote by T_i the one containing v_i, where $0 \leq i \leq d$.

Let $T^{(2)}$ be the tree obtained from $T^{(1)}$ by a series of generalized deleting pendant P_2 operations such that one vertex of T_i is left if $|V(T_i)|$ is odd and two vertices of T_i are left if $|V(T_i)|$ is even for all $2 \leq i \leq d-2$. Let $n^{(2)} = |V(T^{(2)})|$.

Now by Lemmas 2.3, 2.4, and 4.1, we have

$$\begin{aligned}\left\lceil \frac{n^{(1)}}{2} \right\rceil = m_{T^{(1)}}[0,2) &\geq m_{T^{(2)}}[0,2) + \frac{n^{(1)} - n^{(2)}}{2} \\ &\geq \left\lceil \frac{n^{(2)}}{2} \right\rceil + \frac{n^{(1)} - n^{(2)}}{2} \\ &= \left\lceil \frac{n^{(1)}}{2} \right\rceil .\end{aligned}$$

Thus $m_{T^{(2)}}[0,2) = \left\lceil \frac{n^{(2)}}{2} \right\rceil$.

Note that $P = v_0 v_1 \ldots v_d$ is still a diametrical path of $T^{(2)}$, v_1 and v_{d-1} are both of degree at least three, and the vertices $v_2, v_3, \ldots, v_{d-2}$ are all of degrees two or three. This implies that the diameter, say $\bar{d}$, of $T^{(2)}$ satisfies that $4 \leq \bar{d} \leq n^{(2)} - 3$.

If the vertices $v_2, v_3, \ldots, v_{d-2}$ in $T^{(2)}$ are all of degree two, then $T^{(2)}$ is a double broom, and by Lemma 2.2, we have

$$\left\lceil \frac{n^{(2)}}{2} \right\rceil = m_{T^{(2)}}[0,2) = \left\lfloor \frac{2n^{(2)} - \bar{d}}{2} \right\rfloor \geq \left\lfloor \frac{2n^{(2)} - (n^{(2)} - 3)}{2} \right\rfloor = \left\lfloor \frac{n^{(2)} + 3}{2} \right\rfloor ,$$

which is a contradiction.

Suppose that there is a vertex v_i of degree three in $T^{(2)}$, where $2 \leq i \leq d-2$. Denote by v_i^* the pendant neighbor of v_i in $T^{(2)}$ outside P. Let $T' = T^{(2)} - v_i v_{i+1} + v_i^* v_{i+1}$. Note that T' has one less vertex of degree three than $T^{(2)}$. By Lemma 4.1, we have $m_{T^{(2)}}[0,2) \geq m_{T'}[0,2)$. Repeating the transformation from $T^{(2)}$ to T', we can finally get a double broom T^* with $n^{(2)}$ vertices such that the degrees of v_1 and v_{d-1} in T^* are the same as those in $T^{(2)}$, the vertices $v_2, v_3, \ldots, v_{d-2}$ and their pendant neighbors in $T^{(2)}$ are all of degree two in T^*, and $\left\lceil \frac{n^{(2)}}{2} \right\rceil = m_{T^{(2)}}[0,2) \geq m_{T^*}[0,2)$. Note that T^* has diameter at most $n^{(2)} - 3$ (since v_1 and v_{d-1} are both of degree at least three). As above, we can deduce a contradiction.

Thus $n^{(1)} = 1$ or 2, and Claim 1 follows.

Obviously, each operation of adding a pendant P_2 increases the matching number by one. By Claim 1, $\beta(T) = \beta(P_2) + \frac{n-2}{2} = \frac{n}{2}$ if n is even, and $\beta(T) = \beta(P_1) + \frac{n-1}{2} = \frac{n-1}{2}$ if n is odd. Thus $\beta(T) = \left\lfloor \frac{n}{2} \right\rfloor$. ∎

5. Remark

Recall that for a tree T on $n \geq 2$ vertices, $\left\lceil \frac{n}{2} \right\rceil \leq m_T[0,2) \leq n-1$.

Theorem 5.1. *For positive integers n, k with $n \geq 2$ and $\left\lceil \frac{n}{2} \right\rceil \leq k \leq n-1$, there exists a tree T on n vertices such that $m_T[0,2) = k$.*

Proof. Observe that $m_{S_{2k-n+2}}[0,2) = 2k - n + 1$. Let T be the n-vertex tree obtained by attaching a path on $2n - 2k - 2$ vertices to a vertex of S_{2k-n+2}. By Lemma 2.3, we have

$$m_T[0,2) = m_{S_{2k-n+2}}[0,2) + \frac{2n-2k-2}{2} = k\,,$$

as desired. ■

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Lingling Zhou and Bo Zhou
Department of Mathematics
South China Normal University
Guangzhou 510631
P. R. China
E-mail: zhoubo@scnu.edu.cn

Zhibin Du
School of Mathematics and Information Sciences
Zhaoqing University
Zhaoqing 526061
P. R. China