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EQUITABLE VERTEX ARBORICITY OF PLANAR GRAPHS

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Abstract. Let G_1 be a planar graph such that all cycles of length at most 4 are independent and let G_2 be a planar graph without 3-cycles and adjacent 4-cycles. It is proved that the set of vertices of G_1 and G_2 can be equitably partitioned into t subsets for every $t \geq 3$ so that each subset induces a forest. These results partially confirm a conjecture of Wu, Zhang and Li [5].

1. INTRODUCTION

All graphs considered in this paper are finite, simple and undirected. By $V(G)$, $E(G)$, $\delta(G)$ and $\Delta(G)$, we denote the set of vertices, the set of edges, the minimum degree and the maximum degree of a graph G, respectively. For a plane graph G, $F(G)$ denotes its set of faces. A k-, k^+- and k^--$vertex$ (resp.*face*) in G is a vertex (resp. face) of degree k, at least k and at most k, respectively. By $N(v)$, we denote the set of neighbors of v. We call u the *k-neighbor* or *k^+-neighbor* of v if $uv \in E(G)$ and u is a k-vertex or a k^+-vertex, respectively. Two cycles are *independent* in G if they share no common vertices in G. For other undefined notations, we refer the readers to [1].

The *vertex arboricity*, or *point arboricity* $a(G)$ of a graph G is the minimum number of subsets into which the set of vertices can be partitioned so that each subset induces a forest. This chromatic parameter of graphs was extensively studied since it was first introduced by Chartrand and Kronk in [3], where is proved that $a(G) \leq 3$ for every planar graph.

As we know, there are many variations of vertex arboricity of graphs, such as linear vertex arboricity [4], fractional vertex arboricity [6], fractional linear vertex

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arboricity [8] and tree arboricity [2]. Naturally, we can also consider the equitable version of vertex arboricity when we restrict the partition in its original definition to be an equitable one, that is, a partition so that the size of each subset is either $\lceil |G|/k \rceil$ or $\lfloor |G|/k \rfloor$. If the set of vertices of a graph G can be equitably partitioned into k subsets such that each subset of vertices induce a forest of G, then we call that G admits an *equitable k-tree-coloring*. The minimum integer k such that G has an equitable k-tree-coloring is the *equitable vertex arboricity* $a_{eq}(G)$ of G. The notion of equitable vertex arboricity was first introduced by Wu, Zhang and Li [5]. In their paper, the authors proved that the complete bipartite graph $K_{n,n}$ has an equitable k-tree-coloring for every $k \geq 2\lfloor(\sqrt{8n+9}-1)/4\rfloor$ and showed that the bound is sharp when $2n = t(t+3)$ and t is odd. Note that $K_{n,n}$ admits an equitable 2-tree-coloring. Hence a graph admitting an equitable k-tree-coloring may has no equitable $(k+1)$-tree-colorings. This motivates us to introduce another chromatic parameter. The *strong equitable vertex arboricity* of G, denoted by $a^*_{eq}(G)$, is the smallest t such that G has an equitable t'-tree-coloring for every $t' \geq t$. It is easy to see that $a^*_{eq}(G) \geq a_{eq}(G)$. Concerning $a^*_{eq}(G)$, there are two interesting conjectures.

Conjecture 1. $a^*_{eq}(G) \leq \lceil \frac{\Delta(G)+1}{2} \rceil$ for every graph G.

Conjecture 2. There is a constant ζ such that $a^*_{eq}(G) \leq \zeta$ for every planar graph G.

Until now, Conjecture 1 was confirmed for complete bipartite graphs, planar graphs with girth at least 6, planar graphs with maximum degree at least 4 and girth 5, outerplanar graphs [5] and graphs G with $\Delta(G) \geq |G|/2$ [7], and Conjecture 2 was settled for planar graphs with girth at least 5 and outerplanar graphs [5]. In particular, Wu, Zhang and Li [5] proved that $a^*_{eq}(G) \leq 3$ for every planar graph with girth at least 5. In this paper, we will confirm Conjecture 2 for planar graphs with all cycles of length at most 4 being independent and planar graphs without 3-cycles and adjacent 4-cycles.

2. Main Results and Their Proofs

Lemma 3. (Wu, Zhang and Li [5]) *Let $S = \{x_1, \cdots, x_t\}$, where $x_1, \cdots, x_t$ are distinct vertices in G. If $G-S$ has an equitable t-tree-coloring and $|N(x_i)\backslash S| \leq 2i-1$ for every $1 \leq i \leq t$, then G has an equitable t-tree-coloring.*

Lemma 4. *If G is a planar graph such that all cycles of length at most 4 are independent, then $\delta(G) \leq 3$.*

Proof. Suppose, to the contrary, that $\delta(G) \geq 4$. By Euler's formula, we have $\sum_{x \in V(G) \cup F(G)} \big(d(x)-4\big) = -8$. Assign every element $x \in V(G) \cup F(G)$ an initial charge $c(x) = d(x) - 4$ and define a discharging rule as follows.

Rule. Every 5^+-face transfer $\frac{1}{3}$ to each of its adjacent 3-faces.

Let c' be the final charge function after discharging according to the rule. Since every 3-face is adjacent only to 5^+-faces by the definition of G, $c'(f) = 3 - 4 + 3 \times \frac{1}{3} = 0$ for $d(f) = 3$. On the other hand, every 5^+-face f is adjacent to at most $\lfloor \frac{d(f)}{2} \rfloor$ 3-faces, which implies that $c'(f) \geq d(f) - 4 - \frac{1}{3}\lfloor \frac{d(f)}{2} \rfloor > 0$ for $d(f) \geq 5$. Therefore, $\sum_{x \in V(G) \cup F(G)} c'(x) \geq 0$, contradicting the fact that $\sum_{x \in V(G) \cup F(G)} c'(x) = \sum_{x \in V(G) \cup F(G)} c(x) = -8$. ■

Theorem 5. *If G is a planar graph such that all cycles of length at most 4 are independent, then $a^*_{eq}(G) \leq 3$.*

Proof. Let G be the minimal counterexample to this result and let $t \geq 3$ be an integer. To begin with, we introduce some useful structural properties of G.

Proposition 1. *Every 2-vertex in G is adjacent only to 7^+-vertices.*

Proof. If there is a 2-vertex u that is adjacent to a 6^--vertex v, then label u and v by x_1 and x_t, respectively. We now construct the set $S = \{x_1, \ldots, x_t\}$ as in Lemma 3 by filling the remaining unspecified positions in S from highest to lowest indices properly. Actually one can easily complete it by choosing at each step a vertex of degree at most 3 in the graph obtained from G by deleting the vertices already chosen for S. Lemma 4 guarantees that such vertices always exist. By the minimality of G, $G - S$ has an equitable t-tree-coloring for every $t \geq 3$. Therefore, G also has such a desired coloring by Lemma 3. ■

Proposition 2. *Every 3-vertex in G is either adjacent to three 5^+-vertices or adjacent to one 4^--vertex and two 7^+-vertices.*

Proof. If there is a 3-vertex u that is adjacent to a 4^--vertex v and a 6^--vertex w, then label u, v and w by x_1, x_{t-1} and x_t, respectively. By similar argument as in the proof of Proposition 1, we can construct the set $S = \{x_1, \ldots, x_t\}$ as in Lemma 3 and then deduce that G has an equitable t-tree-coloring for every $t \geq 3$, a contradiction. ■

Similarly, we have the following:

Proposition 3. *If there is a 3-face f that is incident with a 3-vertex, then f is either incident with two 6^+-vertices or incident with another one 5^--vertex and a 8^+-vertex.* ■

Proposition 4. *If there is a 4-face f that is incident with a 3-vertex, then f is either incident with three 4^+-vertices, or incident with two 5^+-vertices, or incident with a 4-vertex and a 7^+-vertex.*

Proof. Let $f = u_1u_2u_3u_4$ and $d(u_1) = 3$. If f is not incident with three 4^+-vertices, then there is at least one 3^--vertex among u_2, u_3 and u_4. If $\min\{d(u_2),$

$d(u_3), d(u_4)\} = 2$, then by Proposition 1, $d(u_3) = 2$ and $\min\{d(u_2), d(u_4)\} \geq 7$. If $d(u_2) = 3$ or $d(u_4) = 3$, then by Proposition 2, $\min\{d(u_3), d(u_4)\} \geq 7$ or $\min\{d(u_2), d(u_3)\} \geq 7$, respectively. If $d(u_3) = 3$, then by Proposition 2, either $\min\{d(u_2), d(u_4)\} \geq 5$ or $\min\{d(u_2), d(u_4)\} = 4$ and $\min\{d(u_2), d(u_4)\} \geq 7$. ■

Proposition 5. *Every* 7*-vertex is adjacent to at most one* 2*-vertex.*

Proof. If there is a 7-vertex u that is adjacent to two 2-vertices v and w, then label v, w and u by x_1, x_{t-1} and x_t, respectively. By the similar arguments asin the proof of Proposition 1, we can construct the set $S = \{x_1, \ldots, x_t\}$ as in Lemma 3. Therefore, $G - S$ has an equitable t-tree-coloring by the minimality of G, which implies that G also has such a desired coloring for every $t \geq 3$ by Lemma 3. ■

Proposition 6. *Every* 8*-vertex and every* 9*-vertex is adjacent to at most four* 2*-vertices.*

Proof. Let u be a k-vertex with $8 \leq k \leq 9$ and let $v_1, \ldots, v_k$ be its neighbors in G. Without loss of generality, assume that v_1, v_2, v_3, v_4 and v_5 are 2-vertices. Let w_i be the other neighbor of v_i for every $1 \leq i \leq 5$.

If $t \geq 4$, then label v_1, v_2, v_3 and u with x_1, x_{t-2}, x_{t-1} and x_t, respectively, and construct the set $S = \{x_1, \ldots, x_t\}$ as in Lemma 3 by the similar arguments as in the proof of Proposition 1. Therefore, $G - S$ has an equitable t-tree-coloring by the minimality of G, which implies that G also has such a desired coloring for every $t \geq 4$ by Lemma 3.

We now prove that G has an equitable 3-tree-coloring. By the minimality of G, the graph $H = G - \{u, v_1, v_2, v_3, v_4, v_5\}$ has an equitable 3-tree-coloring φ. If there is one color, say 3, that does not appear on $N(u) \setminus \{v_1, v_2, v_3, v_4, v_5\}$, then color u and v_1 with 3, v_2 and v_3 with 1, and v_4 and v_5 with 2. One can check that the resulted coloring of G is just an equitable 3-tree-coloring.

We now assume that all of the three colors appear on $N(u) \setminus \{v_1, v_2, v_3, v_4, v_5\}$. If $d(u) = 8$, then we assume that $\varphi(v_6) = 1$, $\varphi(v_7) = 2$ and $\varphi(v_8) = 3$. If $d(u) = 9$, then we assume, without loss of generality, that $\varphi(v_6) = 1$, $\varphi(v_7) = 2$ and $\varphi(v_8) = \varphi(v_9) = 3$. The following argument is independent of the degree of u. First, we color u with 1. If the color on one of the vertices among w_1, w_2, w_3, w_4 and w_5, say w_1, is not 1, then color v_1 with 1, v_2 and v_3 with 2, and v_4 and v_5 with 3. If $\varphi(w_i) = 1$ for every $1 \leq i \leq 5$, then recolor u with 2, and color v_1 with 2, v_2 and v_3 with 1, and v_4 and v_5 with 3. In each case, one can easily check that the resulted coloring is an equitable 3-tree-coloring of G. ■

Proposition 7. *Every* 10*-vertex is adjacent to at most seven* 2*-vertices.*

Proof. Let u be a 10-vertex and let $v_1, \ldots, v_{10}$ be its neighbors in G. Without loss of generality, assume that $v_1, \ldots, v_7$ and v_8 are 2-vertices. Let w_i be the other neighbor of v_i for every $1 \leq i \leq 8$. By the same argument as in the proof of Proposition 6, one can confirm that G has an equitable t-tree-coloring for every $t \geq 4$. Thus we just need prove that G admits an equitable 3-tree-coloring.

Let $H = G - \{u, v_1, \ldots, v_8\}$. By the minimality of G, H has an equitable 3-tree-coloring φ. Suppose that the color 3 does not appear on v_9 or v_{10}. If there is a vertex among $w_1, \ldots, w_8$, say w_1, that is not colored by 3, then we can extend φ to an equitable 3-tree-coloring of G by coloring u, v_1, v_2 with 3, v_3, v_4, v_5 with 1, and v_6, v_7, v_8 with 2. If $\varphi(w_i) = 3$ for every $1 \leq i \leq 8$, then color u with a color, say 1, that appears on v_9 and v_{10} at most once, color v_1 and v_2 with 1, v_3, v_4, v_5 with 2, and v_6, v_7, v_8 with 3. One can easily check that the resulted coloring is an equitable 3-tree-coloring of G. ■

We now prove the theorem by discharging. First, assign each vertex v of G an initial charge $c(v) = 3d(v) - 10$ and each face f of G an initial charge $c(v) = 2d(f) - 10$. By Euler's formula, $\sum_{x \in V(G) \cup F(G)} c(x) = -20$. It is easy to see that there is no 1-vertices in G. The discharging rules we are applying are defined as follows.

R1. Every 2-vertex receives 2 from each of its neighbors.

R2. If u be a 3-vertex and $uv \in E(G)$, then v sends to u a charge of $\frac{1}{3}$ if $5 \leq d(v) \leq 6$ and $\frac{1}{2}$ if $d(v) \geq 7$.

R3. Let f be a 3-face that is incident with no 2-vertices and let v be a vertex that is incident with f. If $4 \leq d(v) \leq 7$, then v sends 2 to f, and if $d(v) \geq 8$, then v sends 4 to f.

R4. If f is a 3-face that is incident with a 2-vertex, then f receives 2 from each of its incident 7^+-vertices.

R5. Every 4-face receives 1 from each of its incident 4^+-vertices.

Let c' be the final charge after discharging. We now prove that $c'(x) \geq 0$ for every $x \in V(G) \cup F(G)$, which contradicts the fact that $\sum_{x \in V(G) \cup F(G)} c'(x) = \sum_{x \in V(G) \cup F(G)} c(x) = -20$.

If f is a 3-face that is incident with a 2-vertex, then by Proposition 1, f is incident with two 7^+-vertices, which implies that $c'(v) = -4 + 2 \times 2 = 0$ by R4. Suppose that f is a 3-face that is incident with no 2-vertices. If f is incident with at least a 8^+-vertex, then $c'(f) \geq -4 + 4 = 0$ by R3. If f is incident only with 7^--vertices, then by Propositions 3, f is incident with at least two 4^+-vertices, which implies that $c'(f) \geq -4 + 2 \times 2 = 0$ by R3. If f is a 4-face, then by Propositions 1 and 2, f is incident with at least two 4^+-vertices, thus by R5 we have $c'(f) \geq -2 + 2 \times 1 = 0$. If f is a 5^+-face, then it is easy to see that $c'(f) = c(f) \geq 0$.

If v is a 2-vertex, then by Proposition 1, v is adjacent to two 7^+-vertices form which v receives $2 \times 2 = 4$ by R1, therefore $c'(v) = -4 + 4 = 0$. If v is a 3-vertex, then by Proposition 2, v is either adjacent to three 5^+-vertices which implies $c'(v) \geq -1 + 3 \times \frac{1}{3} = 0$ or adjacent to two 7^+-vertices implying $c'(v) \geq -1 + 2 \times \frac{1}{2} = 0$

by R2. Note that every vertex in G is incident with at most one 4^{-}-face by the definition of G. If v is a 4-vertex, then $c'(v) \geq 2-2=0$ by R3 and R5. If v is a 5-vertex or a 6-vertex, then by R2, R3 and R5, $c'(v) \geq 3d(v)-10-\frac{1}{3}d(v)-2>0$. If v is a 7-vertex, then v is adjacent to at most one 2-vertex by Proposition 5, thus $c'(v) \geq 11-2-6\times\frac{1}{2}-2>0$ by R1–R5. If v is a 8-vertex or a 9-vertex, then by Proposition 6 and R1–R5, $c'(v) \geq 3d(v)-10-4\times 2-(d(v)-4)\times\frac{1}{2}-4=\frac{1}{2}\big(5d(v)-40\big) \geq 0$. If v is a 10-vertex, then by Proposition 7 and R1–R5, $c'(v) \geq 20-7\times 2-3\times\frac{1}{2}-4>0$.

At last, we consider the vertex v with $d(v) \geq 11$. If v is adjacent only to 2-vertices, then v is incident with no 3-faces because otherwise there would be two adjacent 2-vertices in G, a contradiction. Therefore, by R1 and R5, we have $c'(v) \geq 3d(v)-10-2d(v)-1 \geq 0$. If v is adjacent to at most $d(v)-2$ vertices of degree 2, then by R1–R5, $c'(v) \geq 3d(v)-10-2\big(d(v)-2\big)-2\times\frac{1}{2}-4=d(v)-11 \geq 0$. Suppose that v is adjacent to $d(v)-1$ vertices of degree 2. If v is incident with no 4^{-}-faces, then $c'(v) \geq 3d(v)-10-2\big(d(v)-1\big)-\frac{1}{2}=d(v)-\frac{17}{2}>0$ by R1 and R2. If v is incident with a 4^{-}-face f, then either f is a 4-face or a 3-face that is incident with a 2-vertex. In the former case we have $c'(v) \geq 3d(v)-10-2\big(d(v)-1\big)-\frac{1}{2}-1=d(v)-\frac{19}{2}>0$ by R1, R2 and R5, and in the latter case we have $c'(v) \geq 3d(v)-10-2\big(d(v)-1\big)-\frac{1}{2}-2=d(v)-\frac{21}{2}>0$ by R1, R2 and R4. ∎

Theorem 6. *If G is a planar graph with girth at least 4 such that no two 4-cycles are adjacent, then $a^{*}_{eq}(G) \leq 3$.*

Proof. Let G be the minimal counterexample to this result and let $t \geq 3$ be an integer. Since every planar graph with girth at least 4 contains a 3^{-}-vertex, Propositions 1,2,4–7 still hold here. Therefore, the order of the following propositions we are to prove are naturally labeled from 8.

Proposition 8. *Every 11-vertex is adjacent to at most seven 2-vertices.*

Proof. Let u be a 11-vertex and let $v_1,\ldots,v_{11}$ be its neighbors in G. Without loss of generality, assume that $v_1,\ldots,v_7$ and v_8 are 2-vertices. Let w_i be the other neighbor of v_i for every $1 \leq i \leq 8$.

If $t \geq 5$, then label v_1,v_2,v_3,v_4 and u with $x_1,x_{t-3},x_{t-2},x_{t-1}$ and x_t, respectively, and construct the set $S=\{x_1,\ldots,x_t\}$ as in Lemma 3 by the similar arguments as in the proof of Proposition 1. Therefore, $G-S$ has an equitable t-tree-coloring by the minimality of G, which implies that G also has such a desired coloring for every $t \geq 5$ by Lemma 3.

We now prove that G has an equitable 4-tree-coloring. Let $H_1=G-\{u,v_1,\ldots,v_7\}$. By the minimality of G, H_1 has an equitable 4-tree-coloring φ_1. It is easy to see that there are at least two colors, say 1 and 2, that are used at most once on v_8,v_9,v_{10} and v_{11}. Color u with 1. If there is one vertex among $w_1,\ldots,w_7$, say w_1, that is not colored with 1 under φ_1, then color v_1 with 1, v_2,v_3 with 2, v_4,v_5 with 3, and v_6,v_7

with 4. If $\varphi_1(w_i) = 1$ for every $1 \le i \le 7$, then recolor u with 2, color v_1 with 2, v_2, v_3 with 1, v_4, v_5 with 3, and v_6, v_7 with 4. In each case we obtain an equitable 4-tree-coloring of G.

At last, we show that G also admits an equitable 3-tree-coloring. By the minimality of G, $H_2 = G - \{u, v_1, \ldots, v_8\}$ has an equitable 3-tree-coloring φ_2. Without loss of generality, let 1 and 2 be the colors used at most once on v_9, v_{10} and v_{11}. Color u with 1. If there are two vertices among $w_1, \ldots, w_8$, say w_1 and w_2, that are not colored with 1 under φ_2, then color v_1, v_2 with 1, v_3, v_4, v_5 with 2, and v_6, v_7, v_8 with 3. On the other hand, we can assume, without loss of generality, that $\varphi_2(w_i) = 1$ for every $1 \le i \le 7$. We now recolor u with 2, color v_1, v_2 with 2, v_3, v_4, v_5 with 1, and v_6, v_7, v_8 with 3. In each case, one can check that the resulted coloring is an equitable 3-tree-coloring of G. ■

Proposition 9. *Every* 12*-vertex and every* 13*-vertex is adjacent to at most ten* 2*-vertices.*

Proof. Let u be a k-vertex with $12 \le k \le 13$ and let $v_1, \ldots, v_k$ be its neighbors in G. Without loss of generality, assume that $v_1, \ldots, v_{10}$ and v_{11} are 2-vertices. Let w_i be the other neighbor of v_i for every $1 \le i \le 11$.

By the same argument as in the proof of the above proposition, one can show that G has an equitable t-tree-coloring for every $t \ge 5$. Let $H = G - \{u, v_1, \ldots, v_{11}\}$. By the minimality of G, H has an equitable 4-tree-coloring φ_1 and an equitable 3-tree-coloring φ_2. It is easy to see that there is a color, say 1, that has not used on $\{w_1\} \cup N(u) \setminus \{v_1, \ldots, v_{11}\}$ under φ_1. Hence we can extend φ_1 to an equitable 4-tree-coloring of G by coloring u, v_1, v_2 with 1, v_3, v_4, v_5 with 2, v_6, v_7, v_8 with 3, and v_9, v_{10}, v_{11} with 4. On the other hand, there exists a color, say 1, that is used on $N(u) \setminus \{v_1, \ldots, v_{11}\}$ at most once, and with which three vertices among $w_1, \ldots, w_{11}$, say w_1, w_2 and w_3, are not colored under φ_2. Therefore, φ_2 can be extended to an equitable 3-tree-coloring of G by coloring u, v_1, v_2, v_3 with 1, v_4, v_5, v_6, v_7 with 2, and v_8, v_9, v_{10}, v_{11} with 3. Hence, G admits an equitable t-tree-coloring for every $t \ge 3$, a contradiction. ■

Proposition 10. *Every* 14*-vertex and every* 15*-vertex is adjacent to at most thirteen* 2*-vertices.*

Proof. Let u be a k-vertex with $14 \le k \le 15$ and let $v_1, \ldots, v_k$ be its neighbors in G. Without loss of generality, assume that $v_1, \ldots, v_{13}$ and v_{14} are 2-vertices. Let w_i be the other neighbor of v_i for every $1 \le i \le 14$.

If $t \ge 6$, then label v_1, v_2, v_3, v_4, v_5 and u with $x_1, x_{t-4}, x_{t-3}, x_{t-2}, x_{t-1}$ and x_t, respectively, and construct the set $S = \{x_1, \ldots, x_t\}$ as in Lemma 3 by the similar arguments as in the proof of Proposition 1. Therefore, $G - S$ has an equitable t-tree-coloring by the minimality of G, which implies that G also has such a desired coloring for every $t \ge 6$ by Lemma 3.

Let $H = G - \{u, v_1, \ldots, v_{14}\}$. One can see that H has an equitable 5-tree coloring φ_1 and an equitable 3-tree coloring φ_2 by the minimality of G. Without loss of generality, let 1 be the color that is not used on $\{w_1\} \cup N(u) \setminus \{v_1, \ldots, v_{14}\}$ under φ_1. We extend φ_1 to an equitable 5-tree-coloring of G by coloring u, v_1, v_2 with 1, v_3, v_4, v_5 with 2, v_6, v_7, v_8 with 3, v_9, v_{10}, v_{11} with 4, and v_{12}, v_{13}, v_{14} with 5. On the other hand, since there is a color, say 1, that is not used on $N(u) \setminus \{v_1, \ldots, v_{14}\}$, and with which four vertices among $w_1, \ldots, w_{14}$, say w_1, w_2, w_3 and w_4, are not colored under φ_2, we can extend φ_2 to an equitable 3-tree-coloring of G by coloring u, v_1, v_2, v_3, v_4 with 1, v_5, v_6, v_7, v_8, v_9 with 2, and $v_{10}, v_{11}, v_{12}, v_{13}, v_{14}$ with 3. Let $H' = G - \{u, v_1, \ldots, v_{11}\}$. By the minimality of G, H' admits an equitable 4-tree-coloring φ_3. Note that there is a color, say 1, that has been used on $N(u) \setminus \{v_1, \ldots, v_{11}\}$ at most once, and with which two vertices among $w_1, \ldots, w_{11}$, say w_1 and w_2, are not colored under φ_3. Therefore, we extend φ_3 to an equitable 4-tree-coloring of G by coloring u, v_1, v_2 with 1, v_3, v_4, v_5 with 2, v_6, v_7, v_8 with 3, and v_9, v_{10}, v_{11} with 4. Hence, G has an equitable t-tree-coloring for every $t \geq 3$, a contradiction. ∎

We now prove the theorem by discharging. First, assign each vertex v of G an initial charge $c(v) = d(v) - 4$ and each face f of G an initial charge $c(v) = d(f) - 4$. By Euler's formula, $\sum_{x \in V(G) \cup F(G)} c(x) = -8$. It is easy to see that there is no 1-vertices in G. The discharging rules we are applying are defined as follows.

R1. Each 2-vertex receives $\frac{3}{4}$ from each of its neighbors, and $\frac{1}{2}$ from each of its incident 5^+-faces.

R2. Each 3-vertex receives $\frac{1}{6}$ from each of its 5-neighbors or 6-neighbors, $\frac{1}{4}$ from each of its 7^+-neighbors, and $\frac{1}{4}$ from each of it incident 5^+-faces.

Let c' be the final charge after discharging. If f is a 5^+-face that is incident with $\lfloor \frac{d(f)}{2} \rfloor$ vertices of degree 2, then f is incident with no 3-vertices, since 2-vertices are not adjacent to any 3^--vertices by Proposition 1. Hence, $c'(f) \geq d(f) - 4 - \frac{1}{2} \lfloor \frac{d(f)}{2} \rfloor \geq 0$ by R1 and R2. If f is a 5^+-face that is incident with $n < \lfloor \frac{d(f)}{2} \rfloor$ vertices of degree 2, then f is incident with at most $d(f) - 2n - 1$ vertices of degree 3. Hence, $c'(f) \geq d(f) - 4 - \frac{1}{2}n - \frac{1}{4}\big(d(f) - 2n - 1\big) = \frac{3}{4}\big(d(f) - 5\big) \geq 0$ by R1 and R2. If v is a 2-vertex, then v is incident with at least one 5^+-face by the definition of G, so $c'(v) \geq -2 + 2 \times \frac{3}{4} + \frac{1}{2} = 0$ by R1. If v is a 3-vertex, then v is incident with at least two 5^+-faces, because otherwise there would be two adjacent 4-cycles in G. If v is adjacent to three 5^+-vertices, then by R2, $c'(v) \geq -1 + 3 \times \frac{1}{6} + 2 \times \frac{1}{4} = 0$. If v is adjacent to a 4^--vertex, then by Proposition 2, v is adjacent to two 7^+-vertices, which implies that $c'(v) \geq -1 + 2 \times \frac{1}{4} + 2 \times \frac{1}{4} = 0$ by R2. If v is a 5-vertex or a 6-vertex, then $c'(v) \geq d(v) - 4 - \frac{1}{6}d(v) > 0$ by R2, since v has no 2-neighbors. If v is a 7-vertex, then by Proposition 5, v has at most one 2-neighbor, which implies that $c'(v) \geq 3 - \frac{3}{4} - 6 \times \frac{1}{4} > 0$ by R1 and R2. If v is a 8-vertex or a 9-vertex, then by Proposition 6, R1 and R2, $c'(v) \geq d(v) - 4 - 4 \times \frac{3}{4} - \frac{1}{4}\big(d(v) - 4\big) = \frac{3}{4}\big(d(v) - 8\big) \geq 0$.

If v is a 10-vertex, then by Proposition 7, R1 and R2, $c'(v) \geq 6 - 7 \times \frac{3}{4} - 3 \times \frac{1}{4} = 0$. If v is a 11-vertex, then by Proposition 8, R1 and R2, $c'(v) \geq 7 - 7 \times \frac{3}{4} - 4 \times \frac{1}{4} > 0$. If v is a 12-vertex or a 13-vertex, then by Proposition 9, R1 and R2, $c'(v) \geq d(v) - 4 - 10 \times \frac{3}{4} - \frac{1}{4}\big(d(v) - 10\big) = \frac{3}{4}\big(d(v) - 12\big) \geq 0$. If v is a 14-vertex or a 15-vertex, then by Proposition 10, R1 and R2, $c'(v) \geq d(v) - 4 - 13 \times \frac{3}{4} - \frac{1}{4}\big(d(v) - 13\big) = \frac{3}{4}\big(d(v) - 14\big) \geq 0$. If v is a 16^{+}-vertex, then $c'(v) \geq d(v) - 4 - \frac{3}{4}d(v) = \frac{1}{4}\big(d(v) - 16\big) \geq 0$ by R1 and R2. Therefore, $\sum_{x \in V(G) \cup F(G)} c'(x) \geq 0$, a contradiction completing the proof. ■

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