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SUZUKI-WARDOWSKI TYPE FIXED POINT THEOREMS FOR α-GF-CONTRACTIONS

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Abstract. Recently, Wardowski [Fixed Point Theory Appl. 2012:94, 2012] introduced and studied a new contraction called F-contraction to prove a fixed point result as a generalization of the Banach contraction principle. Abbas et al. [2] further generalized the concept of F-contraction and proved certain fixed and common fixed point results. In this paper, we introduce an α-GF-contraction with respect to a general family of functions G and establish Wardowski type fixed point results in metric and ordered metric spaces. As an application of our results we deduce Suzuki type fixed point results for GF-contractions. We also derive certain fixed and periodic point results for orbitally continuous generalized F-contractions. Moreover, we discuss some illustrative examples to highlight the realized improvements.

1. Introduction and Preliminaries

The Banach contraction principle is a popular tool in solving existence problems in many branches of mathematics. This result has been extended in many directions (see [1-18]). In 2008, in order to characterize the completeness of underlying metric spaces, Suzuki [17] introduced a weaker notion of contraction. Recently, Wardowski [19] introduced a new contraction called F-contraction and proved a fixed point result as a generalization of the Banach contraction principle. Abbas et al. [2] further generalized the concept of F-contraction and proved certain fixed and common fixed point results. In this paper, we introduce an α-GF-contraction with respect to a more general family of functions G and obtain fixed point results in metric space and partially ordered metric space. As an application of our results we deduce Suzuki type results for GF-contractions. In the last section, we derive fixed and periodic point results for orbitally continuous generalized F-contractions. We begin with some basic definitions and results which will be used in the sequel.

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In 2012, Samet et al. [13] introduced the concepts of α-ψ-contractive and α-admissible mappings and established various fixed point theorems for such mappings defined on complete metric spaces. Afterwards Salimi et al. [12] and Hussain et al. [6, 8, 9] modified the notions of α-ψ-contractive and α-admissible mappings and established certain fixed point theorems.

Definition 1.1. [13]. Let T be a self-mapping on X and $\alpha : X \times X \to [0, +\infty)$ be a function. We say that T is an α-admissible mapping if

$$x, y \in X, \quad \alpha(x, y) \geq 1 \quad \Longrightarrow \quad \alpha(Tx, Ty) \geq 1.$$

Definition 1.2. [12]. Let T be a self-mapping on X and $\alpha, \eta : X \times X \to [0, +\infty)$ be two functions. We say that T is an α-admissible mapping with respect to η if

$$x, y \in X, \quad \alpha(x, y) \geq \eta(x, y) \quad \Longrightarrow \quad \alpha(Tx, Ty) \geq \eta(Tx, Ty).$$

Note that if we take $\eta(x, y) = 1$ then this definition reduces to Definition 1.1. Also, if we take, $\alpha(x, y) = 1$ then we say that T is an η-subadmissible mapping.

Definition 1.3. [8]. Let (X, d) be a metric space. Let $\alpha, \eta : X \times X \to [0, \infty)$ and $T : X \to X$ be functions. We say T is an α-η-continuous mapping on (X, d), if, for given $x \in X$ and sequence $\{x_n\}$ with

$$x_n \to x \text{ as } n \to \infty, \ \alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1}) \text{ for all } n \in \mathbb{N} \Longrightarrow \ Tx_n \to Tx.$$

Example 1.1. [8]. Let $X = [0, \infty)$ and $d(x, y) = |x - y|$ be a metric on X. Assume, $T : X \to X$ and $\alpha, \eta : X \times X \to [0, +\infty)$ be defined by

$$Tx = \begin{cases} x^5, & \text{if } x \in [0, 1] \\ \sin \pi x + 2, & \text{if } (1, \infty) \end{cases}, \ \alpha(x, y) = \begin{cases} x^2 + y^2 + 1, & \text{if } x, y \in [0, 1] \\ 0, & \text{otherwise} \end{cases}$$

and $\eta(x, y) = x^2$. Clearly, T is not continuous, but T is α-η-continuous on (X, d).

A mapping $T : X \to X$ is called orbitally continuous at $p \in X$ if $lim_{n\to\infty} T^n x = p$ implies that $lim_{n\to\infty} TT^n x = Tp$. The mapping T is orbitally continuous on X if T is orbitally continuous for all $p \in X$.

Remark 1.1. [8]. Let $T : X \to X$ be a self-mapping on an orbitally T-complete metric space X. Define, $\alpha, \eta : X \times X \to [0, +\infty)$ by

$$\alpha(x, y) = \begin{cases} 3, & \text{if } x, y \in O(w) \\ 0, & \text{otherwise} \end{cases} \quad \text{and } \eta(x, y) = 1$$

where $O(w)$ is an orbit of a point $w \in X$. If, $T : X \to X$ is an orbitally continuous map on (X, d), then T is α-η-continuous on (X, d).

2. Fixed Point Results for α-GF-Contractions

Consistent with Wordowsky [19], we denote by Δ_F the set of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ satisfying following conditions:

(F_1) F is strictly increasing;

(F_2) for all sequence $\{\alpha_n\} \subseteq R^+$, $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;

(F_3) there exists $0 < k < 1$ such that $\lim_{n\to 0^+} \alpha^k F(\alpha) = 0$.

Now we introduce the following family of new functions.

Let Δ_G denotes the set of all functions $G : R^{+^4} \to R^+$ satisfying:

(G) for all $t_1, t_t, t_3, t_4 \in R^+$ with $t_1t_2t_3t_4 = 0$ there exists $\tau > 0$ such that $G(t_1, t_2, t_3, t_4) = \tau$.

Example 2.1. if $G(t_1, t_2, t_3, t_4) = L \min\{t_1, t_2, t_3, t_4\} + \tau$ where $L \in \mathbb{R}^+$ and $\tau > 0$, then $G \in \Delta_G$.

Example 2.2. if $G(t_1, t_2, t_3, t_4) = \tau e^{L \min\{t_1, t_2, t_3, t_4\}}$ where $L \in \mathbb{R}^+$ and $\tau > 0$, then $G \in \Delta_G$.

Example 2.3. if $G(t_1, t_2, t_3, t_4) = L \ln(\min\{t_1, t_2, t_3, t_4\} + 1) + \tau$ where $L \in \mathbb{R}^+$ and $\tau > 0$, then $G \in \Delta_G$.

Definition 2.1. Let (X, d) be a metric space and T be a self-mapping on X. Also suppose that $\alpha, \eta : X \times X \to [0, \infty)$ be two functions. We say T is an α-η-GF-contraction if for $x, y \in X$ with $\eta(x, Tx) \leq \alpha(x, y)$ and $d(Tx, Ty) > 0$ we have,

$$G\big(d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\big) + F\big(d(Tx, Ty)\big) \leq F\big(d(x, y)\big) \tag{2.1}$$

where $G \in \Delta_G$ and $F \in \Delta_F$.

Now we state and prove our main result of this section.

Theorem 2.1. *Let (X, d) be a complete metric space. Let $T : X \to X$ be a self-mapping satisfying the following assertions:*

(i) *T is an α-admissible mapping with respect to η;*

(ii) *T is an α-η-GF-contraction;*

(iii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$;*

(iv) *T is an α-η-continuous.*

Then T has a fixed point. Moreover, T has a unique fixed point when $\alpha(x, y) \geq \eta(x, x)$ for all $x, y \in Fix(T)$.

Proof. Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$. For $x_0 \in X$, we define the sequence $\{x_n\}$ by $x_n = T^n x_0 = Tx_n$. Now since, T is an α-admissible mapping with respect to η then, $\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0) = \eta(x_0, x_1)$. By continuing this process we have,

$$\eta(x_{n-1}, Tx_{n-1}) = \eta(x_{n-1}, x_n) \leq \alpha(x_{n-1}, x_n)$$

for all $n \in \mathbb{N}$. Also, let there exists $n_0 \in \mathbb{N}$ such that, $x_{n_0} = x_{n_0+1}$. Then x_{n_0} is fixed point of T and we have nothing to prove. Hence, we assume, $x_n \neq x_{n+1}$ or $d(Tx_{n-1}, Tx_n) > 0$ for all $n \in \mathbb{N} \cup \{0\}$. Since, T is an α-η-GF-contraction, so we derive,

$$\begin{aligned} &G\big(d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), d(x_{n-1}, Tx_n), d(x_n, Tx_{n-1})\big) \\ &\quad + F\big(d(Tx_{n-1}, Tx_n)\big) \leq F\big(d(x_{n-1}, x_n)\big) \end{aligned}$$

which implies,

$$\begin{aligned} &G\big(d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1}), 0\big) + F\big(d(x_n, x_{n+1})\big) \\ &\leq F\big(d(x_{n-1}, x_n)\big). \end{aligned} \tag{2.2}$$

Now since, $d(x_{n-1}, x_n).d(x_n, x_{n+1}).d(x_{n-1}, x_{n+1}).0 = 0$, so from (G) there exists $\tau > 0$ such that,

$$G\big(d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1}), 0\big) = \tau.$$

From (2.2) we deduce that,

$$F\big(d(x_n, x_{n+1})\big) \leq F\big(d(x_{n-1}, x_n)\big) - \tau.$$

Therefore,

$$\begin{aligned} F\big(d(x_n, x_{n+1})\big) &\leq F\big(d(x_{n-1}, x_n)\big) - \tau \\ &\leq F\big(d(x_{n-2}, x_{n-1})\big) - 2\tau \leq \ldots \leq F(d(x_0, x_1)) - n\tau. \end{aligned} \tag{2.3}$$

By taking limit as $n \to \infty$ in (2.3) we have, $\lim_{n\to\infty} F\big(d(x_n, x_{n+1})\big) = -\infty$, and since, $F \in \Delta_F$ we obtain,

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.4}$$

Now from (F3), there exists $0 < k < 1$ such that,

$$\lim_{n\to\infty} [d(x_n, x_{n+1})]^k F\big(d(x_n, x_{n+1})\big) = 0. \tag{2.5}$$

By (2.3) we have,

$$\lim_{n\to\infty}[d(x_n,x_{n+1})]^k[F\big(d(x_n,x_{n+1})\big)-F(d(x_0,x_1))] \\ \leq -n\tau[d(x_n,x_{n+1})]^k \leq 0. \tag{2.6}$$

By taking limit as $n \to \infty$ in (2.6) and applying (2.4) and (2.5) we have,

$$\lim_{n\to\infty} n[d(x_n,x_{n+1})]^k = 0. \tag{2.7}$$

It follows from (2.7) that there exists, $n_1 \in \mathbb{N}$ such that,

$$n[d(x_n,x_{n+1})]^k \leq 1$$

for all $n > n_1$. This implies,

$$d(x_n,x_{n+1}) \leq \frac{1}{n^{1/k}}$$

for all $n > n_1$. Now for $m > n > n_1$ we have,

$$d(x_n,x_m) \leq \sum_{i=n}^{m-1} d(x_i,x_{i+1}) \leq \sum_{i=n}^{m-1} \frac{1}{i^{1/k}}.$$

Since, $0 < k < 1$, then $\sum_{i=1}^{\infty} \frac{1}{i^{1/k}}$ converges. Therefore, $d(x_n,x_m) \to 0$ as $m,n \to \infty$. Thus we proved that $\{x_n\}$ is a Cauchy sequence. Completeness of X ensures that there exist $x^* \in X$ such that, $x_n \to x^*$ as $n \to \infty$. Now since, T is an α-η-continuous and $\eta(x_{n-1},x_n) \leq \alpha(x_{n-1},x_n)$ then, $x_{n+1} = Tx_n \to Tx^*$ as $n \to \infty$. That is, $x^* = Tx^*$. Thus T has a fixed point.

Let $x, y \in Fix(T)$ where $x \neq y$. Then from

$$G\big(d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\big) + F\big(d(Tx,Ty)\big) \leq F\big(d(x,y)\big)$$

we get,

$$\tau + F\big(d(x,y)\big) \leq F\big(d(x,y)\big)$$

which is a contradiction. Hence, $x = y$. Therefore, T has a unique fixed point. ∎

Combining Theorem 2.1 and Example 2.1 we deduce the following Corollary.

Corollary 2.1. *Let (X,d) be a complete metric space and $T : X \to X$ be a self-mapping satisfying the following assertions:*

(*i*) *T is an α-admissible mapping with respect to η;*

(*ii*) *if for $x, y \in X$ with $\eta(x,Tx) \leq \alpha(x,y)$ and $d(Tx,Ty) > 0$ we have,*

$$\tau + F\big(d(Tx,Ty)\big) \leq F\big(d(x,y)\big) \tag{2.8}$$

where $\tau > 0$ *and* $F \in \Delta_F$.

(iii) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$;

(iv) T *is an* α-η-*continuous function.*

Then T *has a fixed point. Moreover,* T *has a unique fixed point when* $\alpha(x, y) \geq \eta(x, y)$ *for all* $x, y \in Fix(T)$.

Theorem 2.2. *Let* (X, d) *be a complete metric space. Let* $T : X \to X$ *be a self-mapping satisfying the following assertions:*

(i) T *is a* α-*admissible mapping with respect to* η;

(ii) T *is an* α-η-GF-*contraction;*

(iii) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$;

(iv) *if* $\{x_n\}$ *is a sequence in* X *such that* $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ *with* $x_n \to x$ *as* $n \to \infty$, *then either*

$$\eta(Tx_n, T^2x_n) \leq \alpha(Tx_n, x) \text{ or } \eta(T^2x_n, T^3x_n) \leq \alpha(T^2x_n, x)$$

holds for all $n \in \mathbb{N}$.

Then T *has a fixed point. Moreover,* T *has a unique fixed point whenever* $\alpha(x, y) \geq \eta(x, x)$ *for all* $x, y \in Fix(T)$.

Proof. Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$. As in proof of Theorem 2.1 we can conclude that

$$\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1}) \text{ and } x_n \to x^* \text{ as } n \to \infty$$

where, $x_{n+1} = Tx_n$. So, from (iv), either

$$\eta(Tx_n, T^2x_n) \leq \alpha(Tx_n, x^*) \text{ or } \eta(T^2x_n, T^3x_n) \leq \alpha(T^2x_n, x^*)$$

holds for all $n \in \mathbb{N}$. This implies,

$$\eta(x_{n+1}, x_{n+2}) \leq \alpha(x_{n+1}, x) \text{ or } \eta(x_{n+2}, x_{n+3}) \leq \alpha(x_{n+2}, x)$$

holds for all $n \in \mathbb{N}$. Equivalently, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\eta(x_{n_k}, Tx_{n_k}) = \eta(x_{n_k}, x_{n_k+1}) \leq \alpha(x_{n_k}, x^*)$$

and so from (2.1) we deduce that,

$$G\big(d(x_{n_k}, Tx_{n_k}), d(x^*, Tx^*), d(x_{n_k}, Tx^*), d(x^*, Tx_{n_k})\big) + F\big(d(Tx_{n_k}, Tx^*)\big)$$
$$\leq F\big(d(x_{n_k}, x^*)\big)$$

which implies,

$$F\big(d(Tx_{n_k}, Tx^*)\big) \le F\big(d(x_{n_k}, x^*)\big). \tag{2.9}$$

From (F1) we have,

$$d(x_{n_k+1}, Tx^*) < d(x_{n_k}, x^*).$$

By taking limit as $k \to \infty$ in the above inequality we get, $d(x^*, Tx^*) = 0$. i.e., $x^* = Tx^*$. Uniqueness follows similarly as in Theorem 2.1. ■

Combining Theorem 2.2 and Example 2.1 we deduce the following Corollary.

Corollary 2.2. *Let (X, d) be a complete metric space. Let $T : X \to X$ be a self-mapping satisfying the following assertions:*

(i) *T is a α-admissible mapping with respect to η;*

(ii) *if for $x, y \in X$ with $\eta(x, Tx) \le \alpha(x, y)$ and $d(Tx, Ty) > 0$ we have,*

$$\tau + F\big(d(Tx, Ty)\big) \le F\big(d(x, y)\big)$$

where $\tau > 0$ and $F \in \Delta_F$.

(iii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \ge \eta(x_0, Tx_0)$;*

(iv) *if $\{x_n\}$ be a sequence in X such that $\alpha(x_n, x_{n+1}) \ge \eta(x_n, x_{n+1})$ with $x_n \to x$ as $n \to \infty$, then either*

$$\eta(Tx_n, T^2x_n) \le \alpha(Tx_n, x) \text{ or } \eta(T^2x_n, T^3x_n) \le \alpha(T^2x_n, x)$$

holds for all $n \in \mathbb{N}$.

Then T has a fixed point. Moreover, T has a unique fixed point when $\alpha(x, y) \ge \eta(x, x)$ for all $x, y \in Fix(T)$.

If in Corollary 2.2 we take $\alpha(x, y) = \eta(x, y) = 1$ for all $x, y \in X$, then we deduce the following Corollary.

Corollary 2.3. (Theorem 2.1 of [19]). *Let (X, d) be a complete metric space and $T : X \to X$ be a self-mapping. If for $x, y \in X$ with $d(Tx, Ty) > 0$ we have,*

$$\tau + F\big(d(Tx, Ty)\big) \le F\big(d(x, y)\big)$$

where $\tau > 0$ and $F \in \Delta_F$. Then T has a fixed point.

Example 2.4. Let $X = [0, +\infty)$. We endow X with usual metric. Define, $T : X \to X$, $\alpha, \eta : X \times X \to [0, \infty)$, $G : {R^+}^4 \to R^+$ and $F : R^+ \to R$ by,

$$Tx = \begin{cases} \frac{1}{2}e^{-\tau}x^2, & \text{if } x \in [0, 1] \\ 3x & \text{if } x \in (1, \infty) \end{cases}$$

$\alpha(x,y)=\begin{cases}\dfrac{1}{2}, & \text{if } x,y\in[0,1]\\ \dfrac{1}{9}, & \text{otherwise}\end{cases}$ and $\eta(x,y)=\frac{1}{4}$, $G(t_1,t_2,t_3,t_4)=\tau$ where $\tau>0$ and $F(r)=\ln r$.

Let, $\alpha(x,y)\geq\eta(x,y)$, then $x,y\in[0,1]$. On the other hand, $Tw\in[0,1]$ for all $w\in[0,1]$. Then, $\alpha(Tx,Ty)\geq\eta(Tx,Ty)$. That is, T is an α-admissible mapping with respect to η. If $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1})\geq\eta(x_n,x_{n+1})$ with $x_n\to x$ as $n\to\infty$. Then, $Tx_n,T^2x_n,T^3x_n\in[0,1]$ for all $n\in\mathbb{N}$. That is,

$$\eta(Tx_n,T^2x_n)\leq\alpha(Tx_n,x) \text{ and } \eta(T^2x_n,T^3x_n)\leq\alpha(T^2x_n,x)$$

hold for all $n\in\mathbb{N}$. Clearly, $\alpha(0,T0)\geq\eta(0,T0)$. Let, $\alpha(x,y)\geq\eta(x,Tx)$. Now, if $x\notin[0,1]$ or $y\notin[0,1]$, then, $\dfrac{1}{9}\geq\dfrac{1}{4}$, which is a contradiction, so $x,y\in[0,1]$ and hence we obtain,

$$d(Tx,Ty)=\frac{1}{2}e^{-\tau}|x^2-y^2| \;=\; \frac{1}{2}e^{-\tau}|x-y||x+y|\leq e^{-\tau}|x-y|=e^{-\tau}d(x,y)$$

which implies,

$$\tau+F(d(Tx,Ty))=\tau+\ln d(Tx,Ty)\leq\tau+\ln e^{-\tau}d(x,y)=\ln d(x,y)=F(d(x,y)).$$

Hence, T is an α-η-GF-contraction mapping. Thus all conditions of Corollary 2.2 (and Theorem 2.2) hold and T has a fixed point. Let $x=0$, $y=2$ and $\tau>0$. Then,

$$\tau+F(d(T0,T2))\geq F(d(T0,T2))=\ln 6>\ln 2=F(d(0,2)).$$

That is Theorem 2.1 of [19] can not be applied for this example.

Recall that a self-mapping T is said to have the property P if $Fix(T^n)=F(T)$ for every $n\in\mathbb{N}$.

Theorem 2.3. *Let (X,d) be a complete metric space and $T:X\to X$ be an α-continuous self-mapping. Assume that there exists $\tau>0$ such that*

$$\tau+F\big(d(Tx,T^2x)\big)\leq F\big(d(x,Tx)\big) \tag{2.10}$$

holds for all $x\in X$ with $d(Tx,T^2x)>0$ where $F\in\Delta_F$. If T is an α-admissible mapping and there exists $x_0\in X$ such that, $\alpha(x_0,Tx_0)\geq 1$, then T has the property P.

Proof. Let $x_0\in X$ such that $\alpha(x_0,Tx_0)\geq 1$. For $x_0\in X$, we define the sequence $\{x_n\}$ by $x_n=T^nx_0=Tx_n$. Now since, T is an α-admissible mapping, so $\alpha(x_1,x_2)=\alpha(Tx_0,Tx_1)\geq 1$. By continuing this process we have,

$$\alpha(x_{n-1},x_n)\geq 1$$

for all $n \in \mathbb{N}$. If there exists $n_0 \in \mathbb{N}$ such that, $x_{n_0} = x_{n_0+1} = Tx_{n_0}$. Then x_{n_0} is fixed point of T and we have nothing to prove. Hence, we assume, $x_n \neq x_{n+1}$ or $d(Tx_{n-1}, T^2x_{n-1}) > 0$ for all $n \in \mathbb{N} \cup \{0\}$. From (2.10) we have,

$$\tau + F\big(d(Tx_{n-1}, T^2x_{n-1})\big) \leq F\big(d(x_{n-1}, Tx_{n-1})\big)$$

which implies,

$$\tau + F\big(d(x_n, x_{n+1})\big) \leq F\big(d(x_{n-1}, x_n)\big) \tag{2.11}$$

and so,

$$F\big(d(x_n, x_{n+1})\big) \leq F\big(d(x_{n-1}, x_n)\big) - \tau.$$

Therefore,

$$\begin{aligned} F\big(d(x_n, x_{n+1})\big) &\leq F\big(d(x_{n-1}, x_n)\big) - \tau \leq F\big(d(x_{n-2}, x_{n-1})\big) - 2\tau \\ &\leq \ldots \leq F(d(x_0, x_1)) - n\tau. \end{aligned} \tag{2.12}$$

By taking limit as $n \to \infty$ in (2.12) we have, $\lim_{n\to\infty} F\big(d(x_n, x_{n+1})\big) = -\infty$, and since, $F \in \Delta_F$ we obtain,

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.13}$$

Now from (F3), there exists $0 < k < 1$ such that,

$$\lim_{n\to\infty} [d(x_n, x_{n+1})]^k F\big(d(x_n, x_{n+1})\big) = 0. \tag{2.14}$$

By (2.12) we have,

$$\begin{aligned} &\lim_{n\to\infty} [d(x_n, x_{n+1})]^k [F\big(d(x_n, x_{n+1})\big) - F(d(x_0, x_1))] \\ &\leq -n\tau[d(x_n, x_{n+1})]^k \leq 0. \end{aligned} \tag{2.15}$$

By taking limit as $n \to \infty$ in (2.15) and applying (2.13) and (2.14) we have,

$$\lim_{n\to\infty} n[d(x_n, x_{n+1})]^k = 0. \tag{2.16}$$

Consequently, there exists, $n_1 \in \mathbb{N}$ such that,

$$n[d(x_n, x_{n+1})]^k \leq 1$$

for all $n > n_1$. This implies,

$$d(x_n, x_{n+1}) \leq \frac{1}{n^{1/k}}$$

for all $n > n_1$. Now for $m > n > n_1$ we have,

$$d(x_n, x_m) \leq \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \leq \sum_{i=n}^{m-1} \frac{1}{i^{1/k}}$$

Since, $0 < k < 1$, then $\sum_{i=n}^{\infty} \frac{1}{i^{1/k}}$ converges. Therefore, $d(x_n, x_m) \to 0$ as $m, n \to \infty$. Thus we proved that $\{x_n\}$ is a Cauchy sequence. Completeness of X ensures that there exists $x^* \in X$ such that, $x_n \to x^*$ as $n \to \infty$. Now since, T is α-continuous and $\alpha(x_{n-1}, x_n) \geq 1$ then, $x_{n+1} = Tx_n \to Tx^*$ as $n \to \infty$. That is, $x^* = Tx^*$. Thus T has a fixed point and $F(T^n) = F(T)$ for $n = 1$. Let $n > 1$. Assume contrarily that $w \in F(T^n)$ and $w \notin F(T)$. Then, $d(w, Tw) > 0$. Now we have,

$$\begin{aligned} F(d(w, Tw)) &= F(d(T(T^{n-1}w)), T^2(T^{n-1}w))) \\ &\leq F(d(T^{n-1}w), T^n w)) - \tau \\ &\leq F(d(T^{n-2}w), T^{n-1}w)) - 2\tau \leq \cdots \\ &\leq d(w, Tw) - n\tau. \end{aligned}$$

By taking limit as $n \to \infty$ in the above inequality we have, $F(d(w, Tw)) = -\infty$. Hence, by (F2) we get, $d(w, Tw) = 0$ which is a contradictions. Therefore, $F(T^n) = F(T)$ for all $n \in \mathbb{N}$. ∎

Let $(X, d, \preceq)$ be a partially ordered metric space. Recall that $T : X \to X$ is nondecreasing if $\forall x\ , y \in X,\ x \preceq y \Rightarrow T(x) \preceq T(y)$. Fixed point theorems for monotone operators in ordered metric spaces are widely investigated and have found various applications in differential and integral equations (see [1, 3, 7, 8, 10, 11] and references therein). From Theorems 2.1-2.3, we derive following new results in partially ordered metric spaces.

Theorem 2.4. *Let $(X, d, \preceq)$ be a complete partially ordered metric space. Assume that the following assertions hold true:*

(i) *T is nondecreasing and ordered GF-contraction;*

(ii) *there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;*

(iii) *either for a given $x \in X$ and sequence $\{x_n\}$*

$$x_n \to x \textit{ as } n \to \infty \textit{ and } x_n \preceq x_{n+1} \textit{ for all } n \in \mathbb{N} \textit{ we have } Tx_n \to Tx$$

or if $\{x_n\}$ is a sequence such that $x_n \preceq x_{n+1}$ with $x_n \to x$ as $n \to \infty$, then either

$$Tx_n \preceq x, \textit{ or } T^2x_n \preceq x$$

holds for all $n \in \mathbb{N}$.

Then T has a fixed point.

Theorem 2.5. *Let $(X, d, \preceq)$ be a complete partially ordered metric space. Assume that the following assertions hold true:*

(i) *T is nondecreasing and satisfies* (2.10) *for all $x \in X$ with $d(Tx, T^2x) > 0$ where $F \in \Delta_F$ and $\tau > 0$;*

(ii) *there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;*

(iii) *for a given $x \in X$ and sequence $\{x_n\}$*

$$x_n \to x \text{ as } n \to \infty \text{ and } x_n \preceq x_{n+1} \text{ for all } n \in \mathbb{N} \text{ we have } Tx_n \to Tx.$$

Then T has a property P.

3. Suzuki-wardowski Type Fixed Point Results

In this section, as an application of our results proved above, we deduce certain Suzuki-Wardowski type fixed point theorems.

Theorem 3.1. *Let (X, d) be a complete metric space and T be a continuous self-mapping on X. If for $x, y \in X$ with $d(x, Tx) \le d(x, y)$ and $d(Tx, Ty) > 0$ we have,*

$$G\big(d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\big) + F\big(d(Tx,Ty)\big) \le F\big(d(x,y)\big) \tag{3.1}$$

where $G \in \Delta_G$ and $F \in \Delta_F$. Then T has a unique fixed point.

Proof. Define, $\alpha, \eta : X \times X \to [0, \infty)$ by

$$\alpha(x,y) = d(x,y) \text{ and } \eta(x,y) = d(x,y)$$

for all $x, y \in X$. Now, since, $d(x,y) \le d(x,y)$ for all $x, y \in X$, so $\eta(x,y) \le \alpha(x,y)$ for all $x, y \in X$. That is, conditions (i) and (iii) of Theorem 2.1 hold true. Since T is continuous, so T is α-η-continuous. Let, $\eta(x,Tx) \le \alpha(x,y)$ with $d(Tx,Ty) > 0$. Equivalently, if $d(x,Tx) \le d(x,y)$ with $d(Tx,Ty) > 0$, then, from (3.1) we have,

$$G\big(d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\big) + F\big(d(Tx,Ty)\big) \le F\big(d(x,y)\big).$$

That is, T is an α-η-GF-contraction mapping. Hence, all conditions of Theorem 2.1 hold and T has a unique fixed point. ■

Corollary 3.1. *Let (X, d) be a complete metric space and T be a continuous self-mapping on X. If for $x, y \in X$ with $d(x, Tx) \le d(x, y)$ and $d(Tx, Ty) > 0$ we have*

$$\tau + F\big(d(Tx,Ty)\big) \le F\big(d(x,y)\big)$$

where $\tau > 0$ and $F \in \Delta_F$. Then T has a unique fixed point.

Corollary 3.2. *Let (X,d) be a complete metric space and T be a continuous self-mapping on X. If for $x,y\in X$ with $d(x,Tx)\le d(x,y)$ and $d(Tx,Ty)>0$ we have,*

$$\tau e^{L\min\{d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\}}+F\big(d(Tx,Ty)\big)\le F\big(d(x,y)\big)$$

where $\tau>0$, $L\ge 0$ and $F\in\Delta_F$. Then T has a unique fixed point.

Theorem 3.2. *Let (X,d) be a complete metric space and T be a self-mapping on X. Assume that there exists $\tau>0$ such that*

$$\frac{1}{2(1+\tau)}d(x,Tx)\le d(x,y)\text{ implies }\tau+F\big(d(Tx,Ty)\big)\le F\big(d(x,y)\big) \tag{3.2}$$

for $x,y\in X$ with $d(Tx,Ty)>0$ where $F\in\Delta_F$. Then T has a unique fixed point.

Proof. Define, $\alpha,\eta:X\times X\to[0,\infty)$ by

$$\alpha(x,y)=d(x,y)\text{ and }\eta(x,y)=\frac{1}{2(1+\tau)}d(x,y)$$

for all $x,y\in X$ where $\tau>0$. Now, since, $\frac{1}{2(1+\tau)}d(x,y)\le d(x,y)$ for all $x,y\in X$, so $\eta(x,y)\le\alpha(x,y)$ for all $x,y\in X$. That is, conditions (i) and (iii) of Theorem 2.2 hold true. Let, $\{x_n\}$ be a sequence with $x_n\to x$ as $n\to\infty$. Assume that $d(Tx_n,T^2x_n)=0$ for some n. Then $Tx_n=T^2x_n$. That is Tx_n is a fixed point of T and we have nothing to prove. Hence we assume, $Tx_n\neq T^2x_n$ for all $n\in\mathbb{N}$. Since, $\frac{1}{2(1+\tau)}d(Tx_n,T^2x_n)\le d(Tx_n,T^2x_n)$ for all $n\in\mathbb{N}$. Then from (3.2) we get,

$$F\big(d(T^2x_n,T^3x_n)\big)\le\tau+F\big(d(T^2x_n,T^3x_n)\big)\le F\big(d(Tx_n,T^2x_n)\big)$$

and so from (F1) we get,

$$d(T^2x_n,T^3x_n)\le d(Tx_n,T^2x_n). \tag{3.3}$$

Assume there exists $n_0\in\mathbb{N}$ such that,

$$\eta(Tx_{n_0},T^2x_{n_0})>\alpha(Tx_{n_0},x)\text{ and }\eta(T^2x_{n_0},T^3x_{n_0})>\alpha(T^2x_{n_0},x)$$

then,

$$\frac{1}{2(1+\tau)}d(Tx_{n_0},T^2x_{n_0})>d(Tx_{n_0},x)\text{ and }\frac{1}{2(1+\tau)}d(T^2x_{n_0},T^3x_{n_0})>d(T^2x_{n_0},x)$$

so by (3.3) we have,

$$\begin{aligned}
d(Tx_{n_0},T^2x_{n_0}) &\le d(Tx_{n_0},x)+d(T^2x_{n_0},x)\\
&< \frac{1}{2(1+\tau)}d(Tx_{n_0},T^2x_{n_0})+\frac{1}{2(1+\tau)}d(T^2x_{n_0},T^3x_{n_0})\\
&\le \frac{1}{2(1+\tau)}d(Tx_{n_0},T^2x_{n_0})+\frac{1}{2(1+\tau)}d(Tx_{n_0},T^2x_{n_0})\\
&= \frac{2}{2(1+\tau)}d(Tx_{n_0},T^2x_{n_0})\le d(Tx_{n_0},T^2x_{n_0})
\end{aligned}$$

which is a contradiction. Hence, either

$$\eta(Tx_n, T^2x_n) \leq \alpha(Tx_n, x) \text{ or } \eta(T^2x_n, T^3x_n) \leq \alpha(T^2x_n, x)$$

holds for all $n \in \mathbb{N}$. That is condition (iv) of Theorem 2.2 holds. Let, $\eta(x, Tx) \leq \alpha(x, y)$. So, $\frac{1}{2(1+\tau)}d(x, Tx) \leq d(x, y)$. Then from (3.2) we get, $\tau + F\big(d(Tx, Ty)\big) \leq F\big(d(x, y)\big)$. Hence, all conditions of Theorem 2.2 hold and T has a unique fixed point. ■

4. Applications to Orbitally Continuous Mappings

Theorem 4.1. *Let (X, d) be a complete metric space and $T : X \to X$ be a self-mapping satisfying the following assertions:*

(i) *for $x, y \in O(w)$ with $d(Tx, Ty) > 0$ we have,*

$$G\big(d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\big) + F\big(d(Tx, Ty)\big) \leq F\big(d(x, y)\big)$$

where $G \in \Delta_G$ and $F \in \Delta_F$;

(ii) *T is an orbitally continuous function.*

Then T has a fixed point. Moreover, T has a unique fixed point when $Fix(T) \subseteq O(w)$.

Proof. Define, $\alpha, \eta : X \times X \to [0, +\infty)$ by

$$\alpha(x, y) = \begin{cases} 3, & \text{if } x, y \in O(w) \\ 0, & \text{otherwise} \end{cases} \quad \text{and } \eta(x, y) = 1$$

where $O(w)$ is an orbit of a point $w \in X$. From Remark 1.1 we know that T is an α-η-continuous mapping. Let, $\alpha(x, y) \geq \eta(x, y)$, then $x, y \in O(w)$. So $Tx, Ty \in O(w)$. That is, $\alpha(Tx, Ty) \geq \eta(Tx, Ty)$. Therefore, T is an α-admissible mapping with respect to η. Since $w, Tw \in O(w)$, then $\alpha(w, Tw) \geq \eta(w, Tw)$. Let, $\alpha(x, y) \geq \eta(x, Tx)$ and $d(Tx, Ty) > 0$. Then, $x, y \in O(w)$ and $d(Tx, Ty) > 0$. Therefore from (i) we have,

$$G\big(d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\big) + F\big(d(Tx, Ty)\big) \leq F\big(d(x, y)\big)$$

which implies, T is an α-η-GF-contraction mapping. Hence, all conditions of Theorem 2.1 hold true and T has a fixed point. If $Fix(T) \subseteq O(w)$, then, $\alpha(x, y) \geq \eta(x, y)$ for all $x, y \in Fix(T)$ and so from Theorem 2.1 T has a unique fixed point. ■

Corollary 4.1. *Let (X, d) be a complete metric space and $T : X \to X$ be a self-mapping satisfying the following assertions:*

(i) *for* $x, y \in O(w)$ *with* $d(Tx, Ty) > 0$ *we have,*

$$\tau + F\big(d(Tx, Ty)\big) \leq F\big(d(x, y)\big)$$

where $\tau > 0$ *and* $F \in \Delta_F$*;*

(ii) T *is orbitally continuous.*

Then T *has a fixed point. Moreover,* T *has a unique fixed point when* $Fix(T) \subseteq O(w)$.

Corollary 4.2. *Let* (X, d) *be a complete metric space and* $T : X \to X$ *be a self-mapping satisfying the following assertions:*

(i) *for* $x, y \in O(w)$ *with* $d(Tx, Ty) > 0$ *we have,*

$$\tau e^{L \min\{d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}} + F\big(d(Tx, Ty)\big) \leq F\big(d(x, y)\big)$$

where $\tau > 0$*,* $L \geq 0$ *and* $F \in \Delta_F$*;*

(ii) T *is orbitally continuous.*

Then T *has a fixed point. Moreover,* T *has a unique fixed point when* $Fix(T) \subseteq O(w)$.

Theorem 4.2. *(Theorem 4 of [2]) Let* (X, d) *be a complete metric space and* $T : X \to X$ *be a self-mapping satisfying the following assertions:*

(i) for $x \in X$ *with* $d(Tx, T^2x) > 0$ *we have,*

$$\tau + F\big(d(Tx, T^2x)\big) \leq F\big(d(x, Tx)\big)$$

where $\tau > 0$ *and* $F \in \Delta_F$*;*

(ii) T *is an orbitally continuous function.*

Then T *has the property* P*.*

Proof. Define, $\alpha : X \times X \to [0, +\infty)$ by

$$\alpha(x, y) = \begin{cases} 1, & \text{if } x \in O(w) \\ 0, & \text{otherwise} \end{cases}$$

where $w \in X$. Let, $\alpha(x, y) \geq 1$, then $x, y \in O(w)$. So $Tx, Ty \in O(w)$. That is, $\alpha(Tx, Ty) \geq 1$. Therefore, T is α-admissible mapping. Since $w, Tw \in O(w)$, so $\alpha(w, Tw) \geq 1$. By Remark 1.1 we conclude that T is an α-continuous mapping. If, $x \in X$ with $d(Tx, T^2x) > 0$, then, from (i) we have,

$$\tau + F\big(d(Tx, T^2x)\big) \leq F\big(d(x, Tx)\big).$$

Thus all conditions of Theorem 2.3 hold true and T has the property P. ∎

We can easily deduce following results involving integral inequalities.

Theorem 4.3. *Let (X,d) be a complete metric space and T be a continuous self-mapping on X. If for $x,y \in X$ with*

$$\int_0^{d(x,Tx)} \rho(t)dt \le \int_0^{d(x,y)} \rho(t)dt \text{ and } \int_0^{d(Tx,Ty)} \rho(t)dt > 0$$

we have,

$$G\Big(\int_0^{d(x,Tx)} \rho(t)dt, \int_0^{d(y,Ty)} \rho(t)dt, \int_0^{d(x,Ty)} \rho(t)dt, \int_0^{d(y,Tx)} \rho(t)dt\Big) + F\big(\int_0^{d(Tx,Ty)} \rho(t)dt\big) \le F\big(\int_0^{d(x,y)} \rho(t)dt\big)$$

where $G \in \Delta_G$, $F \in \Delta_F$ and $\rho : [0,\infty) \to [0,\infty)$ is a Lebesgue-integrable mapping satisfying $\int_0^{\varepsilon} \rho(t)dt > 0$ for $\varepsilon > 0$. Then T has a unique fixed point.

Theorem 4.4. *Let (X,d) be a complete metric space and T be a self-mapping on X. Assume that there exists $\tau > 0$ such that*

$$\frac{1}{2(1+\tau)} \int_0^{d(x,Tx)} \rho(t)dt \le \int_0^{d(x,y)} \rho(t)dt \Rightarrow$$

$$\tau + F\big(\int_0^{d(Tx,Ty)} \rho(t)dt\big) \le F\big(\int_0^{d(x,y)} \rho(t)dt\big)$$

for $x,y \in X$ with $\int_0^{d(Tx,Ty)} \rho(t)dt > 0$ where $F \in \Delta_F$ and $\rho : [0,\infty) \to [0,\infty)$ is a Lebesgue-integrable mapping satisfying $\int_0^{\varepsilon} \rho(t)dt > 0$ for $\varepsilon > 0$. Then T has a unique fixed point.

Theorem 4.5. *Let (X,d) be a complete metric space and $T : X \to X$ be a self-mapping satisfying the following assertions:*

(i) *for $x,y \in O(w)$ with $\int_0^{d(Tx,Ty)} \rho(t)dt > 0$ we have,*

$$G\big(\int_0^{d(x,Tx)} \rho(t)dt, \int_0^{d(y,Ty)} \rho(t)dt, \int_0^{d(x,Ty)} \rho(t)dt, \int_0^{d(y,Tx)} \rho(t)dt\big) + F\big(\int_0^{d(Tx,Ty)} \rho(t)dt\big) \le F\big(\int_0^{d(x,y)} \rho(t)dt\big)$$

where $G \in \Delta_G$, $F \in \Delta_F$ and $\rho : [0,\infty) \to [0,\infty)$ is a Lebesgue-integrable mapping satisfying $\int_0^{\varepsilon} \rho(t)dt > 0$ for $\varepsilon > 0$.

(ii) *T is an orbitally continuous function;*

Then T has a fixed point. Moreover, T has a unique fixed point when $Fix(T) \subseteq O(w)$.

Theorem 4.6. *Let (X, d) be a complete metric space and $T : X \to X$ be a self-mapping satisfying the following assertions:*

(i) *for $x \in X$ with $\int_0^{d(Tx, T^2x)} \rho(t)dt > 0$ we have,*

$$\tau + F\big(\int_0^{d(Tx, T^2x)} \rho(t)dt\big) \le F\big(\int_0^{d(x, Tx)} \rho(t)dt\big)$$

where $\tau > 0$ and $F \in \Delta_F$ and $\rho : [0, \infty) \to [0, \infty)$ is a Lebesgue-integrable mapping satisfying $\int_0^{\varepsilon} \rho(t)dt > 0$ for $\varepsilon > 0$.

(ii) *T is an orbitally continuous function.*

Then T has the property P.

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