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THE CANTOR MANIFOLD THEOREM WITH SYMMETRY AND APPLICATIONS TO PDEs

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Abstract. In this paper we introduce a new Cantor manifold theorem and then apply it to one new type of one-dimensional ($1d$) beam equations

$$u_{tt} + u_{xxxx} + mu - 2u^2 u_{xx} - 2uu_x^2 = 0, \qquad m > 0,$$

with periodic boundary conditions. We show that the above equation admits small-amplitude linearly stable quasi-periodic solutions corresponding to finite dimensional invariant tori of an associated infinite dimensional dynamical system. The proof is based on a partial Birkhoff normal form and an infinite dimensional KAM theorem for Hamiltonians with symmetry(cf. [19]).

1. Introduction and Main Result

1.1. Cantor Manifold Theorem with Symmetry

In [23] Kuksin and Pöschel introduced the famous Cantor manifold theorem, which can be applied to one dimensional Schrödinger equations and wave equations under the Dirichlet boundary conditions. There are further progress in infinite dimensional KAM theory since then. We refer the readers to [3, 12] and [27] for one dimensional wave equations and to [14, 15, 24] and [26] for one-dimensional Schrödinger equations. The above mentioned quasi-periodic solutions are all close to 0. We refer to [19] for the existence of quasi-periodic solutions of defocusing nonlinear Schrödinger equations with large amplitude. More recently, there are also essential progress in nonlinear Schrödinger equations by Eliasson-Kuksin [13] (with Fourier multipliers) in any space

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dimension. We refer to [18] and [31] for further progress based on [13]. All above results obtain reducible elliptic tori.

If one only concerns with the existence of quasi-periodic solutions, many relevant results are first proved by Bourgain by extending the Newton approach introduced by Craig-Wayne [11] for periodic solutions. Its main advantage is to require only the first order Melnikov non-resonance conditions for solving the homological equations. Developing this perspective, Bourgain proved the existence of quasi-periodic solutions for any dimensional wave equations and Schrödinger equations (refer to [7, 8, 9]). Most recently, Berti and Bolle in [6] proved the existence of quasi-periodic solutions for n dimensional nonlinear Schrödinger equations with merely differentiable nonlinearities and a multiplicative and merely differentiable potential. The corresponding reducible quasi-periodic solutions results of nonlinear Schrödinger equations with merely differentiable nonlinearities for any dimension are open. For high dimensional wave equations the existence of reducible quasi-periodic solutions remains unsolved.

In this paper we prove a Cantor manifold theorem with symmetry, which can be applied to one type of one dimensional beam equations. There are two reasons for us to rewrite the Cantor manifold theorem. Firstly, one needs a new Cantor manifold theorem to deal with some PDEs under the periodic boundary conditions while the old version(refer to [23]) only deals with the Dirichlet boundary conditions. The techniques mainly come from [19] and [16]. Secondly, we develop a new idea in obtaining the non-degenerate condition for the first step in KAM iterations. We mention that in many cases it is very difficult to obtain the non-degenerate condition(refer to [24]). For technical reasons we weaken the non-degenerate conditions from A to A'(see below). To obtain the non-degenerate conditions A' we throw away suitable parameter set.

For introducing the theorem we consider a hamiltonian $H = \Lambda + Q + R$ in a neighbourhood of the origin in $\ell^{2,N}$, where R represents some higher order perturbation of an integrable normal form $\Lambda + Q$ and $\ell^{2,N}$ is defined in Section 4. The latter describes a family of linearly stable invariant tori of dimension $b := |A|$ with quasi-periodic motions. The dimension b is fixed, $1 \leq b < \infty$.

In complex coordinates $q = (\tilde{q}, \hat{q})$ on $\ell^{2,N}$, where $\tilde{q} = (q_j)_{j\in A}$ and $\hat{q} = (q_j)_{j\in B}$, and with

$$I = (|q_j|^2)_{j\in A}, \qquad Z = (|q_j|^2)_{j\in B},$$

the normal form consists of the terms

$$\Lambda = \langle \alpha_1, I\rangle + \langle \beta_1, Z\rangle, \qquad Q = \frac{1}{2}\langle A_1 I, I\rangle + \langle B_1 I, Z\rangle,$$

where α_1, β_1 and A_1, B_1 denote vectors and matrices with constant coefficients, respectively. Its equations of motion are

$$\dot{\tilde{q}} = \mathrm{i}diag(\alpha_1 + A_1 I + B_1^T Z)\tilde{q}, \qquad \dot{\hat{q}} = \mathrm{i}diag(\beta_1 + B_1 I)\hat{q}.$$

They admits a complex $b-$dimensional invariant manifold $E = \{\hat{q} = 0\}$, which is completely filled, up to the origin, by the invariant tori

$$\mathcal{T}(I) = \{\tilde{q} : |\tilde{q}_j|^2 = I_j \text{ for } j \in A\}, \qquad I \in \overline{\mathbb{P}^A},$$

where $\mathbb{P}^A = \{I : \ I_j > 0 \text{ for } j \in A\}$. On $\mathcal{T}(I)$ and its normal space, respectively, the flows are given by

$$\dot{\tilde{q}} = \mathrm{i}diag(\omega(I))\tilde{q}, \qquad \omega(I) = \alpha_1 + A_1 I,$$

$$\dot{\hat{q}} = \mathrm{i}diag(\Omega(I))\hat{q}, \qquad \Omega(I) = \beta_1 + B_1 I.$$

They are linear and in diagonal form. In particular, since $\Omega(I)$ is real, $\hat{q} = 0$ is an elliptic fixed point, all the tori are linearly stable, and all their orbits have zero Lyapunov exponents. As in [23], we call $\mathcal{T}(I)$ an elliptic rotational torus with frequencies $\omega(I)$. The following theorem is to show the persistence of a large portion of E forming an invariant Cantor manifold $\mathcal{E}$ for the hamiltonian $H = \Lambda + Q + R$.
That is, there exists a family of $b-$tori

$$\mathcal{T}[\mathcal{C}] = \bigcup_{I \in \mathcal{C}} \mathcal{T}(I) \subset E$$

over a Cantor set $\mathcal{C} \subset \mathbb{P}^A$ and a Lipschitz continuous embedding

$$\Psi : \ \mathcal{T}[\mathcal{C}] \hookrightarrow \ell^{2,N},$$

such that the restriction of Ψ to each torus $\mathcal{T}(I)$ in the family is an embedding of an elliptic rotational $b-$torus for the hamiltonian H. The image $\mathcal{E}$ of $\mathcal{T}(\mathcal{C})$ as [23] we call a Cantor manifold of elliptic rotational $b-$tori given by the embedding $\Psi : \mathcal{T}(\mathcal{C}) \to \mathcal{E}$. In addition, the Cantor set $\mathcal{C}$ has full density at the origin, the embedding Ψ is close to the inclusion map $\Psi_0 : E \hookrightarrow \ell^{2,N}$.

For giving the following theorem we introduce the notations $\nu = (\nu_j)_{j \in \mathbb{Z}}$ where $\nu_j = j$, for any $j \in \mathbb{Z}$. With the notation $\nu_A = (\nu_j)_{j \in A}$ and $\nu_B = (\nu_j)_{j \in B}$ one then has $k \cdot \nu_A = \sum\limits_{j \in A} jk_j$ and $m \cdot \nu_B = \sum\limits_{j \in B} jm_j$. For the existence of $\mathcal{E}$ the following assumptions are made.

A. Nondegeneracy. The normal form $\Lambda + Q$ is nondegenerate in the sense that

$$\det A_1 \neq 0,$$
$$\langle k, \omega(I) \rangle + \langle l, \Omega(I) \rangle \not\equiv 0,$$

for all $(k, l) \in \mathbb{Z}^A \times \mathbb{Z}^B$ with $1 \leq |l| \leq 2$, where $\omega = \alpha_1 + A_1 I$ and $\Omega = \beta_1 + B_1 I$.

A$'$. Nondegeneracy. The normal form $\Lambda + Q$ is nondegenerate in the sense that

$$\det A_1 \neq 0,$$
$$k \cdot \omega(I) + e \cdot \Omega(I) \not\equiv 0,$$

for all $(k, e) \in \mathbb{Z}^A \times \mathbb{Z}^B$ with $1 \leq |e| \leq 2$, satisfying $0 \leq |k| \leq M_*,\ 0 < |e|_1 \leq M_*$ and $k \cdot \nu_A + e \cdot \nu_B = 0$, where $\omega = \alpha_1 + A_1 I$ and $\Omega = \beta_1 + B_1 I$ and the constant M_* is a fixed constant and for integer vectors such as e, the norm $|e|_1$ is given by $|e|_1 = \sum_{j \in B} (|j| \vee 1)|e_j|$ and $|e| = \sum_{j \in B} |e_j|$.

B. Spectral Asympotics. There exists $d > 1$ such that

$$\beta_j = |j|^d + \cdots,$$

where the dots stand for terms of order less than d in $|j|$.

C. Regularity. $X_Q, X_R \in \mathcal{A}(\ell^{2,N}, \ell^{2,N})$, where $\mathcal{A}(\ell^{2,N}, \ell^{2,N})$ denotes the class of all maps from some neighborhood of the origin in $\ell^{2,N}$ into $\ell^{2,N}$, which are real analytic in the real and imaginary parts of the complex coordinate q.

D. Symmetry. The perturbation R satisfies

$$\{R,\ S\} = 0,$$

where

$$S = a + b \sum_{j \in A} j I_j + c \sum_{j \in B} j I_j \tag{1.1}$$

with $a \in \mathbb{R}$ and $b,\ c \in \mathbb{R} \setminus \{0\}$.

Remark 1.1. From the regularity assumption in fact we have $b_{ij} = O(1)$ uniformly in $j \in A$, where $i \in B$ and $B_1 = (b_{ij})_{i \in B, j \in A}$.

Theorem 1.1. (The Cantor Manifold Theorem With Symmetry). *Suppose the hamiltonian $H = \Lambda + Q + R$ satisfies assumptions A or A$'$, B, C and D, and*

$$|R| = O(\|\hat{q}\|_N^3 \|\tilde{q}\|_N) + O(\|\hat{q}\|_N^4) + O(\|q\|_N^{\mathfrak{g}}),$$

with $\mathfrak{g} > 4$. Then there exists a Cantor manifold $\mathcal{E}$ of smooth, elliptic diophantine $b-$tori given by a Lipschitz continuous embedding $\Psi : \mathcal{T}[\mathcal{C}] \to \mathcal{E}$, where $\mathcal{C}$ has full density at the origin, and Ψ is close to the inclusion map Ψ_0 :

$$\|\Psi - \Psi_0\|_{N, B_r \cap \mathcal{T}[\mathcal{C}]} = O(r^\sigma),$$

with $\sigma = \frac{1}{2}\mathfrak{g} - 1$.

1.2. One New Type of Beam Equations

Theorem 1.1 can be applied to one new type of beam equations. Consider the following one-dimensional beam equations

$$u_{tt} + u_{xxxx} + mu - 2u^2 u_{xx} - 2uu_x^2 = 0, \qquad m > 0, \tag{1.2}$$

with periodic boundary conditions

$$u(t, x + 2\pi) = u(t, x). \tag{1.3}$$

We have the following

Theorem 1.2. *Consider one-dimensional beam equations (1.2) with the boundary condition (1.3). Then for each index set* $J = \{(n_1, n_2, \cdots, n_b)\}$ *with* $b \geq 1$ *and* $|n_i| \neq |n_j|$ for $i, j \in \{1, \cdots, b\}$ *and* $n_1 n_2 \cdots n_b \neq 0$*, there exists, for all* $m > 0$ *but a set with Lebesgue measure zero, a Cantor manifold* $\mathcal{E}_J$ *of smooth, linearly stable and Diophantine* $b-$*tori in an associated phase space carrying quasi-periodic solutions of the nonlinear PDEs.*

Remark 1.2. Theorem 1.2 holds true for

$$u_{tt} + u_{xxxx} + mu - 2f(u^2)u_{xx} - 2g(u^2)uu_x^2 = 0, \qquad m > 0, \tag{1.4}$$

with periodic boundary conditions (1.3) where $f(t) = t + \sum\limits_{k \geq 2} a_k t^k$ and f is analytic in $t \in \mathbb{R}$ and $g(t) = f'(t)$. The special form of nonlinearity is to guarantee that the corresponding equations are Hamiltonian. We refer to [2] for details.

Theorem 1.2 confirms that the one dimensional beam equations (1.2)+(1.3) has infinite many linearly stable quasi-periodic solutions. We refer to [16, 17] and [25] for beam equations whose nonlinearities don't involve derivatives. It is worthy of pointing out that there are increasing interests on the PDEs whose nonlinearity involves derivatives since many important PDEs belong to this case. We review some known results. Bourgain [7] announced the existence of quasi-periodic solutions for derivative NLW equation

$$u_{tt} - u_{xx} + V(x)u + Bf(x, u) = 0, \qquad B = \big(-\frac{d^2}{dx^2}\big)^{\frac{1}{2}}.$$

The problem has been reconsidered by Craig in [10] for more general Hamiltonian derivative wave equations like

$$u_{tt} - u_{xx} + g(x)u + f(x, D^\beta u) = 0, \qquad x \in \mathbb{T},$$

where $g(x) \geq 0$ and D is the first order pseudo-differential operator $D := \sqrt{-\partial_{xx} + g(x)}$. Their method for the search of periodic solutions works good for $\beta < 1$. For $\beta = 1$

we refer to the recent work by Berti, Biasco and Procesi [4] - [5]. For KDV equation Kuksin [21] - [22](also refer to Kappeler-Pöschel [20]) smartly got a weak normal form around the torus. Then he proved the existence of quasi-periodic solutions from the strong estimates of solutions of the following equation

$$-\mathrm{i}\partial_\omega u + \lambda u + b(x)u = f(x), \qquad x \in \mathbb{T}^n. \tag{1.5}$$

Actually a weak estimate of (1.5) also works well in the KAM proof at the cost of losing the analyticity. We refer to Liu-Yuan [28] for one-dimensional derivative nonlinear Schrödinger equation and Benjiamin-Ono equation. Since all their results are relevant with a weak normal form, all the quasi-periodic solutions are not proved to be linearly stable.

The rest of the paper is organized as follows. In section 2 the Hamiltonian of the nonlinear beam equation is written in infinitely many coordinates and then transformed into its Birkhoff normal form of order four. In section 3 based on the Cantor Manifold Theorem with symmetry one gets Theorem 1.2. In section 4 we recall an infinite dimensional KAM theorem with symmetry from [19] and also improve it. Then one can use it to prove the Cantor Manifold Theorem with symmetry. Some technical lemmas are deferred in the Appendix.

2. The Hamiltonian Setting of Beam Equations

In this section we will write one-dimensional beam equation mentioned above into infinite Hamiltonian systems and then put the corresponding Hamiltonian into normal form. For convenience, we rewrite (1.2) as follows

$$u_{tt} + A^2 u - 2u^2 u_{xx} - 2uu_x^2 = 0, \qquad m > 0, \tag{2.1}$$

$$u(t, x + 2\pi) = u(t, x), \tag{2.2}$$

where $A = (\partial_{xxxx} + m)^{\frac{1}{2}}$ and the operator A with periodic boundary conditions has an exponential basis $\phi_j(x) = \sqrt{\frac{1}{2\pi}} e^{\mathrm{i}jx}$ and corresponding eigenvalues $\lambda_j = \sqrt{j^4 + m},\ j \in \mathbb{Z}$. One can write equation (2.1)+(2.2) as the infinite dimensional Hamiltonian systems

$$\partial_t q = -\mathrm{i}\partial_{\bar q} H, \qquad \partial_t \bar q = \mathrm{i}\partial_q H \tag{2.3}$$

with Hamiltonian

$$H(q, \bar q) = \langle Aq, \bar q\rangle + \frac{1}{4}\int_0^{2\pi} (A^{-\frac{1}{2}}(q + \bar q))^2 (A^{-\frac{1}{2}}(q_x + \bar q_x))^2 dx = \Lambda + P \tag{2.4}$$

in the complex coordinates

$$q := \frac{1}{\sqrt{2}}A^{\frac{1}{2}}u + \frac{\mathrm{i}}{\sqrt{2}}A^{-\frac{1}{2}}v \qquad \bar{q} := \frac{1}{\sqrt{2}}A^{\frac{1}{2}}u - \frac{\mathrm{i}}{\sqrt{2}}A^{-\frac{1}{2}}v,$$

where $\langle \mathfrak{u}, \mathfrak{v}\rangle = \int_0^{2\pi} \mathfrak{u}\mathfrak{v}dx$ for $\mathfrak{u},\ \mathfrak{v} \in W^1([0, 2\pi])$, which is the Sobolev space of all complex valued L^2-functions on $[0, 2\pi]$ with an L^2-derivative, and the gradient of H is defined with respect to $\langle \cdot,\ \cdot\rangle$. Note that the nonlinearity in (2.1) is x-independent implying the conservation of the momentum

$$H_2(q, \bar{q}) := -\mathrm{i}\int_0^{2\pi} \bar{q}\partial_x q dx. \tag{2.5}$$

This symmetry allows to simplify the KAM proof (see also [4, 18, 16, 19, 31]).

Now setting $q = (q_j)_{j\in\mathbb{Z}}$ through the relations $q(x) = \mathcal{S}q = \sum_{j\in\mathbb{Z}} q_j\phi_j(x)$, we obtain the Hamiltonian in infinitely many coordinates

$$H(q, \bar{q}) = \Lambda(q, \bar{q}) + P(q, \bar{q}),$$

where the coordinates are taken from the Hilbert space $\ell^{2,N}$ of all complex-valued sequences $q = (q_j)_{j\in Z}$ with

$$||q||_N^2 = \sum_{j\in\mathbb{Z}} \langle j\rangle^{2N}|q_j|^2 < \infty,$$

where $\langle j\rangle = 1 \vee |j|$. Further computation shows

$$H = \Lambda + P = \sum_{j\in\mathbb{Z}} \lambda_j|q_j|^2 + \sum_{j=1}^{5} P^j, \tag{2.6}$$

where

$$\begin{aligned}
P^1 &= \sum_{i+j+k+l=0} P^1_{ijkl}q_iq_jq_kq_l = -\frac{1}{8\pi}\sum_{i+j+k+l=0} \frac{kl}{\sqrt{\lambda_i\lambda_j\lambda_k\lambda_l}}q_iq_jq_kq_l,\\
P^2 &= \sum_{i+j+k-l=0} P^2_{ijkl}q_iq_jq_k\bar{q}_l = -\frac{1}{8\pi}\sum_{i+j+k-l=0} \frac{k(i+j-2l)}{\sqrt{\lambda_i\lambda_j\lambda_k\lambda_l}}q_iq_jq_k\bar{q}_l,\\
P^3 &= \sum_{i+j-k-l=0} P^3_{ijkl}q_iq_j\bar{q}_k\bar{q}_l = -\frac{1}{8\pi}\sum_{i+j-k-l=0} \frac{(-k^2+ij-jl-il)}{\sqrt{\lambda_i\lambda_j\lambda_k\lambda_l}}q_iq_j\bar{q}_k\bar{q}_l,\\
P^4 &= \sum_{i-j-k-l=0} P^4_{ijkl}q_i\bar{q}_j\bar{q}_k\bar{q}_l = -\frac{1}{8\pi}\sum_{i-j-k-l=0} \frac{(2kl-il-ki)}{\sqrt{\lambda_i\lambda_j\lambda_k\lambda_l}}q_i\bar{q}_j\bar{q}_k\bar{q}_l
\end{aligned}$$

and

$$P^5 = \sum_{-i-j-k-l=0} P^5_{ijkl}\bar{q}_i\bar{q}_j\bar{q}_k\bar{q}_l = -\frac{1}{8\pi}\sum_{-i-j-k-l=0}\frac{kl}{\sqrt{\lambda_i\lambda_j\lambda_k\lambda_l}}\bar{q}_i\bar{q}_j\bar{q}_k\bar{q}_l.$$

Its equations of motion are

$$\dot{q}_j = -\mathrm{i}\partial_{\bar{q}_j}H, \qquad j\in\mathbb{Z}, \tag{2.7}$$

where we use the symplectic structure $\mathrm{i}\sum\limits_{j\in\mathbb{Z}} dq_j\wedge d\bar{q}_j$. From a straightforward computation we have

$$H_2(q,\bar{q}) = \sum_{j\in\mathbb{Z}} j|q_j|^2. \tag{2.8}$$

As we mention above H_2 satisfies

Lemma 2.1.

$$\{H_2, P\}\equiv 0.$$

Before turning to the normal form analysis of the Hamiltonian (2.6), we state a result concerning the regularity of the gradient $P_{\bar{q}}$. The proof is well known(see [23]).

Lemma 2.2. *For $N>0$, the gradient $P_{\bar{q}}$ is real analytic as a map from some neighbourhood of the origin in $\ell^{2,N}$ into $\ell^{2,N}$, with*

$$||P_{\bar{q}}||_N = \mathcal{O}(||q||_N^3).$$

In the following we will put the Hamiltonian H into normal form. We will use A to stand for the tangent sites and while B for the normal sites and $b:=|A|$. It is clear $\mathbb{Z}=A\cup B$. We denote $C_A := \max\limits_{j\in A}\{|j|\}+1$ and C_a to be an absolute constant which is variant. In the following discussion, the parameter m will be defined in $[M_1,M_2]\subset(0,\infty)$, where $M_1>0$ and $M_2\geq 1$. As in [26], we denote

$$\Delta_0 = \{(i,j,k,l)|\sharp\{(i,j,k,l)\cap B\}=0\},$$

$$\Delta_1 = \{(i,j,k,l)|\sharp\{(i,j,k,l)\cap B\}=1\},$$

$$\Delta_2 = \{(i,j,k,l)|\sharp\{(i,j,k,l)\cap B\}=2\},$$

and

$$\Delta_3 = \{(i,j,k,l)|\sharp\{(i,j,k,l)\cap B\}\geq 3\}.$$

Then we rewrite the terms $P^t,\ t=1,\cdots,5$, in H into $P^t=P^{t,0}+P^{t,1}+P^{t,2}+P^{t,3}$. Here

$$P^{2,\iota} = \sum_{\substack{i+j+k-l=0\\(i,j,k,l)\in\Delta_\iota}} P^2_{ijkl}q_iq_jq_k\bar{q}_l = -\frac{1}{8\pi}\sum_{\substack{i+j+k-l=0\\(i,j,k,l)\in\Delta_\iota}}\frac{k(i+j-2l)}{\sqrt{\lambda_i\lambda_j\lambda_k\lambda_l}}q_iq_jq_k\bar{q}_l,$$

where $\iota = 0, 1, 2, 3$. Similarly, we also similarly define $P^{1,\iota}$, $P^{3,\iota}$, $P^{4,\iota}$ and $P^{5,\iota}$, $\iota = 0, 1, 2, 3$.

Lemma 2.3. *For any* $(i, j, k, l) \in \mathbb{Z}^4$ *and* $m > 0$,

$$\lambda_i + \lambda_j + \lambda_k + \lambda_l \geq 4\sqrt{m}.$$

Lemma 2.4. *If* $(i, j, k, l) \in \Delta_0$, Δ_1 *or* Δ_2, *then for* $\forall\ m \in [M_1, M_2] \setminus \mathcal{R}_0$,

$$|\lambda_i + \lambda_j + \lambda_k - \lambda_l| \geq \min\{2\sqrt{m}, 1, \frac{\mu}{\sqrt{h^4 + m}}\} > 0, \qquad h = \min\{|i|, |j|, |k|, |l|\},$$

where $i + j + k - l = 0$ *and* $meas(\mathcal{R}_0) \leq C_a C_A^4 M_2^4 b^4 \mu$ *and* μ *is a small parameter and will be chosen in the end.*

Lemma 2.5. *If* $i + j - k - l = 0$ *and* $m > 0$,

$$|\lambda_i + \lambda_j - \lambda_k - \lambda_l| \geq c(m)/\sqrt{(2 + h)^4 + m}, \qquad h = \min\{|i|, |j|, |k|, |l|\},$$

except the trivial case $\{i, j\} = \{k, l\}$.

The proof for Lemma 2.3 is clear. The long proofs for Lemma 2.4 and Lemma 2.5 are put into the Appendix.

Next we transform the hamiltonian (2.6) into the partial Birkhoff form. Denote $\mathcal{A}(\ell^{2,N}, \ell^{2,N})$ as the class of all real analytic maps from some neighbourhood of the origin in $\ell^{2,N}$ into $\ell^{2,N}$.

Lemma 2.6. *For each* $m \in [M_1, M_2] \setminus \mathcal{R}_0$, *there exists a real analytic, symplectic change of coordinates* $\Gamma := X_F^1$ *in some neighborhood of the origin that takes the hamiltonian* (2.6) *into*

$$H \circ \Gamma = \Lambda + \bar{P} + \hat{P} + K,$$

where $X_{\bar{P}}, X_{\hat{P}}, X_K \in \mathcal{A}(\ell^{2,N}, \ell^{2,N})$,

$$\bar{P} = \sum_{\substack{i,j \\ \{i,j\} \cap A \neq \emptyset}} P_{ij} |q_i|^2 |q_j|^2$$

with uniquely determined coefficients, and

$$|\hat{P}| = O(\|\hat{q}\|_N^3 \|\tilde{q}\|_N) + O(\|\hat{q}\|_N^4), \ |K| = O(\|q\|_N^6),$$

$\hat{q} = (q_j)_{j \in B}$, $\tilde{q} = (q_j)_{j \in A}$ *and* $P_{ij} = \frac{(2-\delta_{ij})}{8\pi} \frac{i^2 + j^2}{\lambda_i \lambda_j}$. *Moreover,*

$$(2.9) \qquad H_2 \circ \Gamma \equiv H_2, \qquad \{H_2, H \circ \Gamma\} \equiv 0, \qquad \{H_2, \hat{P} + K\} \equiv 0,$$

and the neighborhood can be chosen uniformly and the dependence of Γ *on* m *is real analytic.*

Proof. For the proof it is convenient to introduce the notations $q_j^{\tau_i},\ \tau = \pm 1$, by setting $q_j = q_j^1$ and $\bar{q}_j = q_j^{-1}$. The hamiltonian then reads

$$\begin{aligned} H &= \Lambda + P \\ &= \sum_{j\in\mathbb{Z}} \lambda_j |q_j|^2 + \sum_{\substack{\tau_i i+\tau_j j+\tau_k k+\tau_l l=0 \\ (\tau_i,\tau_j,\tau_k,\tau_l)\in\mathcal{S}}} P_{ijkl}^{\tau_i\tau_j\tau_k\tau_l} q_i^{\tau_i} q_j^{\tau_j} q_k^{\tau_k} q_l^{\tau_l}, \end{aligned}$$

where

$$\mathcal{S} = \{(1,1,1,1),\ (1,1,1,-1),\ (1,1,-1,-1),\ (1,-1,-1,-1),\ (-1,-1,-1,-1)\}.$$

In the following we construct a suitable sympletic transformation, which is defined below, to put the above hamiltonian into normal form. Let $\Gamma := X_F^t|_{t=1}$ be the time 1-map of the flow of the hamiltonian vector field X_F given by the hamiltonian

$$F = \sum_{\substack{\tau_i i+\tau_j j+\tau_k k+\tau_l l=0 \\ (\tau_i,\tau_j,\tau_k,\tau_l)\in\mathcal{S}}} F_{ijkl}^{\tau_i\tau_j\tau_k\tau_l} q_i^{\tau_i} q_j^{\tau_j} q_k^{\tau_k} q_l^{\tau_l},$$

with coefficients

$$\sqrt{-1}F_{ijkl}^{\tau_i\tau_j\tau_k\tau_l} = \begin{cases} \frac{P_{ijkl}^{\tau_i\tau_j\tau_k\tau_l}}{\tau_i\lambda_i+\tau_j\lambda_j+\tau_k\lambda_k+\tau_l\lambda_l},\ (i,j,k,l)\in\mathcal{L}\setminus\mathcal{N} \\ 0 \end{cases},$$

where $\mathcal{L} = \{(i,j,k,l)\in\mathbb{Z}^4 : \{i,j,k,l\}\in\Delta_0\cup\Delta_1\cup\Delta_2,\ \tau_i i+\tau_j j+\tau_k k+\tau_l l = 0,\ (\tau_i,\tau_j,\tau_k,\tau_l)\in\mathcal{S}\}$ and $\mathcal{N}\subset\mathcal{L}$ is the subset of all $\{i,j\} = \{k,l\}$ and $\{\tau_i,\tau_j,\tau_k,\tau_l\} = \{1,1,-1,-1\}$. The definition for F is correct in view of Lemma 2.3, Lemma 2.4 and Lemma 2.5. The remaining proof is a minor change of Main Proposition in [29]. In the end, we go to (2.9). The first one is clear. For the second, note $\{H_2\circ\Gamma, H\circ\Gamma\} = \{H_2, H\}\circ\Gamma = 0$. The last one is obvious. ∎

3. Proof of Theorem 1.2

We prove Theorem 1.2 from deducing it from Theorem 1.1. It is clear that our hamiltonian is $H = \Lambda + P$ with X_P in $\mathcal{A}(\ell^{2,N},\ell^{2,N})$, where we fix $N > 1$ arbitrary. With the help of Lemma 2.6 we put H into its Birkhoff normal form up to order four by a real analytic symplectic map Γ, such that $H\circ\Gamma = \Lambda + \bar{P} + \hat{P} + K$.

Now we choose any finite number $|A|$ of normal modes $(\phi_j)_{j\in A}$, where we choose

$$A = \{(n_1, n_2\cdots, n_b)\}$$

and

$$0 < |n_1| < |n_2| < \cdots < |n_b|. \tag{3.1}$$

We assume (3.1) solely for simplifying the discussion. From (3.1), we have $0 \in B$. With the notation of the previous section we then write

$$\Lambda = \langle \alpha_1, I\rangle + \langle \beta_1, Z\rangle$$

and

$$\bar{P} = \frac{1}{2}\langle A_1 I, I\rangle + \langle B_1 I, Z\rangle,$$

where $\alpha_1 = (\lambda_j)_{j\in A}$, $\beta_1 = (\lambda_j)_{j\in B}$,

$$A_1 = \frac{1}{4\pi}\begin{pmatrix} \frac{(n_1^2+n_1^2)}{\lambda_{n_1}^2} & \frac{2(n_1^2+n_2^2)}{\lambda_{n_1}\lambda_{n_2}} & \cdots & \frac{2(n_1^2+n_b^2)}{\lambda_{n_1}\lambda_{n_b}} \\ \frac{2(n_2^2+n_1^2)}{\lambda_{n_2}\lambda_{n_1}} & \frac{n_2^2+n_2^2}{\lambda_{n_2}^2} & \cdots & \frac{2(n_2^2+n_b^2)}{\lambda_{n_2}\lambda_{n_b}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{2(n_b^2+n_1^2)}{\lambda_{n_b}\lambda_{n_1}} & \frac{2(n_b^2+n_2^2)}{\lambda_{n_b}\lambda_{n_2}} & \cdots & \frac{n_b^2+n_b^2}{\lambda_{n_b}^2} \end{pmatrix},$$

and $B_1 = (b_{ji})_{j\in B, i\in A}$, where $b_{ji} = \frac{i^2+j^2}{\lambda_i\lambda_j}$. It is clear that

$$|\hat{P}| = O(\|\hat{q}\|_N^3\|\tilde{q}\|_N) + O(\|\hat{q}\|_N^4),\ |K| = O(\|q\|_N^6),$$

Thus $H\circ\Gamma = \Lambda + Q + R$ with $Q := \bar{P}$ and $R := \hat{P} + K$, for which we verify the assumptions of Theorem 1.1.

Lemma 3.7. *For $m \in [M_1, M_2]\setminus R^0$ and $0 < |k| \le M_* + 2$, then*

$$|k\cdot\omega_0| \ge \mu > 0,$$

where $\omega_0 = (\lambda_{n_1}, \lambda_{n_2}, \cdots, \lambda_{n_b})$, μ is a small constant and $meas(R^0) \le C(M_*, M_1, M_2, b)\cdot \mu^{\frac{1}{b-1}}$.

Lemma 3.8. *For $m \in [M_1, M_2]\setminus \mathcal{R}_1$ the normal form $\Lambda + Q$ is nondegenerate, where $meas(\mathcal{R}_1) \le C(M_*, M_1, M_2, b)\cdot\mu^{\frac{1}{b+1}}$. So assumption A' is satisfied.*

We put the proof of Lemma 3.8 in Appendix B. Lemma 3.7 is a simple case of Lemma 3.8. We omit the proof.

The conditions B and C are clear. For D, we choose $S := H_2$. From Lemma 2.6, condition D is satisfied. From Lemma 2.6, we have $\mathfrak{g} = 6$. So Theorem 1.1 applies, and we obtain in particular

$$\|\Psi - \Psi_0\|_{N, B_r\cap\mathcal{T}[\mathcal{C}]} = O(r^2).$$

Composing with Γ we obtain a Cantor manifold $\mathcal{E}$ of smooth diophantine $b-$tori in $\ell^{2,N}$ carrying quasi-periodic solutions

$$\gamma_{I,v_0}: \ t \mapsto q(t) = \Gamma \circ \Psi(e^{i\omega(I)t}v_0)$$

for the hamiltonian $H = N + P$. Recall that $\mathcal{S}q = \sum q_j\phi_j(x)$. Now Going back to H^{N+1} by the isomorphism

$$\ell^{2,N} \to H^{N+1}, \qquad q \mapsto u = A^{-\frac{1}{2}}(\frac{1}{\sqrt{2}}(\mathcal{S}q + \bar{\mathcal{S}}q)),$$

$\mathcal{E}$ is mapped into another Cantor manifold of smooth diophantine tori in H^{N+1}, which carry smooth quasi-periodic solution u of the quasi-linear beam equation.

We only need to explain the statement of Theorem 1.2 holds true for a.e. $m > 0$. First we choose $m \in \mathcal{J}_{l,k} = [1/l, k](l = 2, 3, \cdots, k = 2, 3, \cdots)$. In other words we set $M_1 = 1/l$ and $M_2 = k$. Fix k firstly. Note $meas(\mathcal{R}_0^{l,k} \cup \mathcal{R}_1^{l,k}) \leq C(M_*, b, l, k)\mu^{\frac{1}{b+1}}$ and M_* is a fixed constant, then

$$meas(\mathcal{R}_0^{l,k} \cup \mathcal{R}_1^{l,k}) \leq 1/l, \ l = 2, 3, \cdots$$

if $\mu \leq C(M_*, b, l, k)$. Denote $\mathcal{J}_{l,k}^* = \mathcal{R}_0^{l,k} \cup \mathcal{R}_1^{l,k}$. For any $m \in \mathcal{I}_{l,k} = \mathcal{J}_{l,k} \setminus \mathcal{J}_{l,k}^*$ Theorem 1.2 holds true(note the size of the Cantor manifolds is not uniform, but depends on l, k, A) with

$$meas(\mathcal{J}_{l,k}^*) \leq 1/l. \tag{3.2}$$

Note $(0, k] \setminus (\cup_l \mathcal{I}_{l,k}) = \bigcap_l((0, k] \setminus \mathcal{I}_{l,k})$ and (3.2), we then have

$$\begin{aligned} meas\big((0, k] \setminus \cup_l \mathcal{I}_{l,k}\big) &\leq meas((0, k] \setminus \mathcal{I}_{l,k}) \\ &\leq meas((0, k] \setminus \mathcal{J}_{l,k}) + meas(\mathcal{J}_{l,k}^*) \\ &\leq 1/l + 1/l = 2/l \end{aligned}$$

for any $l = 2, 3, \cdots$. This shows us $meas\big((0, k] \setminus (\cup_l \mathcal{I}_{l,k})\big) = 0$. Varying k in the end, we thus have finished the proof.

4. An Infinite Dimensional KAM Theorem with Symmetries

Theorem 1.1 is derived from an infinite KAM Theorem with symmetries in [19], which is based on the KAM theorem from Kuksin [21] and also Pöschel [30] (cf also [20]). Following the exposition in [19, 20] and [30], consider small perturbations of a family of infinite dimensional integrable Hamiltonians $H \equiv H(y, u, v; \xi)$ with parameter ξ in the normal form

$$H = \sum_{j \in A} \omega_j(\xi) y_j + \frac{1}{2} \sum_{j \in B} \Omega_j(\xi)(u_j^2 + v_j^2), \tag{4.1}$$

on the phase space

$$\mathcal{M}^N := \mathbb{T}^A \times \mathbb{R}^A \times \ell^{2,N} \times \ell^{2,N}$$

with coordinates (x, y, u, v) where $A \subseteq \mathbb{Z}$ with $|A| < \infty$, $B = \mathbb{Z} \setminus A$, $N \in \mathbb{Z}_{\geq 1}$ and where $\mathbb{T}^A = \mathbb{R}^A / 2\pi\mathbb{Z}^A$ denotes the $|A|-$dimensional torus, conveniently indexed by the set A. Here $\ell^{2,N} \equiv \ell^{2,N}(B,\ \mathbb{R})$ denotes the Hilbert space of all *real* sequences $u = (u_j)_{j\in B}$ with

$$\|u\|_N^2 = \sum_{j\in B} \langle j\rangle^{2N} |u_j|^2 < \infty,$$

where $\langle j\rangle = 1 \vee |j|$. The 'internal' frequencies, $\omega = (\omega_j)_{j\in A}$, as well as the 'external' ones, $\Omega = (\Omega_j)_{j\in B}$, are real valued and depend on the parameter $\xi \in \Pi \subset \mathbb{R}^A$ and Π is a compact subset of $\mathbb{R}^A$ of positive Lebesgue measure. The symplectic form on $\mathcal{M}^N$ is the standard one given by $\sum_{j\in A} dx_j \wedge dy_j + \sum_{j\in B} du_j \wedge dv_j$. The Hamiltonian equations of motion of H are therefore

$$\dot{x} = \omega(\xi),\ \dot{y} = 0,\ \dot{u} = \Omega(\xi)v,\ \dot{v} = -\Omega(\xi)u,$$

where for any $j \in B$, $\big(\Omega(\xi)u\big)_j = \Omega_j(\xi)u_j$. Hence, for any parameter $\xi \in \Pi$, on the $|A|-$dimensional invariant torus,

$$\mathbb{T}_0 = \mathbb{T}^A \times \{0\} \times \{0\} \times \{0\},$$

the flow is rotational with internal frequencies $\omega(\xi) = (\omega_j(\xi))_{j\in A}$. In the normal space, described by the (u, v) coordinates, we have an elliptic equilibrium at the origin, whose frequencies are $\Omega(\xi) = (\Omega_j(\xi))_{j\in B}$. Hence, for any $\xi \in \Pi$, $\mathbb{T}_0$ is an invariant, rotational, linearly stable torus for the Hamiltonian H. Our aim is to prove the persistence of this torus under small perturbations $H + P$ of the integrable Hamiltonian H for a large Cantor set of parameter values ξ. To this end we make assumptions on the frequencies of the unperturbed Hamiltonian H and on the perturbation P.

Assumption $\mathcal{A}$: Frequencies.

(A1) The map $\xi \mapsto \omega(\xi)$ between Π and its image $\omega(\Pi)$ is a homeomorphism which, together with its inverse, is Lipschitz continuous.

(A2) There exists a real sequence $(\overline{\Omega}_j)_{j\in B}$, independent of $\xi \in \Pi$, of the form

$$\overline{\Omega}_j = |j|^d + a_1|j|^{d_1} + \cdots + a_D|j|^{d_D} \tag{4.2}$$

where $d = d_0 > d_1 > \cdots > d_D \geq 0$ with $D \in \mathbb{Z}_{\geq 0}, d > 1$, and $a_1, \ldots, a_D \in \mathbb{R}$, so that $\xi \mapsto (\Omega_j - \overline{\Omega}_j)_{j\in B}$ is a Lipschitz continuous map on Π with values in $\ell^{\infty,-\delta} \equiv \ell^{\infty,-\delta}(B,\ \mathbb{R})$ for some $0 \leq \delta < 1 \wedge (d-1)$.

(A3) For any (k,e) in $\mathcal{Z} := \{(k,e) \in \mathbb{Z}^A \times \mathbb{Z}^B \setminus (0,0) : |e| \le 2;\ k \cdot \nu_A + e \cdot \nu_B = 0\}$ with $e \neq 0$

$$meas\Big\{\xi \in \Pi : k \cdot \omega(\xi) + e \cdot \Omega(\xi) = 0\Big\} = 0. \tag{4.3}$$

The second set of assumptions concerns the perturbing Hamiltonian P and its vector field, $X_P = (\partial_y P,\ -\partial_x P,\ \partial_v P,\ -\partial_u P)$. We use the notation $i_\xi X_P$ for X_P evaluated at ξ. Finally, we denote by $\mathcal{M}_{\mathbb{C}}^N$ the complexification of the phase space $\mathcal{M}^N$, $\mathcal{M}_{\mathbb{C}}^N = (\mathbb{C}/2\pi\mathbb{Z})^A \times \mathbb{C}^A \times \ell_{\mathbb{C}}^{2,N} \times \ell_{\mathbb{C}}^{2,N}$. Note that at each point of $\mathcal{M}_{\mathbb{C}}^N$, the tangent space is given by

$$\mathcal{P}_{\mathbb{C}}^N := \mathbb{C}^A \times \mathbb{C}^A \times \ell_{\mathbb{C}}^{2,N} \times \ell_{\mathbb{C}}^{2,N}.$$

Assumption $\mathcal{B}$**:** Perturbation.

(B1) There exists a neighborhood V of $\mathbb{T}_0$ in $\mathcal{M}_{\mathbb{C}}^N$ such that P is a function on $V \times \Pi$ and its Hamiltonian vector field defines a map

$$X_P :\ V \times \Pi \to \mathcal{P}_{\mathbb{C}}^N. \tag{4.4}$$

Moreover, $i_\xi X_P$ is real analytic on V for each $\xi \in \Pi$, and $i_w X_P$ is uniformly Lipschitz on Π for each $w \in V$. (Here $i_\xi X_P$ denotes the vector field X_P, evaluated at the parameter value ξ; $i_w X_P$ is defined similarly.)

(B2) $\{P,\ S\} = 0$ where

$$S = a + b\sum_{j\in A} j y_j + c\sum_{j \in B} j(u_j^2 + v_j^2)/2 \tag{4.5}$$

with $a \in \mathbb{R}$ and $b,\ c \in \mathbb{R} \setminus \{0\}$.

To state the KAM theorem we need to introduce some domains and norms. For $s > 0$ and $r > 0$ we introduce the complex $\mathbb{T}_0-$neighborhoods

$$D(s,r) = \{|\Im x| < s\} \times \{|y| < r^2\} \times \{\|u\|_N + \|v\|_N < r\} \subset \mathcal{M}_{\mathbb{C}}^N.$$

Here, for z in $\mathbb{R}^A$ or $\mathbb{C}^A$, $|z| = \max\limits_{j \in A} |z_j|$. For a vector Y in $\mathcal{P}_{\mathbb{C}}^N$ with components (Y_x, Y_y, Y_u, Y_v) introduce the weighted norm

$$\|Y\|_{r,N} = |Y_x| + \frac{1}{r^2}|Y_y| + \frac{1}{r}\|Y_u\|_N + \frac{1}{r}\|Y_v\|_N.$$

Such weights are convenient when estimating the components of a Hamiltonian vector field $X_P = (\partial_y P,\ -\partial_x P,\ \partial_v P,\ -\partial_u P)$ on $D(s,r)$ in terms of r. For a vector field $Y : V \times \Pi \to \mathcal{P}_{\mathbb{C}}^N$ we then define the norms

$$\begin{aligned} \|Y\|_{r,N;V\times\Pi}^{\sup} &= \sup_{(w,\xi)\in V\times\Pi} \|Y(w,\xi)\|_{r,N}, \\ \|Y\|_{r,N;V\times\Pi}^{lip} &= \sup_{\substack{\xi,\zeta\in\Pi \\ \xi\neq\zeta}} \frac{\|\Delta_{\xi\zeta} Y\|_{r,N;V}^{\sup}}{|\xi - \zeta|}, \end{aligned}$$

where $\Delta_{\xi\zeta} Y = i_\xi Y - i_\zeta Y$, and

$$\|i_\xi Y\|^{\sup}_{r,N;V} = \sup_{w\in V} \|Y(w,\xi)\|_{r,N}.$$

In a completely analogous way, the Lipschitz semi-norm of the map $F:\ \Pi \to \ell^{\infty,-\delta}$ is defined as

$$|F|^{lip}_{\Pi,\ell^{\infty,-\delta}} = \sup_{\substack{\xi,\zeta\in\Pi \\ \xi\neq\zeta}} \frac{\|\Delta_{\xi\zeta} F\|_{\ell^{\infty,-\delta}}}{|\xi-\zeta|}.$$

Finally, let $1 \le M < \infty$ be a constant satisfying

$$|\omega|^{lip}_{\Pi} + |\Omega|^{lip}_{\Pi,\ell^{\infty,-\delta}} \le M. \tag{4.6}$$

Note that if Assumption $\mathcal{A}$ and Assumption $\mathcal{B}$ hold such an M exists. By Assumption (A1), there exists a constant $1 \le L < \infty$ satisfying

$$L \ge |\omega^{-1}|^{lip}_{\omega(\Pi)}. \tag{4.7}$$

Theorem 4.1. *Suppose H is a family of Hamiltonians of the form (4.1) defined on the phase space $\mathcal{M}^N$, $N \in \mathbb{Z}_{\ge 1}$, and depending on parameters in Π so that Assumption $\mathcal{A}$ is satisfied with d and δ. Furthermore, assume that $s > 0$. Then there exist a positive constant γ depending on the finite subset $A \subset \mathbb{Z}$ of (4.1), d, δ, the frequencies ω and Ω of H, and s such that for any perturbed Hamiltonian $H+P$ with P satisfying Assumption $\mathcal{B}$ on a neighborhood V of $\mathbb{T}_0$ in $\mathcal{M}^N_{\mathbb{C}}$, with $D(s,r) \subseteq V$ for some $r > 0$, and the smallness condition*

$$\varepsilon :=\|X_P\|^{\sup}_{r,N;D(s,r)\times\Pi} + \frac{\alpha}{M}\|X_P\|^{lip}_{r,N;D(s,r)\times\Pi} \le \alpha\gamma \tag{4.8}$$

for some $0 < \alpha < 1$, the following holds. There exist

- (i) *a closed subset $\Pi_* \subset \Pi$, depending on the perturbation P, with $meas(\Pi\backslash\Pi_*) \to 0$ as $\alpha \to 0$,*
- (ii) *a Lipschitz family of real analytic torus embeddings $\Psi:\ \mathbb{T}^A \times \Pi_* \to \mathcal{M}^N$,*
- (iii) *a Lipschitz map $f:\ \Pi_* \to \mathbb{R}^A$,*

such that for any $\xi \in \Pi_*$, $\Psi(\mathbb{T}^A \times \{\xi\})$ is an invariant torus of the perturbed Hamiltonian $H+P$ at ξ and the flow of $H+P$ on this torus is given by

$$\mathbb{T}^A \times \mathbb{R} \to \mathcal{M}^N,\ (x,t) \mapsto \Psi(x+tf(\xi),\ \xi).$$

Thus for any $x \in \mathbb{T}^A$ and $\xi \in \Pi_*$, the curve $t \mapsto \Psi(x + tf(\xi), \xi)$ is a quasi-periodic solution for the Hamiltonian $i_\xi(H + P)$. Moreover, for any $\xi \in \Pi_*$, the embedding $\Psi(\cdot, \xi) : \mathbb{T}^A \to \mathcal{M}^N$ is real analytic on $D(s/2) = \{|Imx| < s/2\}$, and

$$\|\Psi - \Psi_0\|^{\sup}_{r,N;\ D(s/2)\times\Pi_*} + \frac{\alpha}{M}\|\Psi - \Psi_0\|^{lip}_{r,N;\ D(s/2)\times\Pi_*} \le \frac{c\varepsilon}{\alpha},$$
$$|f - \omega|^{\sup}_{\Pi_*} + \frac{\alpha}{M}|f - \omega|^{lip}_{\Pi_*} \le c\varepsilon,$$

where

$$\Psi_0 :\ \mathbb{T}^A \times \Pi \to \mathbb{T}_0, \qquad (x, \xi) \mapsto (x,\ 0,\ 0,\ 0)$$

is the trivial embedding, and c is a positive constant which depends on the same parameters as γ. If the unperturbed frequencies are affine functions of the parameter ξ, then

$$meas(\Pi \setminus \Pi_\alpha) \le \tilde{c}\rho^{|A|-1}\alpha, \tag{4.9}$$

where $\rho = diam\Pi$. The constant $\tilde{c}$ depend on the finite subset $A \subset \mathbb{Z}$ of (4.1), $d,\ L,\ M$ and the frequencies ω and Ω in a 'monotone' way. That is, $\tilde{c}$ do not increase for a closed subsets of Π.

Remark 4.1. From the proof(see [19]), one can see that (A3) can be weakened to the following

(A3′) For any $(k, e) \in \mathbb{Z}^A \times \mathbb{Z}^B,\ 1 \le |e| \le 2$, satisfying $0 \le |k| \le M_*,\ 0 < |e|_{d-1-\delta} \le M_*$,

$$meas\Big\{\xi : k \cdot \omega(\xi) + e \cdot \Omega(\xi) = 0,\ k \cdot \nu_A + e \cdot \nu_B = 0\Big\} = 0,$$

where the constant M_* depends on $|A|,\ d,\ L,\ M$ and the frequencies ω and Ω in a 'monotone' way. For integer vectors such as e, the norm $|e|_{d-1-\delta}$ is given by $|e|_{d-1-\delta} = \sum\limits_{j\in B}\langle j\rangle^{d-1-\delta}|e_j|$. For our application to beam equation, we choose $d-1-\delta = 1$ since $d = 2$ and $\delta = 0$.

Remark 4.2. We delay the proof of (4.9) in the Appendix.

We finally prove the Cantor Manifold Theorem with symmetry based on Theorem 4.1. We only give a sketch since the method is similar as the proof of the Cantor Manifold Theorem in [23]. For readers' convenience, we follow the most of notations, which are appeared in [23]. We are given a hamiltonian $H = \Lambda + Q + R$ in complex coordinates $q = (\tilde{q}, \hat{q})$, where R is some perturbation of the normal form

$$\Lambda + Q = \langle \alpha_1, I\rangle + \langle \beta_1, Z\rangle + \frac{1}{2}\langle A_1 I, I\rangle + \langle B_1 I, Z\rangle,$$

with $I = (|q_j|^2)_{j\in A}$ and $Z = (|q_j|^2)_{j\in B}$. Assumptions A', B, C and D are supposed to hold.

Step 1. New coordinates. We introduce symplectic polar and real coordinates by setting

$$q_j = \begin{cases} \sqrt{\xi_j + y_j}\, e^{-\mathrm{i}x_j}, & j \in A \\ \frac{1}{\sqrt{2}}(u_j + \mathrm{i}v_j), & j \in B \end{cases}$$

depending on parameters $\xi \in \Pi = [0,1]^b$. It is clear that the symplectic form now is

$$\mathrm{i}\sum_{j\in\mathbb{Z}} dq_j \wedge d\bar{q}_j = \sum_{j\in A} dx_j \wedge dy_j + \sum_{j\in B} du_j \wedge dv_j,$$

$I = \xi + y$ and $Z = \frac{1}{2}(u^2 + v^2)$, with the componentwise interpretation. The normal form becomes

$$\Lambda + Q = \langle \omega(\xi), y\rangle + \frac{1}{2}\langle \Omega(\xi), u^2 + v^2\rangle + \tilde{Q}$$

with frequencies $\omega(\xi) = \alpha_1 + A_1\xi$, $\Omega(\xi) = \beta_1 + B_1\xi$ and remainder $\tilde{Q} = O(\|y\|^2) + O(\|u^2 + v^2\| \cdot \|y\|)$. The total hamiltonian is $H = N + P$ with $P = \tilde{Q} + R$.

Step 2. Checking assumptions $A1, A2, A3'$ and $B1, B2$. The map $\xi \mapsto \omega(\xi)$ is a homeomorphism which, together with its inverse, is Lipschitz continuous since $\det A_1 \neq 0$. So the condition A1 is satisfied. The condition A2 follows from Assumption B and Remark 1.1($\delta = 0$). $A3'$ is clear from Assumption A'. The condition $B1$ is from Assumption C, while $B2$ is directly from Assumption D.

Step 3. Domains and estimates. Let $r > 0$ and consider the phase domain

$$D(2,r) : |Imx| < 2,\ |y| < r^2,\ \|u\|_N + \|v\|_N < r,$$

and the parameter domain

$$\Xi_r^- = U_{-4r^2}\Xi_r,\ \Xi_r = \{\xi : 0 < \xi < r^{2\lambda}\},\ 0 < \lambda < 1,$$

where $U_{-\rho}\Xi$ is the subset of all points in Ξ with boundary distance greater then ρ. Then as [23], we have $|\tilde{Q}| = O(r^4)$ as well as

$$|R| = O(r^{3+\lambda} + r^4 + r^{\lambda\mathfrak{g}}) = O(r^{3+\lambda})$$

on $D(2, 2r)$, where we choose $0 < \lambda = \frac{3}{\mathfrak{g}-1} < 1$. It follows that

$$|X_P|_{r,D(1,r)} + \alpha |X_P|^{lip}_{r,D(1,r)} = O(r^{1+\lambda})$$

with respect to the parameter domain $\Pi_r = U_{-\alpha}\Xi_r,\ \alpha \geq 8r^2$, where α will be chosen as a function of r later.

Step 4. Application of Theorem 4.1. To apply Theorem 4.1, it suffices to require

$$\alpha(r) \geq c_1 r^{1+\lambda}$$

for all small r with a sufficiently large constant c_1 which depends on the parameters indicated in Theorem 4.1. Then we obtain a Cantor set $\Pi_{r,\alpha} \subset \Pi_r$ of parameters, a Lipschitz continuous family of real analytic torus embeddings $\Phi_r : \mathbb{T}^b \times \Pi_{r,\alpha} \to D(1,r)$, and a Lipschtiz continuous frequency map $\tilde{\omega}_r : \Pi_{r,\alpha} \to \mathbb{R}^b$, such that for each $\xi \in \Pi_{r,\alpha}$ the map Φ_r restricted to $\mathbb{T}^b \times \{\xi\}$ is a real analytic embedding of an elliptic, rotational torus with frequencies $\tilde{\omega}_r(\xi)$ for the hamiltonian H at ξ. The following estimates

$$|\Phi_r - \Phi_0|_r + \alpha|\Phi_r - \Phi_0|_r^{lip} \leq cr^{1+\lambda}/\alpha,$$

$$|\tilde{\omega} - \omega| + \alpha|\tilde{\omega}_r - \omega|^{lip} \leq cr^{1+\lambda},$$

hold on $|Imx| < \frac{1}{2}$ and $\Pi_{r,\alpha}$, where the generic constant c depends on the same parameters as c_1. Moreover, we have the measure estimate

$$meas(\Xi_r \setminus \Pi_{r,\alpha}) \leq \frac{c\alpha}{r^{2\lambda}} meas(\Xi_r).$$

Hence, to obtain a nonempty Cantor set we also need $\alpha(r) \leq c_1^{-1} r^{2\lambda}$.

Step 5. This step is the same as [23].

Step 6. Estimates. We can prove that if $r^{1+\lambda}/\alpha(r)$ is a nondecreasing function of r, then on $|Im\phi| < 1/2$ and $\mathcal{C} \cap \Xi_{r_k}$ one has

$$|\Phi - \Phi_0|_{r_k},\ \alpha(r_k)\frac{|\Delta_{IJ}(\Phi - \Phi_0)|_{r_k}}{|I-J|} \leq \frac{cr_k^{1+\lambda}}{\alpha(r_k)}, \tag{4.10}$$

provided $I \in \mathcal{C}_{r_k}$. This holds for all $k \geq 0$. If also $\alpha(r)/r^{2\lambda}$ is a nondecreasing function of r, then

$$\frac{meas(\mathcal{C} \cap \Xi_{r_k})}{meas(\Xi_{r_k})} \geq 1 - \frac{c\alpha(r_k)}{r_k^{2\lambda}}.$$

Step 7. The embedding Ψ. We can show that

$$\|\Psi - \Psi_0\|_N \leq \frac{cr_k^{1+\lambda}}{\alpha(r_k)} \cdot r_k$$

uniformly on $\mathcal{T}[\mathcal{C} \cap \Xi_{r_k}]$ for $k \geq 0$.

For the proof, consider $\tilde{q} = \sqrt{I}e^{-\mathrm{i}\phi}$ and $\hat{q} = \frac{1}{\sqrt{2}}(u + \mathrm{i}v)$, understood component-by-component. On $\mathcal{T}[\mathcal{C} \cap R_{r_k}]$ we have

$$
\begin{aligned}
|\tilde{q}' - \tilde{q}| &= |\sqrt{I+y} - \sqrt{I}| \\
&\le |y| (\min_{I \in R_{r_k}} |\sqrt{I_j}|)^{-1} \\
&\le r_k^2 |\Phi - \Phi_0|_{r_k} \frac{1}{\sqrt{\alpha(r_k)}} \\
&\le \frac{c r_k^{1+\lambda}}{\alpha(r_k)} \cdot r_k^{\frac{3-\lambda}{2}} \le \frac{c r_k^{1+\lambda}}{\alpha(r_k)} \cdot r_k
\end{aligned}
$$

using $\alpha \ge c_1 r^{1+\lambda}$ and (4.10). Similarly, we have $\|\hat{q}'\|_N \le \frac{c r_k^{1+\lambda}}{\alpha(r_k)} \cdot r_k$. The right-hand sides decrease as k increases, so this bound holds also on $\mathcal{T}[\mathcal{C} \cap R_{r_l}]$ with $l > k$ and thus on all of $\mathcal{T}[\mathcal{C} \cap \Xi_{r_k}]$.

Step 8. Choice of $\alpha(r)$. Finally we choose $\alpha = r^{\kappa} = r^{\frac{1}{2}(3\lambda+1)}$, which clearly satisfies $c_1 r^{1+\lambda} \le \alpha(r) \le c_1^{-1} r^{2\lambda}$ since $0 < \lambda < 1$. Then

$$
meas(\mathcal{C} \cap \Xi_{r_k}) \ge 1 - c r_k^{\kappa - 2\lambda} = 1 - c r_k^{\frac{\mathfrak{g}-4}{2(\mathfrak{g}-1)}}.
$$

It is clear $\frac{\mathfrak{g}-4}{2(\mathfrak{g}-1)} > 0$. This means that Cantor set $\mathcal{C}$ has full density at the origin, and

$$
\|\Psi - \Psi_0\|_N \le c r_k^{1+\lambda-\kappa} \cdot r_k = c r_k^{\frac{1}{2}(3-\lambda)}.
$$

Thus

$$
\|\Psi - \Psi_0\| \le c r^{\sigma\lambda}, \qquad \sigma = \frac{1}{2}(\mathfrak{g} - 2),
$$

on $\mathcal{T}[\mathcal{C} \cap \Xi_r]$. The latter contains the set $\mathcal{T}[\mathcal{C}] \cap B_{r^\lambda}$, and so the estimate of Theorem 1.1 is obtained.

5. APPENDIX

In this section we fix $m \in [M_1, M_2]$, where $M_1 > 0$ and $M_2 \ge 1$. We define

$$
\Delta = \begin{pmatrix}
\lambda_{i_1} & \lambda_{i_2} & \cdots & \lambda_{i_{r+1}} \\
\frac{d\lambda_{i_1}}{dm} & \frac{d\lambda_{i_2}}{dm} & \cdots & \frac{d\lambda_{i_{r+1}}}{dm} \\
\vdots & \vdots & \vdots & \vdots \\
\frac{d^r\lambda_{i_1}}{dm^r} & \frac{d^r\lambda_{i_2}}{dm^r} & \cdots & \frac{d^r\lambda_{i_{r+1}}}{dm^r}
\end{pmatrix},
$$

where $|i_1| < |i_2| < \cdots < |i_{r+1}| \le M_0$. Set $x_1 = \sqrt{i_1^4 + m}$, $x_2 = \sqrt{i_2^4 + m}$, $\cdots$, $x_{r+1} = \sqrt{i_{r+1}^4 + m}$. From a straightforward computation we have

$$
\det \Delta = \frac{1}{2} \cdots \frac{(2r-3)!(-1)^{r+1}}{2^{r-2}(r-2)!2^r} \cdot x_1 \cdots x_{r+1} \cdot \prod_{r+1 \ge i > j \ge 1} (\frac{1}{x_i^2} - \frac{1}{x_j^2}).
$$

Denote

$$f_k(m) = k_1\lambda_{i_1} + k_2\lambda_{i_2} + \cdots + k_{r+1}\lambda_{i_{r+1}}, \qquad m \in [M_1, M_2],$$

where we suppose that $k \neq 0$ and $1 \leq r \leq b+1, r \in \mathbb{Z}$. Following [1], we have

Lemma 5.1. *For $k \neq 0$, there exists $i_0 \in \{1, \cdots, r+1\}$ such that*

$$\Big|k_1 \frac{d^{(i_0-1)}\lambda_{i_1}}{dm^{(i_0-1)}} + \cdots + k_{r+1}\frac{d^{(i_0-1)}\lambda_{i_{r+1}}}{dm^{(i_0-1)}}\Big| \geq \frac{c_b(\min\{M_1,1\})^{(b+\frac{1}{2})(b+1)}}{(M_0^4+M_2)^{(b+1)^2+\frac{1}{2}(b+1)}}. \tag{5.1}$$

The following lemma and its proof can be found in [32].

Lemma 5.2. *Suppose that $g(x)$ be rth differentiable function on the closure $\bar{I}$ of I, where $I \subset \mathbb{R}$ is an interval. Let $I_h := \{x \in I : \ |g(x)| < h\}$, $h > 0$. If for some constant $d > 0$, $|g^{(r)}(x)| \geq d$ for any $x \in I$, then $meas(I_h) \leq 2(2+3+\cdots+r+d^{-1})h^{\frac{1}{r}}$.*

Proof of (4.9). In fact, in the proof of Theorem 4.1 (i)(see [19]) we have

$$meas(\Xi_\alpha^2 \cup \Xi_\alpha^3 \cup \Xi_\alpha^4) \leq \tilde{c}_1\rho^{|A|-1}\alpha.$$

We only need to give a new measure estimate of Ξ_α^1 under the condition that the unperturbed frequencies are affine functions of the parameters. We recall that $\Xi_\alpha^1 = \bigcup_{\substack{|k|<K_* \\ (k,e)\in\mathcal{Z}, e\neq 0}} \mathcal{R}_{ke}^0(\alpha_0)$. Rewrite

$$\begin{aligned}\Xi_\alpha^1 &= \Xi_\alpha^{1,1}\bigcup \Xi_\alpha^{1,2} \\ &= (\bigcup_{\substack{|k|<K_*, |e|_{d-1-\delta}\geq E_* \\ (k,e)\in\mathcal{Z}, e\neq 0}} \mathcal{R}_{ke}^0(\alpha_0))\bigcup(\bigcup_{\substack{|k|<K_*, |e|_{d-1-\delta}<E_* \\ (k,e)\in\mathcal{Z}, e\neq 0}} \mathcal{R}_{ke}^0(\alpha_0)).\end{aligned}$$

From Corollary 6.2 in [19] we have

$$meas(\Xi_\alpha^{1,1}) \leq \sum_{|k|<K_*} 12L(LM\rho)^{|A|-1}\alpha|k|^{-\frac{1}{2}}A_k^{-1} \leq \tilde{c}_2\rho^{|A|-1}\alpha.$$

We only need to estimate the measure of the set $\Xi_\alpha^{1,2}$. Denote $\omega(\xi) = \overline{\omega} + \hat{\omega}(\xi)$. Similarly $\Omega(\xi) = \overline{\Omega} + \hat{\Omega}(\xi)$. Denote

$$f_{k,e}(\xi) = (k\cdot\overline{\omega} + e\cdot\overline{\Omega}) + (k\cdot\hat{\omega}(\xi) + e\cdot\hat{\Omega}(\xi)), \ \xi \in \Pi.$$

If $k\cdot\hat{\omega}(\xi)+e\cdot\hat{\Omega}(\xi)\not\equiv 0$, note the unperturbed frequencies are affine functions of ξ, then $\exists\ i_0\in\{1,2,\cdots,|A|\}$ such that $\left|\frac{\partial f_{k,e}(\xi)}{\partial\xi_{i_0}}\right|\geq\tilde{c}_3>0$. Therefore we get

$$meas(\mathcal{R}^0_{ke}(\alpha_0))\leq\frac{2\alpha|e|^{\frac{1}{2}}_{\delta}}{A_k\tilde{c}_3}\cdot\rho^{|A|-1}$$

for any (k,e) satisfying $|k|<K_*$, $|e|_{d-1-\delta}<E_*$, $(k,e)\in\mathcal{Z}$ and $e\neq 0$. Counting the number of (k,e), we have $meas(\Xi^{1,2}_{\alpha})\leq\tilde{c}_4\rho^{|A|-1}\alpha$.

If $k\cdot\hat{\omega}(\xi)+e\cdot\hat{\Omega}(\xi)\equiv 0$ and $k\cdot\overline{\omega}+e\cdot\overline{\Omega}\neq 0$, note $|k|<K_*$ and $|e|_{d-1-\delta}<E_*$, we have $meas(\Xi^{1,2}_{\alpha})=0$ if $\alpha\ll 1$.

If $k\cdot\omega(\xi)+e\cdot\Omega(\xi)\equiv 0$ and $|k|<K_*$, $|e|_{d-1-\delta}<E_*$, $(k,e)\in\mathcal{Z}$ and $e\neq 0$, then $meas(\mathcal{R}^0_{ke}(\alpha))=meas(\Pi)\neq 0$. It contradicts with Assumption $A3'$(choosing M_* large enough).

In the following we will prove Lemma 2.4, Lemma 2.5 and Lemma 3.8. Lemma 2.4 directly follows from the following Lemmas.

Lemma 5.3. *If $(i,j,k,l)\in\Delta_0$ or Δ_1 or Δ_2 and $|l|=|i|+|j|+|k|$ then for any $m\in[M_1,M_2]\setminus\mathcal{R}^1_0$,*

$$|\lambda_i+\lambda_j+\lambda_k-\lambda_l|\geq\min\{\frac{\mu}{\sqrt{h^4+m}},2\sqrt{m},1\},$$

where $meas(\mathcal{R}^1_0)=meas(\bigcup_{j=1}^{6}\mathcal{R}^{1,j}_0)\leq C_aC_A^4b^4M_2^4\mu$.

Proof. Denote

$$f_{ijkl}(m)=\sqrt{i^4+m}+\sqrt{j^4+m}+\sqrt{k^4+m}-\sqrt{l^4+m},\qquad m\in[M_1,M_2].$$

Then by a straightforward computation we have $f'_{ijkl}(m)\geq\frac{1}{2\sqrt{h^4+m}}>0$. In the following we discuss three subcases.

Subcase 1: $(i,j,k,l)\in\Delta_1$.

We count the number of the following set

$$\sharp\{(i,j,k,l)\in\Delta_1||l|=|i|+|j|+|k|\}\leq C_aC_Ab^3.$$

Denote the set

$$\mathcal{R}^{1,1}_0=\{m\in[M_1,M_2]:|f_{ijkl}(m)|<\frac{\mu}{\sqrt{h^4+m}},\ (i,j,k,l)\in\Delta_1\text{ and }|l|=|i|+|j|+|k|\}.$$

Then $meas(\mathcal{R}^{1,1}_0)\leq C_aC_Ab^3\mu$.

Subcase 2: $(i,j,k,l) \in \Delta_2$. There are two cases. One is $l \in A$. In this case one has $|i|, |j|, |k| \le C_A$. It follows that $\sharp\{(i,j,k,l) \in \Delta_1 | |l| = |i| + |j| + |k|\} \le C_a b^2 C_A^2$. As above denote the set

$$\begin{aligned}\mathcal{R}_0^{1,2} = \{& m \in [M_1, M_2] : |f_{ijkl}(m)| \\ & < \frac{\mu}{\sqrt{h^4+m}},\ (i,j,k,l) \in \Delta_2 \text{ and } |l| = |i| + |j| + |k|,\ l \in A\}.\end{aligned}$$

Then $meas(\mathcal{R}_0^{1,2}) \le C_a C_A^2 b^2 \mu$. The other is $l \in B$. Since $l \in B$ and $(i,j,k,l) \in \Delta_2$, then i or j or k belongs to the set B. We first consider $l \in B$ and $i \in B$. It results in $j, k \in A$. In the following discussion we use the simple fact:

$$1 + \frac{1}{2}x - \frac{1}{8}x^2 \le (1+x)^{\frac{1}{2}} \le 1 + \frac{1}{2}x \qquad (x \ge 0). \tag{5.2}$$

Denote $f_{ijkl}(m) = \lambda_i + \lambda_j + \lambda_k - \lambda_l$. From (5.2), if $il \neq 0$, then

$$f_{ijkl}(m) \le (i^2 - l^2) + (I) + (II), \tag{5.3}$$

where $(I) = \sqrt{j^4+m} + \sqrt{k^4+m}$ and $(II) = \frac{1}{2}\frac{m}{i^2} - \frac{1}{2}\frac{m}{l^2} + \frac{1}{8}\frac{m^2}{l^6}$. Clearly $|(I)| \le 2\sqrt{C_A^4 + M_2}$ and $|(II)| \le 2M_2^2$ since $M_2 \ge 1$. If $|j| + |k| = 0$, then $|f_{ijkl}(m)| = 2\sqrt{m} > 0$. If $|j| + |k| \neq 0$, then $i^2 - l^2 \le -2|i|$. Hence $f_{ijkl}(m) \le -2|i| + 2\sqrt{C_A^4 + M_2} + 2M_2^2$. If $|i| \ge 12C_A^2 M_2^2$, then $|f_{ijkl}(m)| \ge 1$. If $|i| \le 12C_A^2 M_2^2$, from $|l| = |i| + |j| + |k|$, then $|l| \le 14C_A^2 M_2^2$. Thus

$$\begin{aligned}&\sharp\big\{(i,j,k,l) \in \Delta_2 \big| |l| \\ &= |i| + |j| + |k|, l \in B,\ i \in B, |i| \le 12C_A^2 M_2^2, |l| \le 14C_A^2 M_2^2\big\} \le C_a b^2 C_A^4 M_2^4.\end{aligned}$$

We now introduce the set

$$\begin{aligned}\mathcal{R}_0^{1,3} = \big\{& m \in [M_1, M_2] : |f_{ijkl}(m)| < \frac{\mu}{\sqrt{h^4+m}}, \\ & (i,j,k,l) \in \Delta_2 \text{ and } |l| = |i| + |j| + |k|,\ l \in B,\ i \in B,\ |i| \le 12C_A^2 M_2^2\big\}.\end{aligned}$$

Then $meas(\mathcal{R}_0^{1,3}) \le C_a C_A^4 b^2 M_2^4 \mu$. Similarly

$$\begin{aligned}\mathcal{R}_0^{1,4} = \{& m \in [M_1, M_2] : |f_{ijkl}(m)| < \frac{\mu}{\sqrt{h^4+m}}, \\ & (i,j,k,l) \in \Delta_2 \text{ and } |l| = |i| + |j| + |k|,\ l \in B,\ j \in B,\ |j| \le 12C_A^2 M_2^2\}.\end{aligned}$$

One gets $meas(\mathcal{R}_0^{1,4}) \le C_a C_A^4 b^2 M_2^4 \mu$.
Introduce the set

$$\begin{aligned}\mathcal{R}_0^{1,5} = \{& m \in [M_1, M_2] : |f_{ijkl}(m)| < \frac{\mu}{\sqrt{h^4+m}}, \\ & (i,j,k,l) \in \Delta_2 \text{ and } |l| = |i| + |j| + |k|,\ l \in B,\ k \in B,\ |k| \le 12C_A^2 M_2^2\}.\end{aligned}$$

Then $meas(\mathcal{R}_0^{1,5}) \le C_a C_A^4 b^2 M_2^4 \mu$.

Subcase 3: $(i,j,k,l) \in \Delta_0$. We throw away a set $\mathcal{R}_0^{1,6}$ whose measure is no more than $C_a b^4 \mu$. ∎

Similarly

Lemma 5.4. *If* $(i,j,k,l) \in \Delta_0$ *or* Δ_1 *or* Δ_2 *and* $|i|+|j| = |k|+|l|$ *then for any* $m \in [M_1, M_2] \setminus \mathcal{R}_0^2$,

$$|\lambda_i + \lambda_j + \lambda_k - \lambda_l| \ge \min\{\frac{\mu}{\sqrt{h^4+m}}, 2\sqrt{m}, 1\},$$

where $meas(\mathcal{R}_0^2) \le C_a C_A^4 b^4 M_2^4 \mu$.

Lemma 5.5. *If* $(i,j,k,l) \in \Delta_0$ *or* Δ_1 *or* Δ_2 *and* $|i|+|k| = |j|+|l|$ *then for any* $m \in [M_1, M_2] \setminus \mathcal{R}_0^3$,

$$|\lambda_i + \lambda_j + \lambda_k - \lambda_l| \ge \min\{\frac{\mu}{\sqrt{h^4+m}}, 2\sqrt{m}, 1\},$$

where $meas(\mathcal{R}_0^3) \le C_a C_A^4 b^4 M_2^4 \mu$.

Lemma 5.6. *If* $(i,j,k,l) \in \Delta_0$ *or* Δ_1 *or* Δ_2 *and* $|i|+|l| = |j|+|k|$ *then for any* $m \in [M_1, M_2] \setminus \mathcal{R}_0^4$,

$$|\lambda_i + \lambda_j + \lambda_k - \lambda_l| \ge \min\{\frac{\mu}{\sqrt{h^4+m}}, 2\sqrt{m}, 1\},$$

where $meas(\mathcal{R}_0^4) \le C_a C_A^4 b^4 M_2^4 \mu$.

To obtain Lemma 2.4 we choose $\mathcal{R}_0 = \bigcup_{j=1}^{4} \mathcal{R}_0^j$.

Proof of Lemma 2.5. Since $i+j-k-l=0$, we have 8 cases. From the symmetry we only need to consider the following three cases:

Case 1: $|l| = |i|+|j|+|k|$. Denote $g_{ijkl}(m) = \lambda_l + \lambda_k - \lambda_i - \lambda_j$. Suppose $|k| \le |j| \le |i|$(the case $|k| \le |i| \le |j|$ is similar). Note $\tilde{h}(t) = \sqrt{t^4+m}$ is convex, it follows

$$\lambda_{|l|} - \lambda_{|i|} \ge \lambda_{|l|-p} - \lambda_{|i|-p} \tag{5.4}$$

for any $0 \le p \le |i|$. Choose $p = |i| - |k|$, then

$$\lambda_{|l|} - \lambda_{|i|} \ge \lambda_{|j|+2|k|} - \lambda_{|k|}. \tag{5.5}$$

Suppose $k \neq 0$ in the first discussion. Thus if $j \neq 0$ from (5.5) and mean value theorem we have

$$\begin{aligned} g_{ijkl}(m) &= \lambda_{|l|} + \lambda_{|k|} - \lambda_{|i|} - \lambda_{|j|} \\ &\geq \lambda_{|j|+2|k|} - \lambda_{|j|} \\ &\geq 2(\lambda_{|j|+|k|} - \lambda_{|j|}) \\ &\geq \frac{4|k||j|^3}{\sqrt{|j|^4+m}} \geq \frac{4h}{\sqrt{1+m}}. \end{aligned}$$

If $j = 0$ it results in $k = 0$. It contradicts with $k \neq 0$. The remained case is $h = k = 0$. It follows $|l| = |i| + |j|$. From a straightforward computation we have $\lambda_{|l|} + \lambda_0 \geq \lambda_{|i|} + \lambda_{|j|} + \frac{1}{4\sqrt{m}}$ if $m > 1$. If $0 < m \leq 1$ one has $\lambda_{|l|} + \lambda_0 \geq \lambda_{|i|} + \lambda_{|j|} + \frac{1}{4}$. In the both cases we have

$$\lambda_{|i|+|j|} + \lambda_0 - \lambda_{|i|} - \lambda_{|j|} \geq \min\{\frac{1}{4\sqrt{m}}, \frac{1}{4}\} = c(m) > 0, \qquad ij \neq 0.$$

If $k = 0$ and $ij = 0$ it is trivial.

The remained case is $|i| = \min\{|l|, |k|, |i|, |j|\}$ (the case for $|j| = \min\{|l|, |k|, |i|, |j|\}$ is similar). In this case one has

$$\begin{aligned} \lambda_{|l|} + \lambda_{|k|} &\geq \lambda_{|i|+|j|} + \lambda_0 \\ &\geq \lambda_{|i|} + \lambda_{|j|} + c(m), \end{aligned}$$

if $ij \neq 0$. If $i = 0$, then $|l| = |j| + |k|$. From the convexity, one has $\lambda_{|l|} - \lambda_{|j|} \geq \lambda_{|k|} - \lambda_0$. Therefore if $k \neq 0$

$$\lambda_{|l|} + \lambda_{|k|} - \lambda_0 - \lambda_{|j|} \geq 2(\lambda_{|k|} - \lambda_0) \geq 2(\sqrt{1+m} - \sqrt{m}) = c(m) > 0.$$

If $k = 0$, combing with $i = 0$ we have $j = l$. It is trivial.

Case 2: $|i| + |j| = |k| + |l|$. Note the symmetry one only needs to consider $|i| \leq |l| \leq |k| \leq |j|$ and $|j| - |k| = |l| - |i| \neq 0$. Using (5.4) we have $\lambda_{|j|} - \lambda_{|k|} \geq \lambda_{|l|+1} - \lambda_{|i|+1}$ and $\lambda_{|l|+1} - \lambda_{|l|} \geq \lambda_{|i|+2} - \lambda_{|i|+1}$. Hence,

$$\lambda_{|j|} + \lambda_{|i|} - \lambda_{|k|} - \lambda_{|l|} \geq \lambda_{|i|+2} - 2\lambda_{|i|+1} + \lambda_{|i|}. \tag{5.6}$$

Denote $w(t) = \sqrt{(t+1)^4 + m} - \sqrt{t^4 + m}(t \geq 0)$. Then

$$\begin{aligned} (5.6) &= w'(\theta)|_{\theta \in [|i|,|i+1|]} \\ &= \tilde{h}'(\theta+1) - \tilde{h}'(\theta) \\ &= \tilde{h}''(\theta_1), \end{aligned}$$

where $\theta_1 \in [|i|, |i|+2]$. If $|i| \neq 0$, then

$$\begin{aligned}\tilde{h}''(\theta_1) &= \frac{\theta_1^2}{\sqrt{\theta_1^4+m}}(6-\frac{4\theta_1^4}{\theta_1^4+m}) \\ &\geq \frac{2\theta_1^2}{\sqrt{\theta_1^4+m}} \geq \frac{2}{\sqrt{(h+2)^4+m}}.\end{aligned}$$

If $i=0$ then $|j|=|k|+|l|$. As before one has $\lambda_{|k|+|l|}+\lambda_0-\lambda_{|k|}-\lambda_{|l|} \geq c(m) > 0$ if $kl \neq 0$. The remained cases are trivial.

Case 3: $|i|+|k|=|j|+|l|$. From the symmetry suppose $|i| \leq |j|$ and $|i|-|j| = |l|-|k| \leq 0$. One can further suppose $|i| = \min\{|i|,|j|,|k|,|l|\}$. We have two subcases. The first is $|i| \leq |j| \leq |l| \leq |k|$. The second is $|i| \leq |l| \leq |j| \leq |k|$. Since $\lambda_{|k|}+\lambda_{|l|}-\lambda_{|i|}-\lambda_{|j|} \geq \lambda_{|k|}+\lambda_{|j|}-\lambda_{|i|}-\lambda_{|l|}$. We only need to consider the second one. Note $|k|-|j| = |l|-|i| \neq 0$, then $|i| < |l| \leq |j| < |k|$. It follows $\lambda_{|k|}-\lambda_{|j|} \geq \lambda_{|l|}-\lambda_{|i|}$. Thus

$$\begin{aligned}\lambda_{|k|}+\lambda_{|l|}-\lambda_{|i|}-\lambda_{|j|} &\geq 2(\lambda_{|l|}-\lambda_{|i|}) \\ &\geq 2(\sqrt{(|i|+1)^4+m}-\sqrt{|i|^4+m}) \\ &\geq \frac{1}{\sqrt{(1+h)^4+m}}.\end{aligned}$$

Proof of Lemma 3.8. From a straightforward computation, we have $\det A_1 \neq 0$(note $0 \in B$).

Next we will check the second condition in A'. The set $\mathcal{R}_1$ will be clear in the following.

Case 1: $|e|=1$. Note $k\cdot\omega+\Omega_j = k\cdot\omega_0+\lambda_j+\frac{1}{4\pi}\langle A_1^T k+2e_0, I\rangle$, we discuss the following subcases, where $j \in B$, $e_0 = (\frac{n_1^2+j^2}{\lambda_{n_1}\lambda_j}, \cdots, \frac{n_b^2+j^2}{\lambda_{n_b}\lambda_j})^T$ and $I=(I_j)_{j\in A}$.

Subcase 1: $|j| \notin \{|n_1|,|n_2|,\cdots,|n_b|\}$. We reorder $(n_1,n_2,\cdots,n_b,j)$ as $i_1,\cdots,$ i_{b+1} such that $|i_1|<|i_2|<\cdots<|i_{b+1}|$. Suppose that $|j| \leq M_*$ and $0 \leq |k| \leq M_*$. Without losing generality, suppose that $M_* \geq |n_b|$. It follows $|i_{b+1}| \leq M_*$. We write

$$f_{\bar{k}}(m) = k\cdot\omega_0+\lambda_j = \bar{k}_1\lambda_{i_1}+\bar{k}_2\lambda_{i_2}+\cdots+\bar{k}_{b+1}\lambda_{i_{b+1}}.$$

It is clear that $\bar{k}=(\bar{k}_1,\cdots,\bar{k}_{b+1}) \neq 0$. From Lemma 5.1, there exists $i_0 \in \{1,\cdots,b+1\}$ such that

$$|\bar{k}_1\frac{d^{(i_0-1)}\lambda_{i_1}}{dm^{i_0-1}}+\cdots+\bar{k}_{b+1}\frac{d^{(i_0-1)}\lambda_{i_{b+1}}}{dm^{i_0-1}}| \geq \frac{c_b(\min\{M_1,1\})^{(b+\frac{1}{2})(b+1)}}{(M_*^4+M_2)^{(b+1)^2+\frac{1}{2}(b+1)}} = d_1 > 0.$$

If $i_0 = 1$, it is clear that $k \cdot \omega_0 + \lambda_j > 0$. While for $i_0 \in \{2, \cdots, b+1\}$ we define

$$R^1_{k,j}(m) = \{m \in [M_1, M_2] : \ |f_{\bar{k}}(m)| < \mu\}.$$

From Lemma 5.2, $meas(R^1_{k,j}) \leq C(M_*, M_1, M_2, b) \cdot \mu^{\frac{1}{b}}$. If denote $R^1 = \bigcup_{|k| \leq M_*, |j| \leq M_*} R^1_{k,j}$, counting the number of (k, j), we have $meas(R^1) \leq C(M_*, M_1, M_2, b) \cdot \mu^{\frac{1}{b}}$. Thus, if $m \in [M_1, M_2] \setminus R^1$ we have $k \cdot \omega_0 + \lambda_j \neq 0$ for any $|k| \leq M_*$ and $|j| \leq M_*$.

Subcase 2: $|j| \in \{|n_1|, |n_2|, \cdots, |n_b|\}$. Without losing generality, suppose that $j = -n_1$. As above, we define $f_k(m) = (k_1 + 1)\lambda_{n_1} + k_2\lambda_{n_2} + \cdots + k_b\lambda_{n_b}$. If $k_0 = (k_1, k_2, \cdots, k_b)^T = (-1, 0, \cdots, 0)^T$, then $f_{k_0}(m) = 0$. But $A^T k_0 + 2e_0 = (\frac{2n_1^2}{\lambda^2_{n_1}}, \cdots, *)^T \neq 0$, since $n_1 \neq 0$. For the case $(k_1, k_2, \cdots, k_b) \neq (-1, 0, \cdots, 0)$ and $j = -n_1$, from Lemma 3.7 we have $|(k_1 + 1)\lambda_{n_1} + k_2\lambda_{n_2} + \cdots + k_b\lambda_{n_b}| \geq \mu$ for any $m \in [M_1, M_2] \setminus R^0$. In conclude, in this subcase if $m \in [M_1, M_2] \setminus (R^0 \cup R^1)$, then we have two possibilities: one is $k \cdot \omega_0 + \lambda_j \neq 0$ for any $|k| \leq M_*$ and $|j| \leq M_*$, while the other possibility is $A_1^T k + 2e \neq 0$.

Case 2: $|e| = 2$. We will divide this case into the following three subcases.

Subcase 1: $e = (\cdots, 1, \cdots, 1, \cdots)$. The nonzero sites are j_1th and j_2th sites respectively, where $j_1, \ j_2 \in B$.

(1). $|j_1| \neq |j_2|$ and $|j_1|, \ |j_2| \notin \{|n_1|, |n_2|, \cdots, |n_b|\}$. As above, we will throw a set denoted by R^2 whose measure is smaller than $C(M_*, M_1, M_2, b) \cdot \mu^{\frac{1}{b+1}}$. For any $m \in [M_1, M_2] \setminus R^2$, we have $k \cdot \omega_0 + \lambda_{j_1} + \lambda_{j_2} \neq 0$ for any $|k| \leq M_*$ and $|j_1|, |j_2| \leq M_*$.

(2). $|j_1| \in \{|n_1|, \cdots, |n_b|\}$ or $|j_2| \in \{|n_1|, \cdots, |n_b|\}$. In this subcase, we need to throw away a set denoted by R^3, whose measure is smaller than $C(M_*, M_1, M_2, b) \cdot \mu^{\frac{1}{b}}$.

(3). $|j_1|, \ |j_2| \in \{|n_1| \cdots, |n_b|\}$. To fix our idea we suppose that $j_1 = -n_1, \ j_2 = -n_2$. For the other cases the discussion is similar. If $(k_1+1, k_2+1, k_3, \cdots, k_b) \neq 0$ and $m \in [M_1, M_2] \setminus R^0$, from Lemma 3.7 we have $|(k_1+1)\lambda_{n_1} + (k_2+1)\lambda_{n_2} + \cdots + k_b\lambda_{n_b}| \geq \mu > 0$. The left case is $(k_1, k_2 \cdots, k_b) = (-1, -1, 0, \cdots, 0)$. For this case $A^T k + 2e_0 + 2e_1 = (\frac{2n_1^2}{\lambda^2_{n_1}}, \cdots, *)^T \neq 0$.

Subcase 2: $e = (\cdots, 2, \cdots)$. We discuss the following two cases. The first one is $|j| \notin \{|n_1|, \cdots, |n_b|\}$. We need to throw away a set denoted by R^4, whose measure is smaller than $C(M_*, M_1, M_2, b) \cdot \mu^{\frac{1}{b}}$. The second case is that $|j| \in \{|n_1|, \cdots, |n_b|\}$. In this subcase as above if $m \in [M_1, M_2] \setminus R^0$, we have two possibilities: one is $k \cdot \omega_0 + 2\lambda_j \neq 0$ for any $|k| \leq M_*$ and $|j| \leq M_*$, while the other possibility is $A^T k + 4e_0 \neq 0$.

Subcase 1: $e = (\cdots, 1, \cdots, -1, \cdots)$. The nonzero sites are j_1th and j_2th sites respectively, where $j_1,\ j_2 \in B$. If $|j_1| \neq |j_2|$, then we discuss the following three cases.

(1). $|j_1|,\ |j_2| \notin \{|n_1|, |n_2|, \cdots, |n_b|\}$. As above, we throw a set denoted by R^5, whose measure is smaller than $C(M_*, M_1, M_2, b) \cdot \mu^{\frac{1}{b+1}}$.

(2). Only one of $|j_1|$, $|j_2|$ belongs to $\{|n_1|, \cdots, |n_b|\}$. In this subcase we throw away a set denoted by R^6, whose measure is smaller than $C(M_*, M_1, M_2, b) \cdot \mu^{\frac{1}{b}}$.

(3). $|j_1|,\ |j_2| \in \{|n_1| \cdots, |n_b|\}$. To fix the idea we suppose that $j_1 = -n_1,\ j_2 = -n_2$. For the other cases the discussion is similar. If $(k_1+1, k_2-1, k_3, \cdots, k_b) \neq 0$ and $m \in [M_1, M_2] \setminus R^0$, from Lemma 3.7 we have $|(k_1+1)\lambda_{n_1} + (k_2-1)\lambda_{n_2} + \cdots + k_b\lambda_{n_b}| \geq \mu > 0$. The left case is $(k_1, k_2 \cdots, k_b) = (-1, 1, 0, \cdots, 0)$. For this case, $A^T k + 2e_0 - 2e_1 = (\frac{2n_1^2}{\lambda_{n_1}^2}, \cdots, *)^T \neq 0$.

If $|j_1| = |j_2|$, $j_1, j_2 \in B$. In this case we have $j_2 = -j_1$. If $k \neq 0$, then from Lemma 3.7 we have $k \cdot \omega_0 + \lambda_j - \lambda_{-j} = k \cdot \omega_0 \neq 0$ for any $m \in [M_1, M_2] \setminus R^0$. If $k = 0$ and $j \neq 0$, we have $k_1 n_1 + \cdots + k_b n_b - 2j = -2j \neq 0$. This shows that this subcase doesn't satisfy the symmetry condition.

Finally, we set $\mathcal{R}_1 = \bigcup_{i=0}^{6} R^i$.

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