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MODULES WHOSE CLOSED SUBMODULES WITH ESSENTIAL SOCLE ARE DIRECT SUMMANDS

Septimiu Crivei and Serap Şahinkaya

Abstract. We introduce and study CLESS-modules, which subsume two generalizations of extending modules due to P.F. Smith and A. Tercan. A module M will be called a CLESS-module if every closed submodule N of M (in the sense that M/N is non-singular) with essential socle is a direct summand of M. Various properties concerning direct sums of CLESS-modules are established. We show that, over a Dedekind domain, a module is CLESS if and only if its torsion submodule is a direct summand. We also study the behaviour of CLESS-modules under excellent extensions of rings.

1. Introduction

Extending modules (or CS-modules) have offered a rich topic of research, especially in the last 20 years, due to their important role played in ring and module theory. The monograph by N.V. Dung et al. [6] gives an excellent account on the developments of the theory up to that moment. In parallel, several generalizations of CS-modules have been considered, for instance CESS-modules [3], weak CS-modules [18], C_{11}-modules [19], CLS-modules [21] etc. They generalize the theory of extending modules towards different directions.

The purpose of the present paper is to introduce and study CLESS-modules, which allow us to give a unified approach of CESS-modules and CLS-modules, introduced by P.F. Smith [18] and A. Tercan [21] respectively. Recall that a module M is called a *CS-module* (or *extending module*) if every complement submodule of M is a direct

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summand of M, a *CESS-module* if every complement submodule of M with essential socle is a direct summand of M, and a *CLS-module* if every closed submodule N of M (in the sense that M/N is non-singular, see [7, 16]) is a direct summand of M. Let us emphasize that the terminology of closed submodule which we shall use is not the one employed in classical references on extending modules such as [6], where closed submodules are the same as complements. Every closed submodule in our sense is a complement, and every complement of a non-singular module is a closed submodule [16, Lemma 2.3].

We define a *CLESS-module* by the property that every closed submodule with essential socle is a direct summand. We study these modules, generalizing several results both on CESS-modules and CLS-modules. We show that if M is a module such that $\mathrm{Soc}(Z_2(M)) \trianglelefteq Z_2(M)$, then M is a CLESS-module if and only if $M = Z_2(M) \oplus M'$ for some (non-singular) CESS-submodule M' of M. We emphasize that our properties are of the same type as those for extending modules, sharing similar limitations in studying certain properties, such as the closure of the respective class of modules under direct sums. We analyze when a direct sum of CLESS-modules is a CLESS-module, using the concepts of relative ojectivity and relative ejectivity. We discuss the case of a Dedekind domain, since it offers some further motivation for considering CLESS-modules. An important question for a module over a domain is: when does it split, in the sense that its torsion submodule is a direct summand? Some classical results show that a commutative domain is a: (i) field if and only if every module splits (see J. Rotman [15]); (ii) Dedekind domain if and only if every module whose torsion submodule is of bounded order splits (see S.U. Chase [4] and I. Kaplansky [9]). We prove that, over a Dedekind domain, CLESS-modules coincide with modules that split. Finally, we study the behaviour of CLESS-modules under excellent extensions of rings. Our results on CLESS-modules can be used to derive new corresponding results for CESS-modules and CLS-modules.

Now let us give some basic notation and set the terminology. Throughout this paper, we assume that R is an associative ring with identity and all modules are unitary right R-modules. We shall denote the fact that a submodule N is essential in a module M by $N \trianglelefteq M$. The socle and the singular submodule of a module M will be denoted by $\mathrm{Soc}(M)$ and $Z(M) = \{m \in M \mid \mathrm{ann}_R(m) \trianglelefteq R\}$ respectively. A module M is called *singular* (respectively *non-singular*) if $Z(M) = M$ (respectively $Z(M) = 0$). The class of singular modules is a hereditary pretorsion class (i.e. it is closed under submodules, homomorphic images and direct sums), whereas the class of non-singular modules is a torsionfree class (i.e. it is closed under submodules, direct products and extensions). By [6, 4.6], M is a singular module if and only if $M \cong L/K$ for a module L and $K \trianglelefteq L$. The second singular submodule $Z_2(M)$ of M is defined by the equality $Z_2(M)/Z(M) = Z(M/Z(M))$. A submodule N of a module M will be called *closed* if M/N is non-singular (see [7, 16]). For a positive integer n, we denote

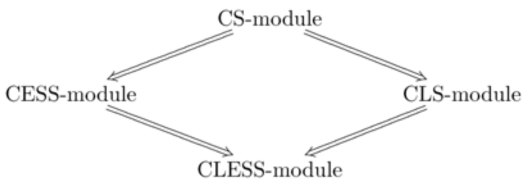
$\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$. We refer to [6, 7] for the undefined notions used in the text.

2. Basic Properties of Cless-modules

We begin with the definition of the main concept of the paper.

Definition 2.1. A module M is called a *CLESS-module* if every closed submodule of M with essential socle is a direct summand of M. The ring R is called a *right CLESS-ring* if it is CLESS as a right R-module.

Let us see how CLESS-modules relate to CS-modules, CESS-modules and CLS-modules, as defined in the introduction. We have the following hierarchy:



There are immediate instances when the above notions coincide. We also note some obvious classes of CESS-modules and CLS-modules.

Remark 2.2. (1) Let M be a module with essential socle. Then M is a CESS-module if and only if M is a CS-module, and M is a CLESS-module if and only if M is a CLS-module.

(2) Let M be a non-singular module. Then M is a CLS-module if and only if M is a CS-module, and M is a CLESS-module if and only if M is a CESS-module.

(3) Every module with zero socle is CESS, and every singular module is CLS.

But in general none of the implications from the above diagram is an equivalence, and CESS-modules and CLS-modules are not directly related. All the necessary examples can be immediately obtained from the following ones. Other classes of examples will emerge from a forthcoming characterization theorem of CLESS-modules over Dedekind domains (see Theorem 4.1).

Example 2.3. (1) The free $\mathbb{Z}$-module of infinite rank $\mathbb{Z}^{(\mathbb{N})}$ is a CESS-module [18, p. 101], but not a CLS-module [21, Example 16].

(2) Let p be a prime. The $\mathbb{Z}$-module $\mathbb{Z}_p \oplus \mathbb{Z}_{p^3}$ is a CLS-module [21, Example 6], but not a CESS-module [18, Example 1.1].

We point out that CLESS-modules also generalize Goldie extending modules, recently introduced and studied in [1]. Recall that a module is called *Goldie extending* if for every submodule X of M there is a direct summand D of M such that $X \cap D$

is essential both in X and D. It is easy to see that every Goldie extending module is CLS, and consequently CLESS.

We collect in the following lemma some frequently used immediate properties on closed submodules and modules with essential socle.

Lemma 2.4. (i) *The relation of being a closed submodule is transitive.*

(ii) *Let M be a module and let K and N be submodules of M such that K is a closed submodule of M. Then $K \cap N$ is a closed submodule of N.*

(iii) *The class of modules with essential socle is closed under submodules, direct sums and essential extensions.*

We continue with a lemma on closed submodules and direct summands of CLESS-modules.

Lemma 2.5. (i) *Let M be a CLESS-module and N a closed submodule of M. Then N is a CLESS-module.*

(ii) *Let $M = M_1 \oplus M_2$ be a CLESS-module such that $\operatorname{Soc}(M_1) \trianglelefteq M_1$. Then M_2 is a CLESS-module.*

Proof. (i) Clear, using Lemma 2.4.

(ii) Let K be a closed submodule of M_2 with $\operatorname{Soc}(K) \trianglelefteq K$. Since we have $M/(M_1 \oplus K) \cong M_2/K$, $M_1 \oplus K$ is a closed submodule of M. Moreover, $\operatorname{Soc}(M_1 \oplus K) \trianglelefteq M_1 \oplus K$. Since M is a CLESS-module, $M_1 \oplus K$ is a direct summand of M. Then it follows that K is a direct summand of M_2. Hence M_2 is a CLESS-module. ■

Theorem 2.6. *Let $M = M_1 \oplus M_2$ be a module. Then M is a CLESS-module if and only if every closed submodule K of M with $\operatorname{Soc}(K) \trianglelefteq K$ such that $K \cap M_1 = 0$ or $K \cap M_2 = 0$ is a direct summand of M.*

Proof. The necessity is clear. Conversely, assume that every closed submodule K of M with $\operatorname{Soc}(K) \trianglelefteq K$ such that $K \cap M_1 = 0$ or $K \cap M_2 = 0$ is a direct summand of M. Let K be a closed submodule of M with $\operatorname{Soc}(K) \trianglelefteq K$. By Lemma 2.4, $K \cap M_1$ is closed in M_1 and $\operatorname{Soc}(K \cap M_1) \trianglelefteq K \cap M_1$. Since $K \cap M_1 \cap M_2 = 0$, by hypothesis $K \cap M_1$ is a direct summand of M, say $M = (K \cap M_1) \oplus L$ for some submodule L of M. By Lemma 2.5, L is a CLESS-module. Since $K \cap L$ is closed in L and $\operatorname{Soc}(K \cap L) \trianglelefteq K \cap L$ by Lemma 2.4, it follows that $K \cap L$ is a direct summand of L. Hence K is a direct summand of M. This shows that M is a CLESS-module. ■

We have the following direct sum decomposition theorem for CLESS-modules whose second singular submodule has essential socle.

Theorem 2.7. *Let M be a module such that $\operatorname{Soc}(Z_2(M)) \trianglelefteq Z_2(M)$. Then M is a CLESS-module if and only if $M = Z_2(M) \oplus M'$ for some (non-singular) CESS-submodule M' of M.*

Proof. Assume first that M is a CLESS-module. Since $Z_2(M)$ is a closed submodule of M with essential socle, we have a decomposition $M = Z_2(M) \oplus M'$ for some submodule M' of M. By Lemma 2.5, M' is a CLESS-module. Moreover, since M' is non-singular, M' is a CESS-module by Remark 2.2.

Conversely, assume that $M = Z_2(M) \oplus M'$ for some CESS-submodule M' of M. Let K be a closed submodule of M with essential socle. We must have $Z_2(M) \subseteq K$, and so $K = Z_2(M) \oplus (K \cap M')$. By Lemma 2.4, $\mathrm{Soc}(K \cap M') \trianglelefteq K \cap M'$, and $K \cap M'$ is closed in M', and consequently, $K \cap M'$ is a complement submodule of M'. Since M' is a CESS-module, $K \cap M'$ is a direct summand of M', say $M' = (K \cap M') \oplus K'$ for some submodule K' of M'. Then $M = K \oplus K'$, that is, K is a direct summand of M. Hence M is a CLESS-module. ■

Recall that a module M is called *semiartinian* if every non-zero factor of M has a simple submodule. It is well known that if M is semiartinian, then its socle is essential in M, and every submodule of M is semiartinian.

Corollary 2.8. *Let M be a semiartinian module. Then M is a CLS-module if and only if $M = Z_2(M) \oplus M'$ for some (non-singular) CS-submodule M' of M.*

Proof. This follows by Theorem 2.7 and Remark 2.2. ■

3. Direct Sums of CLESS-Modules

It is well-known that in general the class of extending modules is not closed under direct sums, and this behaviour is also carried on by CESS-modules (see [3, Theorem 2.2]) and CLS-modules (see [21, p. 1560]). Finding necessary and sufficient conditions for ensuring the closure of such classes under direct sums has been one of the most important open problems in the theory of extending modules and their generalizations. In what follows we shall deal with such a problem for CLESS-modules. But first let us give an example which shows that in general the class of CLESS-modules is not closed under direct sums. Efficient tools for such counterexamples have been the rings of trivial extensions. We use a clue from [22, Example 3].

Example 3.1. Let $B = \mathbb{Z} \oplus \mathbb{Z}_p$ for some prime number p, and let R be the trivial extension of $\mathbb{Z}$ and B. The ring $R = \mathbb{Z} \oplus B$ has addition and multiplication defined by $(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_2 + b_2)$ and $(a_1, b_1)(a_2, b_2) = (a_1 a_2, a_1 b_2 + a_2 b_1)$. Then R is CESS by [19, Corollary 1.3], and so R is CLESS. Note that $I = \mathrm{Soc}(R) = 0 \oplus B$ is essential in R, hence R/I is singular, and so R/I is CLESS (see Remark 2.2). We claim that $M = R \oplus R/I$ is not a CLESS-module. To this end, view $\mathbb{Z}_p$ as a submodule K of M. Then $\mathrm{Soc}(K) = K$. Since $\mathbb{Z}$ is non-singular, we have $Z(M/K) = 0$, hence K is a closed submodule of M. As in [22, Example 3], if $M = K \oplus L$ for some submodule L of M, then $L = \{((a_1, b), (a_2, 0) + I) \mid a_1, a_2 \in \mathbb{Z},\ b \in B\}$, which implies that $K \subseteq L$, a contradiction. Hence K is not a direct summand of M, and consequently, M is not a CLESS-module.

In order to see when a direct sum of two CLESS-modules has the same property, the following concept generalizing relative injectivity will be useful. For a module $M = M_1 \oplus M_2$, recall that M_2 is called M_1-*ojective* if for every complement K of M_2 in M, we have $M = K \oplus M_1' \oplus M_2'$ for some submodules M_1' of M_1 and M_2' of M_2 [11]. Also, M_1 and M_2 are called *relatively ojective* if M_1 is M_2-ojective and M_2 is M_1-ojective.

We shall use an approach similar to that in [11] combined with the following technique. First we show some properties of direct sums of non-singular CLESS-modules, and then we use them for proving similar properties of direct sums of arbitrary CLESS-modules. We have seen that complement submodules and closed submodules of a non-singular module coincide, and non-singular CLESS-modules are the same as non-singular CESS-modules.

Lemma 3.2. *Let $M = M_1 \oplus M_2$ be a module such that M_1 is a non-singular CLESS-module and M_2 is M_1-ojective. Then for every closed submodule K of M with $\mathrm{Soc}(K) \trianglelefteq K$ and $K \cap M_2 = 0$, $M = K \oplus M_1' \oplus M_2'$ for some submodules M_1' of M_1 and M_2' of M_2.*

Proof. Let K be a complement (closed) submodule of M with $\mathrm{Soc}(K) \trianglelefteq K$ and $K \cap M_2 = 0$. By Lemma 2.4, $K \cap M_1$ is closed in M_1 and $\mathrm{Soc}(K \cap M_1) \trianglelefteq K \cap M_1$. Since M_1 is a CLESS-module, it follows that $K \cap M_1$ is a direct summand of M_1, say $M_1 = (K \cap M_1) \oplus N_1$ for some submodule N_1 of M_1. By Lemma 2.5, N_1 is a CLESS-module. Now let $L = (K \oplus M_2) \cap M_1$. Let N_1' be a complement closure of $K \cap L \cap N_1$ in N_1. By Lemma 2.4, $K \cap L \cap N_1$ has essential socle and, furthermore, so has its essential extension N_1'. Since N_1 is non-singular, it is a CESS-module, hence N_1' is a direct summand of N_1, say $N_1 = N_1' \oplus N_1''$ for some submodule N_1'' of N_1. Then we have:

$$K \oplus M_2 = L \oplus M_2 = (K \cap M_1) \oplus (L \cap N_1) \oplus M_2 \trianglelefteq (K \cap M_1) \oplus N_1' \oplus M_2.$$

It follows that K is a complement of M_2 in $N = (K \cap M_1) \oplus N_1' \oplus M_2$. By [11, Proposition 8], M_2 is $(K \cap M_1) \oplus N_1'$-ojective. Hence $N = K \oplus M_1'' \oplus M_2'$ for some submodules M_1'' of $(K \cap M_1) \oplus N_1'$ and M_2' of M_2. Consequently, $M = K \oplus M_1' \oplus M_2'$ with $M_1' = M_1'' \oplus N_1'' \subseteq M_1$ and $M_2' \subseteq M_2$. ∎

Lemma 3.3. *Let $M = M_1 \oplus M_2$ be a direct sum of relatively ojective non-singular CLESS-modules. Then for every closed submodule K of M with $\mathrm{Soc}(K) \trianglelefteq K$, $M = K \oplus M_1' \oplus M_2'$ for some submodules M_1' of M_1 and M_2' of M_2. In particular, M is a CLESS-module.*

Proof. Let K be a complement (closed) submodule of M with $\mathrm{Soc}(K) \trianglelefteq K$. Let L be a complement closure of $K \cap M_1$ in K. By Lemma 2.4, L is a closed submodule of M and $\mathrm{Soc}(L) \trianglelefteq L$. Since $L \cap M_2 = 0$, by Lemma 3.2 we have

$M = L \oplus N_1 \oplus N_2$ for some submodules N_1 of M_1 and N_2 of M_2. It follows that $N = N_1 \oplus N_2$ is non-singular. Also, we have $K = L \oplus L'$, where $L' = K \cap N$. By Lemma 2.4, L' is closed in M, and so L' is closed in N. Also, $\mathrm{Soc}(L) \trianglelefteq L$. We have $L' \cap N_1 = L' \cap K \cap N_1 \subseteq L' \cap K \cap M_1 \subseteq L' \cap L = 0$. Note that N_1 is N_2-ojective by [11, Proposition 8]. By [18, Corollary 1.3], N_2 is a CESS-module as a direct summand of M_2. Then by Lemma 3.2 we have $N = L' \oplus M_1' \oplus M_2'$ for some submodules M_1' of N_1 and M_2' of N_2. It follows that $M = L \oplus N = L \oplus L' \oplus M_1' \oplus M_2' = K \oplus M_1' \oplus M_2'$ with $M_1' \subseteq M_1$ and $M_2' \subseteq M_2$. ∎

Now we can return to arbitrary CLESS-modules.

Theorem 3.4. *Let $M = M_1 \oplus M_2$ be a direct sum of relatively ojective CLESS-modules. Then for every closed submodule K of M with $\mathrm{Soc}(K) \trianglelefteq K$, $M = K \oplus M_1' \oplus M_2'$ for some submodules M_1' of M_1 and M_2' of M_2. In particular, M is a CLESS-module.*

Proof. Let K be a closed submodule of M with $\mathrm{Soc}(K) \trianglelefteq K$. Then $K \cap M_1$ is closed in M_1 and $\mathrm{Soc}(K \cap M_1) \trianglelefteq K \cap M_1$ by Lemma 2.4. Since M_1 is a CLESS-module, $K \cap M_1$ is a direct summand of M_1, say $M_1 = (K \cap M_1) \oplus N_1$ for some submodule N_1 of M_1. Then N_1 is a non-singular CLESS-module by Lemma 2.5. Similarly, we may write $M_2 = (K \cap M_2) \oplus N_2$ for some non-singular CLESS-submodule N_2 of M_2. By [11, Proposition 8], N_1 and N_2 are relatively ojective. Now $N_1 \oplus N_2$ is a CLESS-module by Lemma 3.3. We have

$$K = (K \cap M_1) \oplus (K \cap M_2) \oplus (K \cap (N_1 \oplus N_2)).$$

Hence $K \cap (N_1 \oplus N_2)$ is clearly a complement submodule of $N_1 \oplus N_2$ and has essential socle by Lemma 2.4. By Lemma 3.3, we have $N_1 \oplus N_2 = (K \cap (N_1 \oplus N_2)) \oplus M_1' \oplus M_2'$ for some submodules M_1' of N_1 and M_2' of N_2. Then we have

$$\begin{aligned} M &= M_1 \oplus M_2 = (K \cap M_1) \oplus (K \cap M_2) \oplus N_1 \oplus N_2 \\ &= (K \cap M_1) \oplus (K \cap M_2) \oplus (K \cap (N_1 \oplus N_2)) \oplus M_1' \oplus M_2' \\ &= K \oplus M_1' \oplus M_2' \end{aligned}$$

with $M_1' \subseteq M_1$ and $M_2' \subseteq M_2$. ∎

Lemma 3.5. *Let $M = M_1 \oplus M_2$ be a direct sum of CLESS-modules such that M_2 is M_1-injective. Then M is a CLESS-module.*

Proof. Let K be a closed submodule of M with $\mathrm{Soc}(K) \trianglelefteq K$. By Lemma 2.4, $K \cap M_2$ is closed in M_2 and $\mathrm{Soc}(K \cap M_2) \trianglelefteq K \cap M_2$. Since M_2 is a CLESS-module, $K \cap M_2$ is a direct summand of M_2, and so it is a direct summand of M. Hence $K \cap M_2$ is a direct summand of K, say $K = (K \cap M_2) \oplus N$ for some submodule N

of K. Since $N \cap M_2 = 0$ and M_2 is M_1-injective, the proof of [6, Lemma 7.5] yields the existence of a submodule M' of M such that $M = M' \oplus M_2$ and $N \subseteq M' \cong M_1$. The isomorphism $M/K \cong (M_2/(K \cap M_2)) \oplus (M'/N)$ implies that N is closed in M'. Since $\mathrm{Soc}(N) \trianglelefteq N$ and $M' \cong M_1$ is a CLESS-module, N is a direct summand of M'. It follows that K is a direct summand of M. Hence M is a CLESS-module. ■

Corollary 3.6. *Let $M = M_1 \oplus M_2$ be a module. If either M_1 is a CLESS-module and M_2 is injective, or M_1 is semisimple and M_2 is a CLESS-module, then M is a CLESS-module.*

Proof. Note that M_2 is M_1-injective in both situations. Now use Lemma 3.5. ■

Another notion generalizing relative injectivity was recently considered in [1]. Recall that, for a module $M = M_1 \oplus M_2$, M_2 is called M_1-*ejective* if for every submodule K of M with $K \cap M_2 = 0$, we have $M = M_2 \oplus M_3$ for some submodule M_3 of M such that $K \cap M_3 \trianglelefteq K$ [1, Theorem 2.7].

Theorem 3.7. *Let $M = M_1 \oplus M_2$ be a direct sum of CLESS-modules such that M_2 is M_1-ejective. Then M is a CLESS-module.*

Proof. Let K be a closed submodule of M with $\mathrm{Soc}(K) \trianglelefteq K$. Then $K \cap M_1$ is closed in M_1 and $\mathrm{Soc}(K \cap M_1) \trianglelefteq K \cap M_1$ by Lemma 2.4. Since M_1 is a CLESS-module, $K \cap M_1$ is a direct summand of M_1, say $M_1 = (K \cap M_1) \oplus N_1$ for some submodule N_1 of M_1. Then N_1 is a non-singular CLESS-module by Lemma 2.5. Similarly, we may write $M_2 = (K \cap M_2) \oplus N_2$ for some non-singular CLESS-submodule N_2 of M_2. By [23, Lemma 3.4], N_2 is N_1-ejective. Now by [1, Corollary 2.8], N_2 is N_1-injective. Hence $N_1 \oplus N_2$ is a CLESS-module by Lemma 3.5. Now continue as in the proof of Theorem 3.4 in order to deduce that M is a CLESS-module. ■

4. CLESS-Modules over Dedekind Domains

In the case of Dedekind domains, we have the following direct sum decomposition theorem.

Theorem 4.1. *Let R be a Dedekind domain. Then a module M is a:*

(*i*) *CESS-module if and only if $M = M_1 \oplus M_2$ for some torsion CS-module M_1 and torsionfree module M_2.*

(*ii*) *CLS-module if and only if $M = M_1 \oplus M_2$ for some torsion module M_1 and torsionfree CS-module M_2.*

(*iii*) *CLESS-module if and only if $M = M_1 \oplus M_2$ for some torsion module M_1 and torsionfree module M_2.*

Proof. Note that a module X is torsion if and only if $\mathrm{Soc}(X) \trianglelefteq X$, and X is torsionfree if and only if X is non-singular. Also, the torsion part of X is $t(X) = Z_2(X)$ and $\mathrm{Soc}(Z_2(X)) \trianglelefteq Z_2(X)$.

(i) This is [18, Proposition 1.8].

(ii) If M is a CLS-module, then by [21, Proposition 8] we have $M = t(M) \oplus M_2$, where M_2 is a torsionfree CS-module. Conversely, assume that $M = M_1 \oplus M_2$ for some torsion module M_1 and torsionfree CS-module M_2. Moreover, the torsion module M_1 is CLS. Since M_2 is clearly M_1-injective, M is a CLS-module by [21, Theorem 9].

(iii) If M is a CLESS-module, then by Theorem 2.7 we have $M = t(M) \oplus M_2$, where M_2 is obviously torsionfree. Conversely, assume that $M = M_1 \oplus M_2$ for some torsion module M_1 and torsionfree module M_2. Let K be a closed submodule of M with essential socle. Then K is torsion, and so $K \subseteq M_1$. It follows that K is a closed submodule of M_1. But the torsion module M_1 is a CLS-module, and so K is a direct summand of M_1. Hence K is a direct summand of M, which shows that M is a CLESS-module. ■

Recall that a module is called a *UC-module* if every submodule has a unique complement closure [20]. The structure of finite UC-abelian groups ($\mathbb{Z}$-modules) is given in [5].

Corollary 4.2. *Let R be a Dedekind domain. Then:*

(*i*) *A module is CLESS if and only if it splits. Consequently, every module is CLESS if and only if R is a field.*

(*ii*) *The class of CLESS-modules is closed under direct summands and finite direct sums.*

(*iii*) *Every finitely generated module is CLESS.*

(*iv*) *Every module with finite uniform dimension is CLESS.*

(*v*) *Every UC-module is CLESS.*

Proof.

(i) The first part follows by the proof of Theorem 4.1, and the second one by J. Rotman's result mentioned in the introduction.

(ii) Straightforward using (i).

(iii) Note that every finitely generated module is a direct sum of a torsion module and a torsionfree module and use Theorem 4.1.

(iv) Let M be a module with finite uniform dimension. If M is torsionfree, then it is clearly CLESS. If M is not torsionfree, then $M = M_1 \oplus M_2$ for some injective module M_1 and finitely generated module M_2 [9, Theorem 9]. Then M is CLESS by (iii) and Corollary 3.6.

(v) In this case, M is either torsion or torsionfree [3, Lemma 3.2]. Now use Theorem 4.1. ■

The structure of CS-modules over Dedekind domains (see [8] and [11]), and in particular over the ring of integers, together with Theorem 4.1 and Corollary 4.2 provide good sources of examples concerning CESS-modules, CLS-modules and CLESS-modules, and clarify why the modules in Example 2.3 have or do not have the required properties. In view of the above results we also have the following examples.

Example 4.3.

(1) Let p be a prime and let $\mathbb{Z}_{(p)}$ be the localization of $\mathbb{Z}$ at its prime ideal $p\mathbb{Z}$. Then the torsionfree $\mathbb{Z}$-module $\mathbb{Z} \oplus \mathbb{Z}_{(p)}$ is CESS, but not CLS, because $\mathbb{Z} \oplus \mathbb{Z}_{(p)}$ is not CS.

(2) Let p be a prime. The $\mathbb{Z}$-module $\mathbb{Z}_p \oplus \mathbb{Z}_{p^3} \oplus \mathbb{Z}^{(\mathbb{N})}$ is infinitely generated CLESS, and neither CESS nor CLS, because the $\mathbb{Z}$-modules $\mathbb{Z}_p \oplus \mathbb{Z}_{p^3}$ and $\mathbb{Z}^{(\mathbb{N})}$ are not CS.

(3) If $\mathcal{P}$ denotes the set of all primes, then the $\mathbb{Z}$-module $\prod_{p\in\mathcal{P}} \mathbb{Z}_p$ is not CLESS, because its torsion submodule is not a direct summand (e.g. see [17]).

5. Excellent Extensions and CLESS-Modules

In this section we shall generalize to CLESS-modules corresponding results concerning the behaviour of CS-modules with respect to excellent extensions of rings (see [13]).

Let R be a subring of a ring S such that they have the same identity. Recall that S is called a *right excellent extension* of R [14] if the following two conditions are satisfied:

(1) S_R and ${}_RS$ are free modules with a basis $\{1 = a_1, a_2, \cdots, a_n\}$ such that $a_iR = Ra_i$ for every $i \in \{1, 2, \ldots, n\}$.

(2) For every submodule N_S of a module M_S, if N_R is a direct summand of M_R, then N_S is a direct summand of M_S.

In what follows, S will be a right excellent extension of R, and S_R and ${}_RS$ have a basis $\{1 = a_1, a_2, \ldots, a_n\}$. We recall some needed basic properties.

Lemma 5.1.

(*i*) ([12, Proposition 1.1]). *Let N_S be a submodule of a module M_S. Then $N_R \trianglelefteq M_R$ if and only if $N_S \trianglelefteq M_S$.*

(*ii*) ([12, Corollary 1.2]). *Let M be a right S-module. Then* $\mathrm{Soc}(M_S) = \mathrm{Soc}(M_R)$.

(*iii*) ([12, Lemma 2.1]). *Let M be a right S-module. Then* $Z(M_S) = Z(M_R)$.

(iv) ([12, Proposition 2.2]). *Let M be a right R-module. Then $Z((M \otimes_R S)_S) = Z(M_R) \otimes_R S$.*

Lemma 5.2. *Let N_S be a submodule of a module M_S. Then N_S is closed in M_S if and only if N_R is closed in M_R.*

Proof. Clear by Lemma 5.1 (iii). ■

Lemma 5.3. *Let N_R be a submodule of a module M_R. Then:*

(i) *N_R is closed in M_R if and only if $(N \otimes_R S)_S$ is closed in $(M \otimes_R S)_S$.*

(ii) *$\mathrm{Soc}(N_R) \trianglelefteq N_R$ if and only if $\mathrm{Soc}((N \otimes_R S)_S) \trianglelefteq (N \otimes_R S)_S$.*

Proof. (i) Assume first that N_R is closed in M_R, that is, $Z(M_R/N_R) = 0$. Consider the element $\sum_{i=1}^{n} m_i \otimes a_i + N \otimes_R S \in Z((M \otimes_R S)_S/(N \otimes_R S)_S)$ for some elements $m_i \in M$. There exists $I_S \trianglelefteq S_S$ such that

$$\left(\sum_{i=1}^{n} m_i \otimes a_i + N \otimes_R S\right) I_S \subseteq N \otimes_R S,$$

and so $I_R \trianglelefteq S_R$ by Lemma 5.1 (i). Then $H_R = I_R \cap R_R \trianglelefteq R_R$, and so we have

$$\left(\sum_{i=1}^{n} m_i \otimes a_i + N \otimes_R S\right) H_R \subseteq N \otimes_R S.$$

This means that $\left(\sum_{i=1}^{n} m_i \otimes a_i\right) H_R \subseteq N \otimes_R S$. Since ${}_RS$ is a free module with basis $\{1 = a_1, a_2, \ldots, a_n\}$, every element in $M \otimes_R S$ is uniquely written as $\sum_{i=1}^{n} m_i \otimes a_i$ for elements $m_i \in M$. Notice that for every $i \in \{1, \ldots, n\}$, there exists an automorphism σ_i of R such that $\sigma_i(r)a_i = a_i r$ for every $r \in R$. Thus, for every $h \in H$ we have:

$$\begin{aligned}\left(\sum_{i=1}^{n}(m_i \otimes a_i)\right) h &= \sum_{i=1}^{n} m_i \otimes (a_i h) = \sum_{i=1}^{n} m_i \otimes (\sigma_i(h)a_i) \\ &= \sum_{i=1}^{n}(m_i\sigma_i(h)) \otimes a_i \in N \otimes_R S.\end{aligned}$$

Since ${}_RS$ is a free module with basis $\{1 = a_1, a_2, \cdots, a_n\}$, every element of $N \otimes_R S$ is uniquely written as $\sum_{i=1}^{n} n_i \otimes a_i$ for elements $n_i \in N$. It follows that $m_i\sigma_i(H) \subseteq N$ for every $i \in \{1, \ldots, n\}$. Since σ_i is an automorphism of R and $H_R \trianglelefteq R_R$, we have $\sigma_i(H) \trianglelefteq R_R$ for every $i \in \{1, \ldots, n\}$. Then $m_i + N \in Z(M_R/N_R)$ for every $i \in \{1, \ldots, n\}$. Since $Z(M_R/N_R) = 0$, we have $m_i \in N$ for every $i \in \{1, \ldots, n\}$. Thus $\sum_{i=1}^{n} m_i \otimes a_i \in N \otimes_R S$, and so $Z((M \otimes_R S)_S/(N \otimes_R S)_S) = 0$.

Conversely, assume that $(N \otimes_R S)_S$ is closed in $(M \otimes_R S)_S$, that is, $Z((M \otimes_R S)_S/(N \otimes_R S)_S) = 0$. Then $Z((M \otimes_R S)_R/(N \otimes_R S)_R) = 0$ by Lemma 5.1 (iii). Let $m + N \in Z(M_R/N_R)$. There exists $I_R \trianglelefteq R_R$ such that $mI_R \subseteq N_R$. Therefore $(m \otimes 1 + N \otimes_R S)I_R \subseteq N \otimes_R S$. Since $Z((M \otimes_R S)_R/(N \otimes_R S)_R) = 0$, we obtain that $m \otimes 1 \in N \otimes_R S$. From this we have $m \in N$, which shows that $Z(M_R/N_R) = 0$.

(ii) There is an R-isomorphism $\psi : N \otimes_R S \to N^n$, and so $\psi(\mathrm{Soc}(N \otimes_R S)_R) = (\mathrm{Soc}(N_R))^n$.

Assume first that $\mathrm{Soc}(N_R) \trianglelefteq N_R$. Then $(\mathrm{Soc}(N_R))^n \trianglelefteq N_R^n$, and furthermore, $\mathrm{Soc}(N \otimes_R S)_R \trianglelefteq (N \otimes_R S)_R$. By Lemma 5.1 (iii), $\mathrm{Soc}((N \otimes_R S)_S) = \mathrm{Soc}((N \otimes_R S)_R)$, which implies that $\mathrm{Soc}((N \otimes_R S)_S) \trianglelefteq (N \otimes_R S)_S$.

Conversely, assume that $\mathrm{Soc}((N \otimes_R S)_S) \trianglelefteq (N \otimes_R S)_S$. Then $\mathrm{Soc}((N \otimes_R S)_R) \trianglelefteq (N \otimes_R S)_R$ by Lemma 5.1 (i), and so $(\mathrm{Soc}(N_R))^n \trianglelefteq N_R^n$. It follows that $\mathrm{Soc}(N_R) \trianglelefteq N_R$. ■

Theorem 5.4.

(*i*) *Let M be a right S-module. If M_R is a CLESS-module, then so is M_S.*

(*ii*) *Let M be a right R-module. If $(M \otimes_R S)_S$ is a CLESS-module, then so is M_R.*

Proof.

(i) Assume that M_R is a CLESS-module, and let N_S be a closed submodule of M_S with $\mathrm{Soc}(N_S) \trianglelefteq N_S$. By Lemmas 5.1 and 5.2, $\mathrm{Soc}(N_R) \trianglelefteq N_R$ and N_R is a closed submodule of M_R. Then N_R is a direct summand of M_R. Since S is a right excellent extension of R, N_S is a direct summand of M_S. Hence M_S is a CLESS-module.

(ii) Assume that $(M \otimes_R S)_S$ is a CLESS-module, and let N_R be a closed submodule of M_R with $\mathrm{Soc}(N_R) \trianglelefteq N_R$. By Lemma 5.3 (i), $(N \otimes_R S)_S$ is a closed submodule of $(M \otimes_R S)_S$. By Lemma 5.3 (ii), we have $\mathrm{Soc}((N \otimes_R S)_S) \trianglelefteq (N \otimes_R S)_S$. Then $(N \otimes_R S)_S$ is a direct summand of $(M \otimes_R S)_S$. It follows that $(N \otimes_R S)_R$ is a direct summand of $(M \otimes_R S)_R$. Note that $(N \otimes 1)_R$ is a direct summand of $(N \otimes_R S)_R$ and $(N \otimes 1)_R \subseteq (M \otimes 1)_R$. Then $(N \otimes 1)_R$ is a direct summand of $(M \otimes 1)_R$. This implies that N_R is a direct summand of M_R. Hence M_R is a CLESS-module. ■

Now Lemma 5.1 and Theorem 5.4 give the following consequence.

Corollary 5.5. *Every right R-module (respectively singular, non-singular right R-module) is CLESS if and only if every right S-module (respectively singular, non-singular right S-module) is CLESS.*

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Septimiu Crivei
Faculty of Mathematics and Computer Science
"Babeş-Bolyai" University
Str. Mihail Kogalniceanu 1, 400084 Cluj-Napoca
Romania
E-mail: crivei@math.ubbcluj.ro

Serap Şahinkaya
Department of Mathematics
Gebze Institute of Technology
Çayirova Campus
41400 Gebze-Kocaeli
Turkey
E-mail: ssahinkaya@gyte.edu.tr