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THE ASCENDING CHAIN CONDITION FOR PRINCIPAL LEFT IDEALS OF SKEW POLYNOMIAL RINGS

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Abstract. In this note we study the ascending chain conditions on principal left (resp. right) ideals of the skew polynomial ring $R[x;\alpha,\delta]$. We give a characterization of skew polynomial rings $R[x;\alpha,\delta]$ that are domains and satisfy the ascending chain condition on principal left (resp. right) ideals. We also prove that if R is an α-rigid ring that satisfies the ascending chain condition on right annihilators and ascending chain condition on principal right (resp. left) ideals, then the skew polynomial ring $R[x;\alpha,\delta]$ and skew power series ring $R[[x;\alpha]]$ also satisfy the ascending chain condition on principal right (resp. left) ideals.

1. Introduction

Throughout this paper R denotes an associative ring with unity, α is a ring endomorphism and δ an α-derivation of R, that is, δ is an additive map such that $\delta(ab) = \delta(a)b + \alpha(a)\delta(b)$, for all $a, b \in R$. We denote by $R[x;\alpha,\delta]$ the Ore extension (the skew polynomial ring) whose elements are the left polynomials $\sum_{i=0}^{n} a_i x^i$ with $a_i \in R$, the addition is defined as usual and the multiplication subject to the relation $xa = \alpha(a)x + \delta(a)$ for any $a \in R$. The skew power series ring, whose elements are the series $\sum_{i=0}^{\infty} a_i x^i$ with $a_i \in R$, is denoted by $R[[x;\alpha]]$. The addition in the ring $R[[x;\alpha]]$ is defined as usual and the multiplication subject to the relation $xa = \alpha(a)x$ for any $a \in R$.

A ring R satisfies the ascending chain condition for principal left (resp. right) ideals (ACCPL (resp. ACCPR)), if there does not exist an infinite strictly ascending chain of principal left (resp. right) ideals of R. We say that R is an ACCPL-ring (resp. ACCPR-ring) if R satisfies ACCPL (resp. ACCPR). If a domain R satisfies ACCPL (resp. ACCPR) we say that R is an ACCPL-domain (resp. ACCPR-domain). Clearly every left (resp. right) noetherian ring satisfies ACCPL (resp. ACCPR). Also by

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Jonah's Theorem [9], every left perfect ring satisfies ACCPL. In the commutative ring theory the ascending chain condition on principal ideals (ACCP) is very important for studies of factorization. Several authors studied the passing of ACCP to the polynomial ring and power series ring. It is well known and easy to see that if R is a commutative domain satisfying ACCP, then for any family X of indeterminates, the polynomial ring $R[X]$ and power series ring $R[[X]]$ also satisfy ACCP. Heinzer and Lantz in [7] and Frohn in [4], gave examples to show that ACCP does not rise to the polynomial ring and power series ring in general. Frohn in [5, Theorem 4.1] showed that, if a ring R satisfies ACCP and $R[X]$ has acc on annihilator ideals, then $R[X]$ also satisfies ACCP. The ascending chain condition on principal right (resp. left) ideals has been studied in the noncommutative ring theory, in a number of papers, for example, [1, 6] and [13]. Recently Mazurek and Ziembowski in [11] studied the ascending chain condition on principal left (resp. right) ideals of skew generalized power series rings.

In this paper we study this property for the skew polynomial ring $R[x;\alpha,\delta]$ and skew power series ring $R[[x;\alpha]]$. First we show that $R[x;\alpha,\delta]$ is a domain satisfying the ascending chain condition on principal left ideals and α is injective if and only if $R[x;\alpha]$ is a domain satisfying the ascending chain condition on principal left ideals if and only if $R[[x;\alpha]]$ is a domain satisfying the ascending chain condition on principal left ideals if and only if R is a domain, R satisfies the ascending chain condition on principal left ideals and α is injective. We also show that if R is an ACCPR-domain and α is injective and preserves nonunit elements of R, then $R[x;\alpha,\delta]$ is an ACCPR-domain. A commutative domain R is said to be archimedean if $\bigcap_{n\geq 1} a^nR = 0$ for each nonunit element a of R. It is well-known that any domain satisfying ACCP is archimedean, but the converse is not true (for more details see [3]). We prove that $R[x;\alpha,\delta]$ is a left archimedean domain and α is injective if and only if $R[x;\alpha]$ is a left archimedean domain if and only if $R[[x;\alpha]]$ is a left archimedean domain if and only if R is a left archimedean domain and α is injective. Also we prove that if R is a right archimedean domain and α is injective and preserves nonunit elements of R, then $R[x;\alpha,\delta]$ is a right archimedean domain.

In section 3, we study the ACCPR (resp. ACCPL) property for the skew polynomial ring $R[x;\alpha,\delta]$ and skew power series ring $R[[x;\alpha]]$ in the case R is not a domain. We show that if R satisfies the ascending chain condition on principal right (resp. left) ideals, R has acc on right annihilators and α is a rigid automorphism (i.e., for each $a\in R$, $a\alpha(a)=0$ implies $a=0$) of R then $R[x;\alpha,\delta]$ and $R[[x;\alpha]]$ satisfy the ascending chain condition on principal right (resp. left) ideals.
A commutative ring R is called présimplifiable (for more details see [2]) if for each $a,b\in R$, $ab=a$ implies $a=0$ or b is a unit. A présimplifiable ring is a ring with zero divisors which is nearly an integral domain. We show that for a présimplifiable ring R and automorphism α of R, $R[[x;\alpha]]$ satisfies ACCPR if and only if R satisfies ACCP if and only if $R[[x;\alpha]]$ satisfies ACCPL.

2. Skew Polynomial Rings that are ACCPL-Domains

In this section we study when the skew polynomial ring $R[x;\alpha,\delta]$ and skew power series ring $R[[x;\alpha]]$ are ACCPL-domains (resp. ACCPR-domains).

We denote the set of unit elements of a ring R by $U(R)$.

Proposition 2.1. ([11, Proposition 2.7]). *For any domain R the following conditions are equivalent:*

(1) R satisfies ACCPL.

(2) For any sequences $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ of nonzero elements of R such that $a_n = b_n a_{n+1}$ for all $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ with $b_n \in U(R)$ for all $n \geq m$.

(3) For any sequences $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ of nonzero elements of R such that $a_n = b_n a_{n+1}$ for all $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ with $b_m \in U(R)$.

(4) $\bigcap_{n\in\mathbb{N}} r_1 r_2 \cdots r_n R = 0$ for any sequence $(r_n)_{n\in\mathbb{N}}$ of nonunits of R.

Corollary 2.2. ([11, Corollary 2.8]). *Let A be a subring of a domain B such that $U(A) = A \cap U(B)$. If B satisfies ACCPL, then A satisfies ACCPL.*

The degree of a polynomial $f \in R[x;\alpha,\delta]$ will be denoted by $deg(f)$ and the leading coefficient of f will be denoted by $l(f)$.

Theorem 2.3. *Let R be a ring, α an endomorphism of the ring R and δ an α-derivation of R. Then the following are equivalent:*

(1) *$R[x;\alpha,\delta]$ is an ACCPL-domain and α is injective.*

(2) *$R[[x;\alpha]]$ is an ACCPL-domain.*

(3) *$R[x;\alpha]$ is an ACCPL-domain.*

(4) *R is an ACCPL-domain and α is injective.*

Proof. $(1) \Rightarrow (4)$ Assume that $S = R[x;\alpha,\delta]$ is an ACCPL-domain. Since $R[x;\alpha,\delta]$ is a domain, R is a domain and since $U(R) = R \cap U(S)$, R is an ACCPL-domain by Corollary 2.2. $(4) \Rightarrow (1)$ Assume that R is an ACCPL-domain and α is injective. It is easy to see that $S = R[x;\alpha,\delta]$ is a domain. Let $(f_n)_{n\in\mathbb{N}}$, $(g_n)_{n\in\mathbb{N}}$ be any sequences of nonzero elements of S with $f_n = g_n f_{n+1}$ for each $n \in \mathbb{N}$. Since S is a domain and α is injective, $deg(f_n) = deg(g_n) + deg(f_{n+1})$ for each $n \in \mathbb{N}$. If for each $n \in \mathbb{N}$, $deg(f_n) = deg(f_{n+1})$, then $g_n \in R$ and so $l(f_n) = g_n l(f_{n+1})$. Since R is an ACCPL-domain, there exists $m \in \mathbb{N}$ such that $g_m \in U(R)$, by Proposition 2.1. Thus S is an ACCPL-domain. Now assume that there exists $m \in \mathbb{N}$ such that $deg(g_m) \neq 0$. So $deg(f_m) > deg(f_{m+1})$. If for each $n > m$, $deg(g_n) = 0$ then by the same argument as above there exists $m' > m$ such that $g_{m'} \in U(R)$ and the result follows. So we can assume that there exists a sequence of positive integers

$n_1 < n_2 < n_3 < \cdots$, such that for each positive integer i, $deg(g_{n_i}) \neq 0$. Thus we have $deg(f_{n_1}) > deg(f_{n_2}) > deg(f_{n_3}) > \cdots$. Then there exists a positive integer t such that for each $n \geq t$, $deg(f_n) = 0$. Thus for each $n \geq t$, $f_n, g_n \in R$ and so there exists $m > t$ such that $g_m \in U(R)$ and the result follows.

$(2) \Leftrightarrow (4)$ [11, Corollary 3.4].

$(3) \Leftrightarrow (4)$ The proof is similar to that of the proof $(1) \Leftrightarrow (4)$.

We will say that an endomorphism α of a ring R preserves nonunit elements of R if $\alpha(R \backslash U(R)) \subseteq R \backslash U(R)$.

Theorem 2.4. *Let R be a ring, α an endomorphism of the ring R and δ an α-derivation of R. If R is an ACCPR-domain and α is injective and preserves nonunit elements of R, then $R[x;\alpha,\delta]$ is an ACCPR-domain.*

Proof. It is easy to see that $S = R[x;\alpha,\delta]$ is a domain. Let $(f_n)_{n\in\mathbb{N}}$, $(g_n)_{n\in\mathbb{N}}$ be any sequences of nonzero element of S with $f_n = f_{n+1}g_n$ for each $n \in \mathbb{N}$. Since S is a domain and α is injective, $deg(f_n) = deg(f_{n+1}) + deg(g_n)$ for each $n \in \mathbb{N}$. If for each $n \in \mathbb{N}$, $deg(f_n) = deg(f_{n+1}) = t$, then $g_n \in R$ and so $l(f_n) = l(f_{n+1})\alpha^t(g_n)$. Since R is an ACCPR-domain, there exists $m \in \mathbb{N}$ such that $\alpha^t(g_m) \in U(R)$, by the right-sided version of Proposition 2.1. Since α preserves nonunit elements of R, $g_m \in U(R)$ and thus $g_m \in U(S)$. Now assume that there exists $n \in \mathbb{N}$ such that $deg(g_n) \neq 0$. By a similar argument as in the proof of Theorem 2.3 we can see that $g_m \in U(S)$ for some $m \in \mathbb{N}$. Hence the right-sided version of Proposition 2.1 implies that S is an ACCPR-domain.

Note that if $R[x;\alpha,\delta]$ is an ACCPR-domain then by Corollary 2.2 R is an ACCPR-domain. But we do not know whether α preserves nonunit elements of R and α is injective in this case.

Theorem 2.5. *Let R be a ring and α an endomorphism of the ring R. Then the following are equivalent:*

(1) *$R[x;\alpha]$ is an ACCPR-domain.*

(2) *$R[[x;\alpha]]$ is an ACCPR-domain.*

(3) *R is an ACCPR-domain and α is injective and preserves nonunit elements of R.*

Proof. $(1) \Rightarrow (3)$ Assume that $S = R[x;\alpha]$ is an ACCPR-domain. Since $U(R) = R \cap U(S)$, the right-sided version of Corollary 2.2 implies that R is an ACCPR-domain. Moreover, if $a \in R$ and $\alpha(a) = 0$, then in the domain S we have $xa = 0$. Hence $a = 0$, which shows that α is injective. Suppose that $\alpha(r) \in U(R)$ for some $r \in R \backslash U(R)$. For each $n \in \mathbb{N}$, let $f_n = (\alpha(r))^{-n}x$. Then for each $n \in \mathbb{N}$, $f_n = f_{n+1}r$ and so by using right-sided version of Proposition 2.1, $r \in U(R)$, a contradiction.

The equivalence $(2) \Leftrightarrow (3)$ was proved in [11, Corollary 3.4(ii)], whereas the implication $(3) \Rightarrow (1)$ is an immediate consequence of Theorem 2.4.

Let $R[x;\alpha,\delta]$ be a skew polynomial ring. If there exists $d \in R$ such that $\delta(r) = dr - \alpha(r)d$ for all $r \in R$, then δ is called an inner α-derivation of R. In this case, we have $R[x;\alpha,\delta] = R[x-d;\alpha]$.

Corollary 2.6. *Let R be a ring, α an endomorphism of the ring R and δ an inner α-derivation of R. Then $R[x;\alpha,\delta]$ is an ACCPR-domain if and only if R is an ACCPR-domain and α is injective and preserves nonunit elements of R.*

Corollary 2.7. *Let $S = R[x_1;\delta_1][x_2;\delta_2]\cdots[x_n;\delta_n]$ be an iterated differential polynomial ring, where each δ_i is a derivation of $R[x_1;\delta_1]\cdots[x_{i-1};\delta_{i-1}]$. Then S is an ACCPL-domain (resp. ACCPR-domain) if and only if R is an ACCPL-domain (resp. ACCPR-domain).*

A domain R is said to be left (resp. right) archimedean if $\bigcap_{n\geq 1} a^nR = 0$ ($\bigcap_{n\geq 1} Ra^n = 0$) for each nonunit element a of R. By Proposition 2.1, any ACCPL-domain (resp. ACCPR-domain) is left (resp. right) archimedean, but the converse is not true in general (for more details see [3]).

Theorem 2.8. *Let R be a ring, α an endomorphism of the ring R and δ an α-derivation of R. If R is a right archimedean domain and α is injective and preserves nonunit elements of R, then $R[x;\alpha,\delta]$ is a right archimedean domain.*

Proof. Put $S = R[x;\alpha,\delta]$. Suppose that R is a right archimedean domain and α is injective and preserves nonunit elements of R. It is easy to see that S is a domain. Let f be a nonunit element of S and $g \in \bigcap_{n\geq 1} Sf^n$. Then for each $n \in \mathbb{N}$ there exists $h_n \in S$ such that $g = h_nf^n$. Let m denote the degree of g. If $deg(f) = 0$, then for each $n \in \mathbb{N}$, $l(g) = l(h_n)\alpha^m(f^n)$. Thus $l(g) \in \bigcap_{n\geq 1} R(\alpha^m(f))^n$. Since $\alpha^m(f)$ is nonunit, $l(g) = 0$ and so $g = 0$. If $deg(f) \neq 0$, then for each $n \in \mathbb{N}$, $deg(g) = deg(h_n) + ndeg(f)$. Thus $g = 0$ and the result follows.

Note that if $R[x;\alpha,\delta]$ is a right archimedean domain then R is a right archimedean domain. But we do not know whether α preserves nonunit elements of R and α is injective in this case.

Let $f = f_0 + f_1x + f_2x^2 + \cdots \in R[[x;\alpha]]\backslash\{0\}$. We denote by $\pi(f)$ the smallest $i \geq 0$ such that $f_i \neq 0$.

Theorem 2.9. *Let R be a ring and α an endomorphism of the ring R. Then the following are equivalent:*

(1) $R[x;\alpha]$ *is a right archimedean domain.*

(2) $R[[x;\alpha]]$ *is a right archimedean domain.*

(3) *R is a right archimedean domain and α is injective and preserves nonunit elements of R.*

Proof. $(1) \Rightarrow (3)$ Assume that $S = R[x;\alpha]$ is a right archimedean domain. It is easy to see that R is domain and α is injective. Let a be a nonunit element of R and $b \in \bigcap_{n\geq 1} Ra^n$. Then $b \in \bigcap_{n\geq 1} Sa^n$ and so $b = 0$. Thus R is a right archimedean domain. Suppose that $\alpha(r) \in U(R)$ for some $r \in R\backslash U(R)$. For each $n \in \mathbb{N}$, let $f_n = (\alpha(r))^{-n}x$, then $f_n \in S$ and $f_n r^n = x$. So $x \in \bigcap_{n\geq 1} Sr^n$, a contradiction.

$(3) \Rightarrow (1)$ is an immediate consequence of Theorem 2.8.

$(2) \Rightarrow (3)$ The proof is similar to the proof of $(1) \Rightarrow (3)$.

$(3) \Rightarrow (2)$ Suppose that R is a right archimedean domain and α is injective and preserves nonunit elements of R. It is easy to see that $T = R[[x;\alpha]]$ is domain. Assume that $f = f_0 + f_1x + f_2x^2 + \cdots$ is a nonunit element of T and $0 \neq g = g_0 + g_1x + g_2x^2 + \cdots \in \bigcap_{n\geq 1} Tf^n$. Then, for each $n \in \mathbb{N}$, there exists $h_n = h_{n0} + h_{n1}x + h_{n2}x^2 + \cdots \in T$ such that $g = h_nf^n$. Put $\pi(g) = m$. If $\pi(f) = 0$, then for each $n \in \mathbb{N}$, $g_m = h_{nm}\alpha^m(f_0^n)$. Thus $g_m \in \bigcap_{n\geq 1} R(\alpha^m(f_0))^n$. Since f is nonunit, f_0 is nonunit and so $\alpha^m(f_0)$ is nonunit. Thus $g_m = 0$ and so $g = 0$, a contradiction. If $\pi(f) \neq 0$, then for each $n \in \mathbb{N}$, $m = \pi(h_n) + n\pi(f)$. Thus $g = 0$, a contradiction.

Corollary 2.10. *Let R be a ring, α an endomorphism of the ring R and δ an inner α-derivation of R. Then $R[x;\alpha,\delta]$ is a right archimedean domain if and only if R is a right archimedean domain and α is injective and preserves nonunit elements of R.*

Theorem 2.11. *Let R be a ring, α an endomorphism of the ring R and δ an α-derivation of R. Then the following are equivalent:*

(1) *$R[x;\alpha,\delta]$ is a left archimedean domain and α is injective.*
(2) *$R[[x;\alpha]]$ is a left archimedean domain.*
(3) *$R[x;\alpha]$ is a left archimedean domain.*
(4) *R is a left archimedean domain and α is injective.*

Proof. The proof is similar to the proof of Theorems 2.8 and 2.9.

Corollary 2.12. *Let $S = R[x_1;\delta_1][x_2;\delta_2]\cdots[x_n;\delta_n]$ be an iterated differential polynomial ring, where each δ_i is a derivation of $R[x_1;\delta_1]\cdots[x_{i-1};\delta_{i-1}]$. Then S is a right (resp. left) archimedean domain if and only if R is a right (resp. left) archimedean domain.*

3. ACCPL Skew Polynomial Rings which are not Domains

Frohn in [5, Theorem 4.1] showed that, if a commutative ring R satisfies ACCP and $R[x]$ has acc on annihilator ideals, then $R[x]$ also satisfies ACCP. In this section we show that under suitable conditions on the ACCPL (resp. ACCPR) ring R, the skew polynomial ring $R[x;\alpha,\delta]$, the skew power series ring $R[[x;\alpha]]$ and the skew polynomial ring $R[x;\alpha]$ are ACCPL-rings (resp. ACCPR-rings).

An endomorphism α of a ring R is called a rigid endomorphism if $r\alpha(r) = 0$ implies $r = 0$ for each $r \in R$. A ring R is called α-rigid if there exists a rigid endomorphism α of ring R (for more details see [10]). Note that each α-rigid ring R is reduced (i.e. has no nonzero nilpotent element). α-rigid rings are characterized in the following.

Proposition 3.1. ([8, Proposition 5 and Corollary 18]). *Let R be a ring, α an endomorphism of R and δ an α-derivation of R. Then the following are equivalent:*

(1) *R is α-rigid.*

(2) *$R[x;\alpha,\delta]$ is reduced and α is a monomorphism of R.*

(3) *$R[[x;\alpha]]$ is reduced and α is a monomorphism of R.*

We need the following lemma in the sequel.

Lemma 3.2. ([8, Lemma 4]). *Let R be an α-rigid ring, δ an α-derivation of R and $a, b \in R$. Then we have the following:*

(1) *If $ab = 0$ then $a\alpha^n(b) = \alpha^n(a)b = 0$ for any positive integer n.*

(2) *If $ab = 0$ then $a\delta^m(b) = \delta^m(a)b = 0$ for any positive integer m.*

(3) *If $a\alpha^k(b) = 0$ for some positive integer k, then $ab = 0$.*

Lemma 3.3. *Let I be an ideal of a ring R. If R is an ACCPL (resp. ACCPR) ring, then R/I is an ACCPL (resp. ACCPR) ring.*

Let A be a subset of ring R. The left (resp. right) annihilator of A will be denoted by $l_R(A)$ (resp. $r_R(A)$). Recall that an ideal P of R is completely prime if $ab \in P$ implies $a \in P$ or $b \in P$ for $a, b \in R$.

Let R be a ring, α an endomorphism of R and δ an α-derivation of R. Recall that an ideal I of R is called an α-ideal if $\alpha(I) \subseteq I$, I is called α-invariant if $\alpha^{-1}(I) = I$ and I is called δ-ideal if $\delta(I) \subseteq I$. If I is an α-ideal and δ-ideal we say I is an (α,δ)-ideal. Note that if I is an (α,δ)-ideal, then $\bar{\alpha} : R/I \longrightarrow R/I$ defined by $\bar{\alpha}(a + I) = \alpha(a) + I$ for $a \in R$ is an endomorphism of the factor ring R/I and $\bar{\delta} : R/I \longrightarrow R/I$ defined by $\bar{\delta}(a + I) = \delta(a) + I$ is an $\bar{\alpha}$-derivation of R/I.

Theorem 3.4. *Let R be an ACCPR ring, α a rigid automorphism of R and δ an α-derivation of R. If R satisfies the ACC on right annihilators, then $R[x;\alpha,\delta]$ is an ACCPR ring.*

Proof. We apply the method of Frohn [5, Theorem 4.1] to prove this theorem. For each $f \in S = R[x;\alpha,\delta]$ let I_f be the set of the leading coefficients of elements of the ideal SfS, together with 0. It is easy to see that I_f is an ideal of R. Assume at the contrary that there exists a nonstabilizing chain of principal right ideals of S. So the set $M = \{l_R(\bigcup_{i\geq 1} I_{g_i}) | g_1S \subseteq g_2S \subseteq \cdots$ *is a nonstabilizing chain of principal right ideal in S*$\}$ is nonempty. Since R is α-rigid, R is reduced and so it is easy to see that since R satisfies the ACC on right annihilators, R satisfies the ACC on left annihilators. Thus M has a maximal element. Let $P = l_R(\bigcup_{i\geq 1} I_{f_i})$ be a maximal element of M, where $f_1S \subseteq f_2S \subseteq \cdots$ is a nonstabilizing chain in S. We show that P is a completely prime ideal of R. Assume that $a, b \in R\backslash P$ and $ab \in P$. Since R is α-rigid, by using Lemma 3.2 we can see that $a \in l_R(\bigcup_{i\geq 1} I_{bf_i})$. Also we have $P \subseteq l_R(\bigcup_{i\geq 1} I_{bf_i})$. So the chain $bf_1S \subseteq bf_2S \subseteq \cdots$ stabilizes. Then there exists a positive integer t such that for each $n \geq t$, $bf_{n+1} = bf_nh_n$ for some $h_n \in S$. For each positive integer n, there exists $g_n \in S$ such that $f_n = f_{n+1}g_n$. Thus for each $n \geq t$, $bf_{n+1}(1 - g_nh_n) = 0$. Let $q_i = f_i(1 - g_{i-1}h_{i-1})$, for each $i > t$. Since R is reduced, $b \in l_R(\bigcup_i I_{q_i})$ and $P \subseteq l_R(\bigcup_i I_{q_i})$. Then the chain $q_1S \subseteq q_2S \subseteq \cdots$ stabilizes. Thus there exists a positive integer t' such that for each $m \geq t'$, $q_{m+1} = q_ml_m$ for some $l_m \in S$. Then $f_{m+1}(1 - g_mh_m) = f_m(1 - g_{m-1}h_{m-1})l_m$ and so $f_{m+1} = f_mh_m + f_m(1 - g_{m-1}h_{m-1})l_m$. Thus we have the contradiction $f_{m+1} \in f_mS$. So P is a completely prime ideal of R. Since R is α-rigid and $P = l_R(\bigcup_{i\geq 1} I_{f_i})$, by using Lemma 3.2 it is easy to see that P is an α-invariant, δ-ideal. Now let $T = (R/P)[x; \bar{\alpha}, \bar{\delta}]$. Since R is ACCPR and P is a completely prime ideal of R, by Lemma 3.3, R/P is an ACCPR-domain. Thus T is an ACCPR-domain by Theorem 2.4. For each positive integer i, $\overline{f_i} = \overline{f_{i+1}}\overline{g_i}$, where $\overline{f} = (a_0 + P) + (a_1 + P)x + \cdots + (a_n + P)x^n \in T$, for each $f = a_0 + a_1x + \cdots + a_nx^n \in S$. If $\overline{f_i} = 0$ for some i, then the leading coefficient a of f, $a \in P = l_R(\bigcup_{i\geq 1} I_{f_i})$. Thus $a^2 = 0$ and since R is reduced, $a = 0$, which is a contradiction. So for each i, $\overline{f_i} \neq 0$ and so $\overline{g_i} \neq 0$. By Proposition 2.1, there exists a positive integer s such that for each $m \geq s$, $\overline{g_m}$ is invertible in T. Then there is a $\overline{h} \in T$ such that $\overline{g_m}\overline{h} = \overline{h}\overline{g_m} = \overline{1}$. $\overline{g_m}\overline{h} - \overline{1} = \overline{0}$ and so for each coefficient b of the polynomial $g_mh - 1$, $b \in P$. We claim that $f_{m+1}(g_mh - 1) = 0$. Assume that $f_{m+1} = a_0 + a_1x + \cdots + a_tx^t$. For any coefficient b of $g_mh - 1$, $ba_t = 0$ and since R is reduced, $a_tb = 0$. By Lemma 3.2 $a_tx^tb = 0$ and so $f_{m+1}b = (a_0 + a_1x + \cdots + a_{t-1}x^{t-1})b$. But $f_{m+1}b \in Sf_{m+1}S$ and so $a_{t-1}\alpha^{t-1}(b) \in I_{f_{m+1}}$. Thus $ba_{t-1}\alpha^{t-1}(b) = 0$ and since R is reduced, $a_{t-1}\alpha^{t-1}(b)b = 0$. So by Lemma 3.2, $a_{t-1}b^2 = 0$ and since R is reduced, $a_{t-1}b = 0$. Thus $a_{t-1}x^{t-1}b = 0$. Continuing in this way we have $a_ix^ib = 0$ for each $0 \leq i \leq t$ and so $f_{m+1}b = 0$. Thus $f_{m+1}(g_mh - 1) = 0$ and so $f_{m+1} = f_{m+1}g_mh = f_mh$. Then the chain $f_1S \subseteq f_2S \subseteq \cdots$ stabilizes, which is a contradiction.

In the following example we show that the α-rigid condition and the ascending chain condition on right annihilators are not superfluous in Theorem 3.4.

Example 3.5.

(1) ([7, Example]) Let k be a field and $A_1, A_2, \cdots$ be indeterminates over k, and set

$S = k[A_1, A_2, \cdots]/(\{A_n(A_n - A_{n-1}) : n \geq 2\})k[A_1, A_2, \cdots]$. Denote by a_n the image of A_n in S and by R the localization of S at the ideal $(a_1, a_2, \cdots)S$. Note that S is a limit of the rings S_n where $S_1 = k[a_1]$ and $S_n = S_{n-1}[a_n] = S_{n-1}[A_n]/A_n(a_{n-1} - A_n)S_{n-1}[A_n]$ for $n \geq 2$. Heinzer and Lantz in [7] proved that R satisfies ACCP but the ring $R[x]$, does not satisfy ACCP. Note that in S we have $a_3^2(a_1 - a_2)^2 = a_3a_2(a_1 - a_2)^2 = 0$, but $a_3(a_1 - a_2) \neq 0$. Thus S is not reduced and since R contains (an isomorphic copy of) S (see [7]), R is not reduced. So the α-rigid condition in Theorem 3.4 is not superfluous.

(2) ([5, Remark after Lemma 4.3]) Let K be a field and $A_1, A_2, \cdots$ indeterminates over K. Set $A := \{A_n : n \geq 1\}$, $I := (\bigcup_{n\in\mathbb{N}}\{A_n(A_{k-1} - A_k) : 1 < k \leq n\})K[A]$ and $S := K[A]/I$. Denote by a_n the image of A_n in S and by R the localization of S at the ideal $(a_1, a_2, \cdots)S$. Frohn in [5] proved that R is a reduced ACCP ring while $R[x]$ is not. So the condition "ACC on right annihilators" in Theorem 3.4 is not superfluous.

Corollary 3.6. *Let R be an ACCPL ring, α a rigid automorphism of R and δ an α-derivation of R. If R satisfies the ACC on left annihilators, then $R[x; \alpha, \delta]$ is an ACCPL ring.*

Proof. It is known that α^{-1} is an automorphism of the opposite ring R^{op} and $-\delta\alpha^{-1}$ is an α^{-1}-derivation of R^{op}, and that $R[x; \alpha, \delta]^{op} \cong R^{op}[x; \alpha^{-1}, -\delta\alpha^{-1}]$. Then the result follows from Theorem 3.4.

Corollary 3.7. *Let R be an ACCPR (resp. ACCPL) ring, $S = R[x_1; \alpha_1, \delta_1][x_2; \alpha_2, \delta_2] \cdots [x_n; \alpha_n, \delta_n]$ be an iterated skew polynomial ring, where each α_i is a rigid automorphism of the ring $R[x_1; \alpha_1, \delta_1] \cdots [x_{i-1}; \alpha_{i-1}, \delta_{i-1}]$ and δ_i an α_i-derivation of $R[x_1; \alpha_1, \delta_1] \cdots [x_{i-1}; \alpha_{i-1}, \delta_{i-1}]$. If R satisfies the ACC on right (resp. left) annihilators, then S is an ACCPR (resp. ACCPL) ring.*

Proof. Let R be an α-rigid ring. If R satisfies the ACC on right annihilators then by [12, Theorem 2.6] and [12, Corollary 3.7], $R[x; \alpha, \delta]$ satisfies the ACC on right annihilators. By using this fact the result follows from Theorem 3.4.

Corollary 3.8. *Let R be a reduced ACCPR (resp. ACCPL) ring, $S = R[x_1; \delta_1][x_2; \delta_2] \cdots [x_n; \delta_n]$ be an iterated differential polynomial ring, where each δ_i is a derivation of $R[x_1; \delta_1] \cdots [x_{i-1}; \delta_{i-1}]$. If R satisfies the ACC on right (resp. left) annihilators, then S is an ACCPR (resp. ACCPL) ring.*

Theorem 3.9. *Let R be an ACCPR (resp. ACCPL) ring and α a rigid automorphism of R. If R satisfies the ACC on right (resp. left) annihilators, then $R[[x; \alpha]]$ is an ACCPR (resp. ACCPL) ring.*

Proof. For each $f = a_0 + a_1x + \cdots \in S = R[[x; \alpha]]$ let I_f be the set of $a_{\pi(g)}$ for each $g \in SfS$, together with 0. Then the proof is similar to the proof of Theorem 3.4.

A commutative ring R is called présimplifiable if for each $a, b \in R$, $ab = a$ implies $a = 0$ or b is a unit. A présimplifiable ring is a ring with zero divisors which is nearly an integral domain.

Theorem 3.10. *Let R be a commutative présimplifiable ring and α an automorphism of R. Then the following are equivalent:*

(1) *R is an ACCP ring.*

(2) *$R[[x; \alpha]]$ is an ACCPR-ring.*

(3) *$R[[x; \alpha]]$ is an ACCPL-ring.*

Proof. $(1) \Rightarrow (2)$ Set $T = R[[x; \alpha]]$. Assume that R is an ACCP-ring. Let $f_1T \subseteq f_2T \subseteq f_3T \subseteq \cdots$ be an increasing sequence of principal right ideals of T. We can simplify by a convenient power of x, and suppose that $f_i(0) \neq 0$ for each i. We obtain the sequence of non zero principal ideals of R, $(f_1(0)) \subseteq (f_2(0)) \subseteq (f_3(0)) \subseteq \cdots$. Since R is an ACCP-ring, there exists a positive integer k such that for each $n \geq k$, $(f_k(0)) = (f_n(0))$. But for each $n \geq k$, there exists $g_n \in T$ such that $f_k = f_n g_n$, so $f_k(0) = f_n(0)g_n(0)$. Also there exists an element $r \in R$ such that $f_n(0) = f_k(0)r$. Thus $f_k(0) = f_n(0)g_n(0) = f_k(0)rg_n(0)$ and since R is présimplifiable and $f_k(0) \neq 0$, $rg_n(0)$ is a unit element of R. Thus $g_n(0)$ is a unit element of R and so g_n is a unit element of T. Thus $f_n = f_k g_n^{-1}$ for $n \geq k$ and so the chain $f_1T \subseteq f_2T \subseteq \cdots$ stabilizes.

$(2) \Rightarrow (1)$ Assume that $T = R[[x; \alpha]]$ is an ACCPR-ring and let $(r_1) \subseteq (r_2) \subseteq \cdots$ be a chain of principal ideals of R. We obtain the chain $r_1T \subseteq r_2T \subseteq \cdots$ in T. But T satisfies ACCPR, so there exists a positive integer k such that for each $n \geq k$, we have $r_nT = r_kT$, which implies that $(r_n) = (r_k)$.

$(1) \Leftrightarrow (3)$ It is known that α^{-1} is an automorphism of the opposite ring R^{op} and $R[[x; \alpha]]^{op} \cong R^{op}[[x; \alpha^{-1}]]$. Then the result follows from $(1) \Leftrightarrow (2)$.

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