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ENTIRE FUNCTIONS AND THEIR HIGHER ORDER DIFFERENCES

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Abstract. In this paper, we prove that for a transcendental entire function $f(z)$ of finite order such that $\lambda(f - a(z)) < \sigma(f)$, where $a(z)$ is an entire function and satisfies $\sigma(a(z)) < 1$, n is a positive integer, if $\Delta_\eta^n f(z)$ and $f(z)$ share entire function $b(z)$ ($b(z) \not\equiv a(z)$) satisfying $\sigma(b(z)) < 1$ CM, where $\eta\,(\in \mathbb{C})$ satisfies $\Delta_\eta^n f(z) \not\equiv 0$, then

$$f(z) = a(z) + ce^{c_1 z},$$

where c, c_1 are two nonzero constants.

1. Introduction and Results

In this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna's value distribution theory of meromorpic functions (see [17, 18, 27]). In addition, we use the notation $\lambda(f)$ to denote the exponent of convergence of the sequence of zeros of a meromorphic function f, and $\sigma(f)$ to denote the order growth of f. For a nonzero constant η, the forward differences $\Delta_\eta^n f(z)$ are defined (see [2, 25]) by

$$\Delta_\eta f(z) = \Delta_\eta^1 f(z) = f(z+\eta) - f(z) \quad \text{and}$$
$$\Delta_\eta^{n+1} f(z) = \Delta_\eta^n f(z+\eta) - \Delta_\eta^n f(z),\, n = 1, 2, \cdots .$$

For a meromorphic function $f(z)$, we use $S(f)$ to denote the family of all meromorphic functions $\alpha(z)$ that satisfy $T(r, \alpha) = S(r, f)$, where $S(r, f) = o(T(r, f))$, as $r \to \infty$ outside of a possible exceptional set of finite logarithmic measure. Functions in the set $S(f)$ are called small functions with respect to $f(z)$.

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Let f and g be two nonconstant meromorphic functions, and let $a \in \mathbb{C}$. We say that f and g share the value a CM (IM) provided that $f - a$ and $g - a$ have the same zeros counting multiplicities (ignoring multiplicities), that f and g share the value ∞ CM (IM) provided that f and g have the same poles counting multiplicities (ignoring multiplicities). Using the same method, we can define f and g share function $a(z)$ CM (IM), where $a(z) \in S(f) \bigcap S(g)$. Nevanlinna's four values theorem [24] says that if two nonconstant meromorphic functions f and g share four values CM, then $f \equiv g$ or f is a Möbius transformation of g. The condition "f and g share four values CM" has been weakened to "f and g share two values CM and two values IM" by Gundersen [10, 11]. But whether the condition can be weakened to "f and g share three values IM and another value CM" is still an open question.

In the special case, we recall a well-known conjecture by Brück [1]:

Conjecture. Let f be a nonconstant entire function such that hyper order $\sigma_2(f) < \infty$ and $\sigma_2(f)$ is not a positive integer. If f and f' share the finite value a CM, then

$$\frac{f' - a}{f - a} = c,$$

where c is a nonzero constant.

The notation $\sigma_2(f)$ denotes hyper-order (see [26]) of $f(z)$ which is defined by

$$\sigma_2(f) = \overline{\lim_{r\to\infty}} \frac{\log\log T(r, f)}{\log r}.$$

The conjecture has been verified in the special cases when $a = 0$ [1], or when f is of finite order [12], or when $\sigma_2(f) < \frac{1}{2}$ [6].

Recently, many authors [13, 15, 16, 21, 22, 23] started to consider sharing values of meromorphic functions with their shifts. Heittokangas et al. proved the following theorems.

Theorem A. (see [15]). *Let f be a meromorphic function with $\sigma(f) < 2$, and let $c \in \mathbb{C}$. If $f(z)$ and $f(z + c)$ share the values $a\,(\in \mathbb{C})$ and ∞ CM, then*

$$\frac{f(z + c) - a}{f(z) - a} = \tau$$

for some constant τ.

In [15], Heittokangas et al. give the example $f(z) = e^{z^2} + 1$ which shows that $\sigma(f) < 2$ cannot be relaxed to $\sigma(f) \le 2$.

Theorem B. (see [16]). *Let f be a meromorphic function of finite order, let $c \in \mathbb{C}$. If $f(z)$ and $f(z + c)$ share three distinct periodic functions $a_1, a_2, a_3 \in \hat{S}(f)$ with period c CM (where $\hat{S}(f) = S(f) \cup \{\infty\}$), then $f(z) = f(z + c)$ for all $z \in \mathbb{C}$.*

Recently, many results of complex difference equations are rapidly obtained (see [4, 5, 8, 14, 19, 20]). In present paper, we will utilize some results on complex difference equations.

The main purpose of this paper is to utilize complex difference equation to study problems on sharing values of meromorphic functions and their differences. It is well known that $\Delta_\eta f(z) = f(z+\eta) - f(z)$ (where $\eta\,(\in \mathbb{C})$ is a constant satisfying $f(z+\eta) - f(z) \not\equiv 0$) is regarded as the difference counterpart of f'. So, Chen [7] considered the problem that $\Delta_\eta f(z)$ and $f(z)$ share one value a CM, and proved the following theorem.

Theorem C. (see [7]). *Let f be a finite order transcendental entire function which has a finite Borel exceptional value a, and let $\eta\,(\in \mathbb{C})$ be a constant such that $f(z+\eta) \not\equiv f(z)$. If $\Delta_\eta f(z) = f(z+\eta) - f(z)$ and $f(z)$ share the value $b\,(\neq a)$ CM, then*

$$\frac{f(z+\eta) - f(z) - b}{f(z) - b} = A,$$

where $A = \frac{b}{b-a}$ is a nonzero constant.

Question 1: What can be said if we consider the forward difference $\Delta_\eta^n f(z)$ and $f(z)$ share one value or one small function?

In this paper, we answer the Question 1 and prove the following theorem.

Theorem 1.1. *Let $f(z)$ be a finite order transcendental entire function such that $\lambda(f - a(z)) < \sigma(f)$, where $a(z)$ is an entire function and satisfies $\sigma(a) < 1$, let n be a positive integer. If $\Delta_\eta^n f(z)$ and $f(z)$ share entire function $b(z)\,(\,b(z) \not\equiv a(z)$ and $\sigma(b) < 1)$ CM, where $\eta\,(\in \mathbb{C})$ satisfies $\Delta_\eta^n f(z) \not\equiv 0$, then*

$$f(z) = a(z) + ce^{c_1 z},$$

where c, c_1 are two nonzero constants.

In the special case, if we take $b(z) \equiv b$ and $a(z) \equiv a$ in Theorem 1.1, we can get the following Corollary.

Corollary 1.1. *Let $f(z)$ be a finite order transcendental entire function which has a finite Borel exceptional value a, let n be a positive integer. If $\Delta_\eta^n f(z)$ and $f(z)$ share value $b\,(b \neq a)$ CM, where $\eta\,(\in \mathbb{C})$ satisfies $\Delta_\eta^n f(z) \not\equiv 0$, then*

$$f(z) = a + ce^{c_1 z},$$

where c, c_1 are two nonzero constants.

Remark 1.1. From Corollary 1.1, we can see that under the hypothesis of Theorem C , $f(z)$ has the expression $f(z) = a + ce^{c_1 z}$. And Corollary 1.1 shows that if $a \neq 0$, then, for any constant η satisfies $e^{c_1\eta} \neq 1$, $b = 0$ is not shared CM by $\Delta_\eta^n f(z)$ and $f(z)$; if $a = 0$, then, for any constant η satisfies $(e^{c_1\eta} - 1)^n = 1$, we have $\Delta_\eta^n f(z) = (e^{c_1\eta} - 1)^n ce^{c_1 z} = ce^{c_1 z} = f(z)$. Thus, any constant $b\,(\neq a)$ is shared CM by $\Delta_\eta^n f(z)$ and $f(z)$. See the following Example 1.1-1.2.

Example 1.1. Suppose that $f(z) = e^{2z} + 5$. Then f has a Borel exceptional value 5. For any $\eta \neq 2k\pi i$, $k \in \mathbb{Z}$, the value $0\,(\neq 5)$ is not shared CM by $\Delta_\eta^n f(z)$ and $f(z)$. Observe that

$$\Delta_\eta^n f(z) = \sum_{j=0}^{n} (-1)^j C_n^j f(z + (n-j)\eta),$$

where C_n^j are the binomial coefficients. Thus, for any $\eta \neq k\pi i$, $k \in \mathbb{Z}$, we have

$$\begin{aligned}\Delta_\eta^n f(z) &= (e^{2(z+n\eta)} + 5) - C_n^1(e^{2(z+(n-1)\eta)} + 5) + \cdots + (-1)^n(e^{2z} + 5)\\ &= \left(e^{2n\eta} - C_n^1 e^{2(n-1)\eta} + \cdots + (-1)^n\right) \cdot e^{2z} + 5 \cdot \sum_{j=0}^{n} (-1)^j C_n^j\\ &= (e^{2\eta} - 1)^n \cdot e^{2z}.\end{aligned}$$

Thus, we can see that $f(z) - 0 = e^{2z} + 5$ has infinitely many zeros, but $\Delta_\eta^n f(z) - 0 = (e^{2\eta} - 1)^n e^{2z}$ has no zeros. Hence, the value 0 is not shared CM by $\Delta_\eta^n f(z)$ and $f(z)$.

Example 1.2. Suppose that $f(z) = e^z$. Then f has a Borel exceptional value $a = 0$. If we take $\eta = \log 2$, then, using the same method as Example 1.1, we can get that $\Delta_\eta^n f(z) = (e^\eta - 1)^n e^z = e^z$, that is $\Delta_\eta^n f(z) \equiv f(z)$. Hence, $\Delta_\eta^n f(z)$ and $f(z)$ share every nonzero constant b CM or every nonzero function $b(z)$ ($\sigma(b(z)) < 1$) CM.

2. Lemmas for the Proof of Theorem

Lemma 2.1. (see [8]). *Let f be a meromorphic function with a finite order σ, η be a nonzero constant. Let $\varepsilon > 0$ be given, then there exists a subset $E \subset (1, \infty)$ with finite logarithmic measure such that for all z satisfying $|z| = r \notin E \cup [0, 1]$, we have*

$$\exp\{-r^{\sigma-1+\varepsilon}\} \leq \left|\frac{f(z+\eta)}{f(z)}\right| \leq \exp\{r^{\sigma-1+\varepsilon}\}.$$

Lemma 2.2. (see [9, 26]). *Suppose that $n \geq 2$ and let $f_1(z), \cdots, f_n(z)$ be meromorphic functions and $g_1(z), \cdots, g_n(z)$ be entire functions such that*

(*i*) $\Sigma_{j=1}^n f_j(z) \exp\{g_j(z)\} = 0$;

(*ii*) *when* $1 \leq j < k \leq n$, $g_j(z) - g_k(z)$ *is not constant;*

(*iii*) *when* $1 \leq j \leq n$, $1 \leq h < k \leq n$,

$$T(r, f_j) = o\{T(r, \exp\{g_h - g_k\})\}\ (r \to \infty, r \notin E),$$

where $E \subset (1, \infty)$ has finite linear measure or logarithmic measure.

Then $\quad f_j(z) \equiv 0, \quad j = 1, \cdots, n.$

ε**-set**. Following Hayman [17], we define an ε-set to be a countable union of open discs not containing the origin and subtending angles at the origin whose sum is finite. If E is an ε-set then the set of $r \geq 1$ for which the circle $S(0, r)$ meets E has finite logarithmic measure, and for almost all real θ the intersection of E with the ray $\arg z = \theta$ is bounded.

Lemma 2.3. (see [2]). *Let f be a function transcendental and meromorphic in the plane of order < 1. Let $h > 0$. Then there exists an ε-set E such that*

$$f(z + c) - f(z) = cf'(z)(1 + o(1)) \quad \textit{as } z \to \infty \textit{ in } \mathbb{C}\backslash E,$$

uniformly in c for $|c| \leq h$.

In what follows, we shall establish a Lemma which will play an important role in our proof of Theorem. To this end, we will introduce some notations. The difference polynomial $U(z, f)$ is defined by

$$U(z, f) = \sum_{\lambda \in J} \alpha_\lambda(z) U_\lambda(z, f) = \sum_{\lambda \in J} \alpha_\lambda(z) \left(\prod_{j=1}^{\tau_\lambda} f(z + \delta_{\lambda,j})^{\mu_{\lambda,j}} \right),$$

where J is a finite index set, $\delta_{\lambda,j}$ are distinct complex constants, $\mu_{\lambda,j}$ are nonnegative integers, and $\alpha_\lambda(z)\ (\not\equiv 0)$ are small meromorphic functions of $f(z)$. The degree of $U_\lambda(z, f)$ and $U(z, f)$ in $f(z)$ and the shifts of $f(z)$ are defined by

$$\deg_f U_\lambda(z, f) = \sum_{j=1}^{\tau_\lambda} \mu_{\lambda,j} \quad \text{and} \quad \deg_f U(z, f) = \max_{\lambda \in J} \{\deg_f U_\lambda(z, f)\}$$

respectively. In what follows, we assume that the coefficients of difference polynomials are, unless otherwise stated, small functions in the usual Nevanlinna theory sense, i.e. their characteristic is of type $S(r, f)$.

Lemma 2.4. (see [20]). *Let f be a transcendental meromorphic solution of finite order ρ of a difference equation of the form*

$$U(z, f)P(z, f) = Q(z, f),$$

where $U(z, f)$, $P(z, f)$, $Q(z, f)$ are difference polynomials such that the total degree $\deg U(z, f) = n$ in $f(z)$ and its shifts, and $\deg Q(z, f) \leq n$. Moreover, we assume that $U(z, f)$ contains just one term of maximal total degree in $f(z)$ and its shifts. Then, for each $\varepsilon > 0$,

$$m\left(r, P(z, f)\right) = O(r^{\rho - 1 + \varepsilon}) + S(r, f),$$

possibly outside of an exceptional set of finite logarithmic measure.

Remark 2.1. From the proof of the Lemma 2.4 in [20], we can see that if the coefficients of $U(z, f)$, $P(z, f)$, $Q(z, f)$, namely $\alpha_\lambda(z)$, satisfy $m(r, \alpha_\lambda) = S(r, f)$, then the same conclusion still holds.

Lemma 2.5. (see [3]). *Let $P_n(z), \cdots, P_0(z)$ be polynomials such that $P_nP_0 \not\equiv 0$ and satisfy*

$$P_n(z) + \cdots + P_0(z) \not\equiv 0. \tag{2.1}$$

Then every finite order transcendental meromorphic solution $f(z)(\not\equiv 0)$ of the equation

$$P_n(z)f(z+n) + P_{n-1}(z)f(z+n-1) + \cdots + P_0(z)f(z) = 0 \tag{2.2}$$

satisfy $\sigma(f) \geq 1$, and $f(z)$ assumes every nonzero value $a \in \mathbb{C}$ infinitely often and $\lambda(f-a) = \sigma(f)$.

Remark 2.2. If the equation (2.2) satisfies the condition (2.1) and $P_j(z)\,(j = 0, 1, \cdots, n)$ are constants, we can see that the equation (2.2) does not possess any nonzero polynomial solution. In fact, suppose that $P(z) = a_kz^k + a_{k-1}z^{k-1} + \cdots + a_0\,(k \geq 0,\ a_k \neq 0)$ is a solution of equation (2.2). Then we have

$$a_k\,(P_n(z) + \cdots + P_0(z)) \cdot z^k + (P_n(z) + \cdots + P_0(z)) \cdot O(z^{k-1}) \equiv 0. \tag{2.3}$$

From (2.1) and $a_k \neq 0$, we can see that (2.3) is a contradiction.

Lemma 2.6. (see [3]). *Let $F(z)$, $P_n(z), \cdots, P_0(z)$ be polynomials such that $FP_nP_0 \not\equiv 0$. Then every finite order transcendental meromorphic solution $f(z)(\not\equiv 0)$ of the equation*

$$P_n(z)f(z+n) + P_{n-1}(z)f(z+n-1) + \cdots + P_0(z)f(z) = F \tag{2.4}$$

satisfy $\lambda(f) = \sigma(f) \geq 1$.

Remark 2.3. From the proof of the Lemma 2.5 in [3], we can see that if we replace equation (2.2) by

$$P_n(z)f(z+n\eta) + P_{n-1}(z)f(z+(n-1)\eta) + \cdots + P_0(z)f(z) = 0$$

or equation (2.4) by

$$P_n(z)f(z+n\eta) + P_{n-1}(z)f(z+(n-1)\eta) + \cdots + P_0(z)f(z) = F,$$

then the corresponding conclusion still holds.

Lemma 2.7. *Suppose that n is a positive integer, $f(z)$ is a finite order transcendental entire function such that $\lambda(f-a(z)) < \sigma(f)$, where $a(z)$ is an entire function and satisfies $\lambda(a) < 1$. If $\Delta_\eta^n f(z) \not\equiv 0$ for some constant $\eta\,(\in \mathbb{C})$, and*

$$\frac{\Delta_\eta^n f(z) - b(z)}{f(z) - b(z)} = A \tag{2.5}$$

holds, where A is a nonzero constant and $b(z)$ $(\not\equiv a(z))$ is an entire function satisfying $\sigma(b) < 1$, then

$$A = \frac{\Delta_\eta^n a(z) - b(z)}{a(z) - b(z)} = (e^{c_1\eta} - 1)^n \quad \textit{and} \quad f(z) = a(z) + ce^{c_1 z},$$

where c, c_1 are two nonzero constants.

Proof. Since $f(z)$ is a transcendental entire function of finite order and satisfies $\lambda(f-a(z)) < \sigma(f)$, we can write $f(z)$ in the form

$$f(z) = a(z) + H(z)e^{h(z)}, \tag{2.6}$$

where $H(\not\equiv 0)$ is an entire function, h is a polynomial with $\deg h = k\,(k \geq 1)$, H and h satisfy

$$\lambda(H) = \sigma(H) = \lambda(f - a(z)) < \sigma(f) = \deg h. \tag{2.7}$$

Substituting (2.6) into (2.5), we can get that

$$\frac{\Delta_\eta^n f(z) - b(z)}{f(z) - b(z)} = \frac{\sum\limits_{j=0}^{n} (-1)^j C_n^j H(z + (n-j)\eta) e^{h(z+(n-j)\eta)} + d(z)}{H(z)e^{h(z)} + t(z)} = A, \tag{2.8}$$

where $d(z) = \Delta_\eta^n a(z) - b(z)$, $t(z) = a(z) - b(z)$.

Obviously, since $\sigma(\Delta_\eta^n a(z)) \leq \sigma(a(z)) < 1$, we have

$$\begin{aligned} &\sigma(d(z)) \leq \max\{\sigma(\Delta_\eta^n a(z)),\, \sigma(b(z))\} < 1, \\ &\sigma(t(z)) \leq \max\{\sigma(a(z)),\, \sigma(b(z))\} < 1. \end{aligned} \tag{2.9}$$

Rewrite (2.8) in the form

$$\sum_{j=0}^{n-1} (-1)^j C_n^j H(z+(n-j)\eta) e^{h(z+(n-j)\eta)} + ((-1)^n - A) H(z) e^{h(z)} = At(z) - d(z),$$

that is,

$$\begin{aligned} &\sum_{j=0}^{n-1} (-1)^j C_n^j H(z + (n-j)\eta) e^{h(z+(n-j)\eta) - h(z)} + ((-1)^n - A) H(z) \\ &= (At(z) - d(z)) e^{-h(z)}. \end{aligned} \tag{2.10}$$

First, we assert that $At(z) - d(z) \equiv 0$. On the contrary, we suppose that $At(z) - d(z) \not\equiv 0$. Then, by (2.9), we have $\max\{\sigma(d(z)), \sigma(t(z))\} < 1 \le k$. Hence, $\sigma(At(z) - d(z)) < 1 \le k$. From $\sigma(H(z)) < \deg h(z) = k$ and $\deg(h(z + (n-j)\eta) - h(z)) = k - 1$ ($j = 0, 1, \cdots, n-1$), we can see that the order of growth of the left side of (2.10) is less than k, but the order of growth of the right side of (2.10) is equal to k. This is a contradiction. Hence, $At(z) - d(z) \equiv 0$, that is

$$A = \frac{d(z)}{t(z)} = \frac{\Delta_\eta^n a(z) - b(z)}{a(z) - b(z)}. \tag{2.11}$$

Thus, (2.10) can be written as

$$\sum_{j=0}^{n-1} (-1)^j C_n^j H(z + (n-j)\eta) e^{h(z+(n-j)\eta) - h(z)} + ((-1)^n - A)H(z) = 0. \tag{2.12}$$

Secondly, we prove that $\sigma(f) = k = 1$. On the contrary, we suppose that $\sigma(f) = k \ge 2$. Thus, we will deduce a contradiction for cases $A = (-1)^n$ and $A \ne (-1)^n$ respectively.

Case 1. Suppose that $A = (-1)^n$. Thus, for the positive integer n, there are three subcases: (1) $n = 1$; (2) $n = 2$; (3) $n \ge 3$.

Subcase 1.1. Suppose that $n = 1$. Then, by $A = -1$, we can obtain from (2.12) that

$$e^{h(z+\eta) - h(z)} = (1 + A) \cdot \frac{H(z)}{H(z+\eta)} = (1 - 1) \cdot \frac{H(z)}{H(z+\eta)} \equiv 0,$$

a contradiction.

Subcase 1.2. Suppose that $n = 2$. Then we can obtain from (2.12) that

$$\frac{H(z+2\eta)}{H(z)} e^{h(z+2\eta) - h(z)} - \frac{2H(z+\eta)}{H(z)} e^{h(z+\eta) - h(z)} + (1 - A) = 0.$$

Noting that $A = (-1)^2 = 1$, the equation above can be rewritten as

$$e^{h(z+2\eta) - h(z+\eta)} = \frac{2H(z+\eta)}{H(z+2\eta)}. \tag{2.13}$$

Set $Q_1(z) = \frac{2H(z+\eta)}{H(z+2\eta)}$. Then we can know from (2.13) that $Q_1(z)$ is a nonconstant entire function. Set $\sigma(H) = \sigma_1$. Then $\sigma_1 < \sigma(f) = k$. By Lemma 2.1, we see that for any given ε_1 $(0 < 3\varepsilon_1 < k - \sigma_1)$, there exists a set $E_1 \subset (1, \infty)$ of finite logarithmic measure, such that for all z satisfying $|z| = r \notin [0, 1] \cup E_1$, we have

$$\exp\{-r^{\sigma_1 - 1 + \varepsilon_1}\} \le \left| \frac{H(z+\eta)}{H(z+2\eta)} \right| \le \exp\{r^{\sigma_1 - 1 + \varepsilon_1}\}. \tag{2.14}$$

Since $Q_1(z)$ is an entire function, by (2.14), we have

$$T(r, Q_1(z)) = m(r, Q_1(z)) \leq m\left(r, \frac{H(z+\eta)}{H(z+2\eta)}\right) + O(1) \leq r^{\sigma_1 - 1 + \varepsilon_1},$$

so that,

$$\sigma(Q_1(z)) \leq \sigma_1 - 1 + \varepsilon_1 < k - 1.$$

Thus, by $\deg(h(z+\eta) - h(z)) = k - 1$, we can see that (2.13) is a contradiction.

Subcase 1.3. Suppose that $n \geq 3$. Then we can obtain from (2.12)

$$\begin{aligned} &H(z+n\eta)e^{h(z+n\eta)} - C_n^1 H(z+(n-1)\eta)e^{h(z+(n-1)\eta)} \\ &+ \cdots + (-1)^{n-1} C_n^{n-1} H(z+\eta)e^{h(z+\eta)} = 0, \end{aligned}$$

that is,

$$\begin{aligned} &\frac{H(z+n\eta)}{H(z+\eta)} e^{h(z+n\eta)-h(z+\eta)} - C_n^1 \frac{H(z+(n-1)\eta)}{H(z+\eta)} e^{h(z+(n-1)\eta)-h(z+\eta)} \\ &+ \cdots + (-1)^{n-2} C_n^{n-2} \frac{H(z+2\eta)}{H(z+\eta)} e^{h(z+2\eta)-h(z+\eta)} + (-1)^{n-1} C_n^{n-1} = 0. \end{aligned} \tag{2.15}$$

Set $Q_2(z) = e^{h(z+2\eta)-h(z+\eta)}$. Then $Q_2(z)$ is a transcendental entire function since $\sigma(Q_2(z)) = k - 1 \geq 1$. For $j = 3, 4, \cdots, n$, we have

$$\begin{aligned} &e^{h(z+j\eta)-h(z+\eta)} \\ &= e^{h(z+j\eta)-h(z+(j-1)\eta)} e^{h(z+(j-1)\eta)-h(z+(j-2)\eta)} \cdots e^{h(z+2\eta)-h(z+\eta)} \\ &= Q_2(z+(j-2)\eta) Q_2(z+(j-3)\eta) \cdots Q_2(z). \end{aligned}$$

Thus, (2.15) can be rewritten as

$$U_2(z, Q_2(z)) \cdot Q_2(z) = (-1)^n C_n^{n-1}, \tag{2.16}$$

where

$$\begin{aligned} U_2(z, Q_2(z)) = &\frac{H(z+n\eta)}{H(z+\eta)} Q_2(z+(n-2)\eta) Q_2(z+(n-3)\eta) \cdots Q_2(z+\eta) \\ &- C_n^1 \frac{H(z+(n-1)\eta)}{H(z+\eta)} Q_2(z+(n-3)\eta) Q_2(z+(n-4)\eta) \cdots Q_2(z+\eta) \\ &+ \cdots + (-1)^{n-2} C_n^{n-2} \frac{H(z+2\eta)}{H(z+\eta)}. \end{aligned}$$

Noting that $(-1)^n C_n^{n-1} \neq 0$, we can see that $U_2(z, Q_2(z)) \not\equiv 0$. Set $\sigma(H) = \sigma_2$. Then $\sigma_2 < k$. Since $Q_2(z)$ is of regular growth and $\sigma(Q_2(z)) = k - 1$, for any given $\varepsilon_2\,(0 < 3\varepsilon_2 < k - \sigma_2)$ and all $r > r_0\,(> 0)$, we have

$$T(r, Q_2(z)) > r^{k-1-\varepsilon_2}. \tag{2.17}$$

By Lemma 2.1, we see that for ε_2, there exists a set $E_2 \subset (1,\infty)$ of finite logarithmic measure, such that for all z satisfying $|z| = r \notin [0,1] \cup E_2$, we have

$$\exp\{-r^{\sigma_2-1+\varepsilon_2}\} \leq \left|\frac{H(z+j\eta)}{H(z+\eta)}\right| \leq \exp\{r^{\sigma_2-1+\varepsilon_2}\} \quad (j = 2,3,\cdots,n). \tag{2.18}$$

Thus, from (2.17) and (2.18), we can get that for $j = 2,3,\cdots,n$

$$\frac{m\left(r, \frac{H(z+j\eta)}{H(z+\eta)}\right)}{T(r,Q_2(z))} \leq \frac{r^{\sigma_2-1+\varepsilon_2}}{r^{k-1-\varepsilon_2}} \to 0 \quad (r \to \infty \quad \text{and} \quad r \notin [0,1] \cup E_2),$$

that is,

$$m\left(r, \frac{H(z+j\eta)}{H(z+\eta)}\right) = S(r,Q_2) \quad (j = 2,3,\cdots,n). \tag{2.19}$$

Noting that $\deg_{Q_2} U_2(z,Q_2) = n-2 \geq 1$ and applying Lemma 2.4 and Remark 2.1 to (2.16), we have

$$T(r,Q_2) = m(r,Q_2) = S(r,Q_2),$$

a contradiction.

Case 2. Suppose that $A \neq (-1)^n$. Thus, for the positive integer n, there are two subcases: (1) $n = 1$; (2) $n \geq 2$.

Subcase 2.1. Suppose that $n = 1$. Thus, (2.12) can be rewritten as

$$\frac{H(z+\eta)}{H(z)} = (A+1)e^{h(z)-h(z+\eta)}.$$

Noting that $A+1 \neq 0$, we can use the same method as in the proof of Subcase 1.2 to deduce a contradiction.

Subcase 2.2. Suppose that $n \geq 2$. Then we can obtain from (2.12)

$$\begin{aligned} &\frac{H(z+n\eta)}{H(z)}e^{h(z+n\eta)-h(z)} - C_n^1\frac{H(z+(n-1)\eta)}{H(z)}e^{h(z+(n-1)\eta)-h(z)} \\ &+\cdots+(-1)^{n-1}C_n^{n-1}\frac{H(z+\eta)}{H(z)}e^{h(z+\eta)-h(z)} + (-1)^n - A = 0. \end{aligned} \tag{2.20}$$

Set $Q_3(z) = e^{h(z+\eta)-h(z)}$. Then $Q_3(z)$ is a transcendental entire function since $\sigma(Q_3(z)) = k-1 \geq 1$. For $j = 2,3,\cdots,n$, we have

$$\begin{aligned} &e^{h(z+j\eta)-h(z)} \\ =\ &e^{h(z+j\eta)-h(z+(j-1)\eta)}e^{h(z+(j-1)\eta)-h(z+(j-2)\eta)}\cdots e^{h(z+\eta)-h(z)} \\ =\ &Q_3(z+(j-1)\eta)Q_3(z+(j-2)\eta)\cdots Q_3(z). \end{aligned}$$

Thus, (2.20) can be rewritten as

$$U_3(z, Q_3(z)) \cdot Q_3(z) = A - (-1)^n, \tag{2.21}$$

where

$$\begin{aligned} U_3(z, Q_3(z)) = &\frac{H(z+n\eta)}{H(z)} Q_3(z+(n-1)\eta) Q_3(z+(n-2)\eta) \cdots Q_3(z+\eta) \\ &-C_n^1 \frac{H(z+(n-1)\eta)}{H(z)} Q_3(z+(n-2)\eta) Q_3(z+(n-3)\eta) \cdots Q_3(z+\eta) \\ &+ \cdots + (-1)^{n-1} C_n^{n-1} \frac{H(z+\eta)}{H(z)}. \end{aligned}$$

We can see that $U_3(z, Q_3(z)) \not\equiv 0$ since $A - (-1)^n \neq 0$. Noting that $\deg_{Q_3} U_3(z, Q_3(z)) = n - 1 \geq 1$, we can use the same method as in the proof of Subcase 1.3 to deduce a contradiction.

Thus, we have proved that $\sigma(f) = k = 1$. And $f(z)$ can be written as

$$f(z) = a(z) + H(z) e^{c_1 z}, \tag{2.22}$$

where $c_1\ (\neq 0)$ is a constant and $H(z)\ (\not\equiv 0)$ is an entire function and satisfies

$$\sigma(H(z)) = \lambda(H(z)) = \lambda(f(z) - a(z)) < \sigma(f).$$

Thirdly, we will prove that $H(z)\ (\not\equiv 0)$ is a constant. To this end, we only need to prove $H'(z) \equiv 0$. Thus, substituting (2.22) into (2.12), we obtain

$$\begin{aligned} &e^{nc_1\eta} H(z+n\eta) - C_n^1 e^{(n-1)c_1\eta} H(z+(n-1)\eta) + \cdots \\ &+(-1)^{n-1} C_n^{n-1} e^{c_1\eta} H(z+\eta) + ((-1)^n - A) H(z) = 0. \end{aligned} \tag{2.23}$$

We assert that the sum of all coefficients of (2.23) is equal to zero, that is

$$e^{nc_1\eta} - C_n^1 e^{(n-1)c_1\eta} + \cdots + (-1)^{n-1} C_n^{n-1} e^{c_1\eta} + ((-1)^n - A) = 0.$$

On the contrary, we suppose that

$$e^{nc_1\eta} - C_n^1 e^{(n-1)c_1\eta} + \cdots + (-1)^{n-1} C_n^{n-1} e^{c_1\eta} + ((-1)^n - A) \neq 0.$$

Then (2.23) indicate $H(z)\ (\not\equiv 0)$ is an entire solution of equation

$$\begin{aligned} &e^{nc_1\eta} g(z+n\eta) - C_n^1 e^{(n-1)c_1\eta} g(z+(n-1)\eta) + \cdots \\ &+(-1)^{n-1} C_n^{n-1} e^{c_1\eta} g(z+\eta) + ((-1)^n - A) g(z) = 0. \end{aligned}$$

By Lemma 2.5 and Remark 2.2-2.3, we have $\sigma(H) \geq 1$, a contradiction. Hence,

$$e^{nc_1\eta} - C_n^1 e^{(n-1)c_1\eta} + \cdots + (-1)^{n-1} C_n^{n-1} e^{c_1\eta} + ((-1)^n - A) = 0.$$

Thus,

$$A-(-1)^n = e^{nc_1\eta} - C_n^1 e^{(n-1)c_1\eta} + \cdots + (-1)^{n-1} C_n^{n-1} e^{c_1\eta}.$$

Substituting the expression above into equation (2.23) and we have

$$\begin{aligned} &e^{nc_1\eta}(H(z+n\eta)-H(z)) - C_n^1 e^{(n-1)c_1\eta}(H(z+(n-1)\eta)-H(z)) \\ &+\cdots+(-1)^{n-1}C_n^{n-1}e^{c_1\eta}(H(z+\eta)-H(z))=0. \end{aligned} \tag{2.24}$$

By Lemma 2.3, we can see that, there exists an ε-set E such that for $j=1, 2, \cdots, n$

$$\begin{aligned} &H(z+j\eta)-H(z) = j\eta H'(z)(1+o_j(1)) \\ &= j\eta H'(z) + o_j(1)H'(z) \quad \text{as } z\to\infty \text{ in } \mathbb{C}\backslash E. \end{aligned} \tag{2.25}$$

Substituting (2.25) into (2.24), we can get that

$$\eta H'(z)\cdot K_1 + H'(z)\cdot K_2 = 0 \qquad \text{as } z\to\infty \text{ in } \mathbb{C}\backslash E, \tag{2.26}$$

where K_1 is a constant and satisfies

$$K_1 = ne^{nc_1\eta} - (n-1)C_n^1 e^{(n-1)c_1\eta} + \cdots + (-1)^{n-2}2C_n^{n-2}e^{2c_1\eta} + (-1)^{n-1}C_n^{n-1}e^{c_1\eta},$$

and

$$\begin{aligned} K_2 &= e^{nc_1\eta}o_n(1) - C_n^1 e^{(n-1)c_1\eta}o_{n-1}(1) + \cdots + (-1)^{n-1}C_n^{n-1}e^{c_1\eta}o_1(1) \\ &= o(1) \qquad\qquad \text{as } z\to\infty \text{ in } \mathbb{C}\backslash E. \end{aligned} \tag{2.27}$$

We assume that $K_1 \neq 0$. If $n=1$, then $K_1 = e^{c_1\eta}$ is a nonzero constant; If $n \geq 2$, on the contrary, we suppose that $K_1 = 0$. Then, for $j = 0, 1, \cdots, n-1$, noting that

$$\frac{C_n^j\cdot(n-j)}{n} = \frac{n!\cdot(n-j)}{n(n-j)!j!} = \frac{(n-1)!}{(n-1-j)!j!} = C_{n-1}^j,$$

we have

$$\begin{aligned} &ne^{nc_1\eta} - C_n^1(n-1)e^{(n-1)c_1\eta} + \cdots + (-1)^{n-2}C_n^{n-2}2e^{2c_1\eta} + (-1)^{n-1}C_n^{n-1}e^{c_1\eta} \\ &= ne^{c_1\eta}\left(e^{(n-1)c_1\eta} + (-1)C_{n-1}^1 e^{(n-2)c_1\eta} + \cdots + (-1)^{n-2}C_{n-1}^{n-2}e^{c_1\eta} + (-1)^{n-1}\right) \\ &= ne^{c_1\eta}(e^{c_1\eta}-1)^{n-1} = 0. \end{aligned}$$

Then we can obtain from the equality above that $e^{c_1\eta} = 1$ since $n-1 \geq 1$. Substituting $e^{c_1\eta} = 1$ into (2.23), we have

$$\begin{aligned} &H(z+n\eta) - C_n^1 H(z+(n-1)\eta) + \cdots + (-1)^{n-1}C_n^{n-1}H(z+\eta) \\ &+((-1)^n - A)H(z) = 0. \end{aligned}$$

Thus, we can find that $H(z)$ is a nontrivial entire solution of equation

$$\begin{aligned} & g(z+n\eta) - C_n^1 g(z+(n-1)\eta) \\ & + \cdots + (-1)^{n-1} C_n^{n-1} g(z+\eta) + ((-1)^n - A) g(z) = 0. \end{aligned} \tag{2.28}$$

Noting that $A \neq 0$ and the sum of all coefficients of equation (2.28) is

$$1 - C_n^1 + \cdots + (-1)^{n-1} C_n^{n-1} + (-1)^{n-1} C_n^{n-1} + ((-1)^n - A) = -A,$$

by Lemma 2.5 and Remark 2.2-2.3, we have $\sigma(H) \geq 1$, a contradiction. Hence $K_1 \neq 0$. By (2.27), (2.26) implies $H'(z) \equiv 0$. Thus, we can know that $H(z)$ is a nonzero constant. Hence, $f(z)$ can be written as

$$f(z) = a(z) + ce^{c_1 z}, \tag{2.29}$$

where c, c_1 are two nonzero constants.

Finally, we prove that $A = (e^{c_1\eta} - 1)^n$. Substituting (2.29) into (2.5), we have

$$\Delta_\eta^n a(z) - b(z) + \Delta_\eta^n (ce^{c_1 z}) = A(a(z) - b(z)) + Ace^{c_1 z}.$$

By (2.11), we have

$$\begin{aligned} A \cdot ce^{c_1 z} &= \Delta_\eta^n (ce^{c_1 z}) \\ &= ce^{c_1(z+n\eta)} - C_n^1 ce^{c_1(z+(n-1)\eta)} + \cdots + (-1)^n C_n^n ce^{c_1 z} \\ &= e^{nc_1\eta} \cdot ce^{c_1 z} - C_n^1 e^{(n-1)c_1\eta} \cdot ce^{c_1 z} + \cdots + (-1)^n C_n^n \cdot ce^{c_1 z} \\ &= (e^{c_1\eta} - 1)^n \cdot ce^{c_1 z}. \end{aligned}$$

Hence, $A = (e^{c_1\eta} - 1)^n$.

Thus, Lemma 2.7 is proved.

3. Proof of Theorem 1.1

By the hypotheses of the Theorem 1.1, we can write $f(z)$ in the form (2.6), and (2.7) holds. Since $\Delta_\eta^n f(z)$ and $f(z)$ share entire function $b(z)$ CM, then

$$\frac{\Delta_\eta^n f(z) - b(z)}{f(z) - b(z)} = \frac{\sum\limits_{j=0}^{n} (-1)^{n-j} C_n^j H(z+j\eta) e^{h(z+j\eta)} + d(z)}{H(z)e^{h(z)} + t(z)} = e^{P(z)}, \tag{3.1}$$

where $P(z)$ is a polynomial and

$$d(z) = \Delta_\eta^n a(z) - b(z) = \sum_{j=0}^{n} (-1)^j C_n^j a(z+(n-j)\eta) - b(z), \quad t(z) = a(z) - b(z).$$

Obviously, we have

(3.2) $\sigma(d(z))\leq\max\{\sigma(a(z)),\sigma(b(z))\}<1,\ \sigma(t(z))\leq\max\{\sigma(a(z)),\sigma(b(z))\}<1.$

First step. We prove

$$\frac{\Delta_\eta^n f(z)-b(z)}{f(z)-b(z)}=A,$$

where A is a nonzero constant. If $P(z)\equiv 0$, then, by (3.1), we obtain

$$\frac{\Delta_\eta^n f(z)-b(z)}{f(z)-b(z)}=e^{P(z)}\equiv 1.$$

Now we suppose that $P(z)\not\equiv 0$ and $\deg P(z)=s$. Set

(3.3) $h(z)=a_k z^k+a_{k-1}z^{k-1}+\cdots+a_0,\quad P(z)=b_s z^s+b_{s-1}z^{s-1}+\cdots+b_0,$

where $k=\sigma(f)\geq 1$, $a_k(\neq 0)$, $a_{k-1},\cdots,a_0,b_s(\neq 0),b_{s-1},\cdots,b_0$ are constants. By (3.1), we see that

$$0\leq \deg P=s\leq \deg h=k.$$

In this case, we prove $P(z)$ is a constant, that is $s=0$. To this end, we will deduce a contradiction for cases $s=k$ and $1\leq s<k$ respectively.

Case 1. Suppose that $1\leq s=k$. Thus, there are three subcases: (1) $b_k=a_k$; (2) $b_k=-a_k$; (3) $b_k\neq a_k$ and $b_k\neq -a_k$.

Subcase 1.1. Suppose that $b_k=a_k$. Then (3.1) is rewritten as

$$g_{11}(z)e^{P(z)}+g_{12}e^{-h(z)}+g_{13}e^{h_0(z)}=0, \tag{3.4}$$

where $h_0(z)\equiv 0$ and

$$\begin{cases} g_{11}(z) &= -H(z);\\ g_{12}(z) &= d(z);\\ g_{13}(z) &= \sum\limits_{j=0}^{n}(-1)^{n-j}C_n^j H(z+j\eta)e^{h(z+j\eta)-h(z)}-t(z)e^{P(z)-h(z)}.\end{cases}$$

By $b_k=a_k$, we have $\deg(P(z)-h(z))\leq k-1$. Since $\sigma(H)<k$, $\max\{\sigma(d),\sigma(t)\}<1\leq k$ and $\deg(h(z+j\eta)-h(z))=k-1(\,j=1,2,\cdots,n)$, we can see that

$$\sigma(g_{1m}(z))<k\quad (\,m=1,\,2,3).$$

On the other hand, by $b_k=a_k$, we have

$$\deg(P-(-h))=k,\quad \deg(P-h_0)=k;\quad \deg(-h-h_0)=k.$$

Since $e^{P-(-h)}$, e^{P-h_0} and e^{-h-h_0} are of regular growth, and $\sigma(g_{1m}) < k\,(m = 1,\,2,\,3)$, we can see that for $m = 1,\,2,\,3$

$$\begin{cases} T(r, g_{1m}) = o\left(T\left(r, e^{P-(-h)}\right)\right); \\ T(r, g_{1m}) = o\left(T\left(r, e^{P-h_0}\right)\right); \\ T(r, g_{1m}) = o\left(T\left(r, e^{-h-h_0}\right)\right). \end{cases} \tag{3.5}$$

Thus, applying Lemma 2.2 to (3.4), by (3.5), we obtain

$$g_{1m}(z) \equiv 0 \quad (m = 1,\,2,\,3).$$

Hence, $g_{11}(z) \equiv -H(z) \equiv 0$. A contradiction.

Subcase 1.2. Suppose that $b_k = -a_k$. Then (3.1) is rewritten as

$$g_{21}(z)e^{-h(z)} + g_{22}(z)e^{P(z)-h(z)} + g_{23}(z)e^{h_0(z)} = 0. \tag{3.6}$$

where $h_0(z) \equiv 0$ and

$$\begin{cases} g_{21}(z) = d(z) - H(z)e^{P(z)+h(z)}; \\ g_{22}(z) = -t(z); \\ g_{23}(z) = \sum\limits_{j=0}^{n} (-1)^{n-j} C_n^j H(z+j\eta)e^{h(z+j\eta)-h(z)}. \end{cases}$$

By $b_k = -a_k$, we have $\deg(P(z)+h(z)) \le k-1$. Since $\sigma(H) < k$, $\max\{\sigma(d), \sigma(t)\} < 1 \le k$ and $\deg(h(z+j\eta) - h(z)) = k-1(\,j = 1, 2, \cdots, n)$, we can see that

$$\sigma(g_{2m}(z)) < k \quad (\,m = 1,\,2, 3).$$

On the other hand, by $b_k = -a_k$, we have

$$\deg(-h - (P-h)) = k, \quad \deg(-h - h_0) = k; \quad \deg((P-h) - h_0) = k.$$

Since $e^{-h-(P-h)}$, e^{-h-h_0} and $e^{(P-h)-h_0}$ are of regular growth, and $\sigma(g_{2m}) < k\,(\,m = 1,\,2,\,3)$, we can see that for $m = 1,\,2,\,3$

$$\begin{cases} T(r, g_{2m}) = o\left(T\left(r, e^{-h-(P-h)}\right)\right); \\ T(r, g_{2m}) = o\left(T\left(r, e^{-h-h_0}\right)\right); \\ T(r, g_{2m}) = o\left(T\left(r, e^{(P-h)-h_0}\right)\right). \end{cases} \tag{3.7}$$

Thus, applying Lemma 2.2 to (3.6), by (3.7), we obtain

$$g_{2m}(z) \equiv 0 \quad (m = 1,\,2,\,3).$$

Hence, $g_{22}(z) \equiv -t(z) \equiv b(z) - a(z) \equiv 0$, which contradicts $b(z) \not\equiv a(z)$.

Subcase 1.3. Suppose that $b_k \neq a_k$ and $b_k \neq -a_k$. Then (3.1) is rewritten as

$$g_{31}(z)e^{-h(z)} + g_{32}(z)e^{P(z)} + g_{33}(z)e^{P(z)-h(z)} + g_{34}(z)e^{h_0(z)} = 0. \tag{3.8}$$

where $h_0(z) \equiv 0$ and

$$\begin{cases} g_{31}(z) = d(z); \\ g_{32}(z) = -H(z); \\ g_{33}(z) = -t(z); \\ g_{34}(z) = \sum\limits_{j=0}^{n} (-1)^{n-j} C_n^j H(z+j\eta) e^{h(z+j\eta)-h(z)}. \end{cases}$$

Since $\sigma(H) < k$, $\max\{\sigma(d), \sigma(t)\} < 1 \leq k$ and $\deg(h(z+j\eta) - h(z)) = k-1 (j = 1, 2, \cdots, n)$, we can see that

$$\sigma(g_{3m}(z)) < k \quad (m = 1,\ 2,\ 3,\ 4).$$

On the other hand, by $b_k \neq a_k$ and $b_k \neq -a_k$, we have

$$\deg(-h-P) = k, \quad \deg(-h-(P-h)) = k, \quad \deg(-h-h_0) = k;$$

$$\deg(P-(P-h)) = k, \quad \deg(P-h_0)) = k, \quad \deg((P-h)-h_0) = k.$$

Since e^{-h-P}, $e^{-h-(P-h)}$, $\cdots$, $e^{(P-h)-h_0}$ are of regular growth, and $\sigma(g_{3m}) < k$ ($m = 1,\ 2,\ 3,\ 4$), we can see that for $m = 1,\ 2,\ 3,\ 4$

$$\begin{cases} T(r, g_{3m}) = o\left(T\left(r, e^{-h-P}\right)\right); \\ T(r, g_{3m}) = o\left(T\left(r, e^{-h-(P-h)}\right)\right); \\ \quad \vdots \ = \ \vdots \\ T(r, g_{3m}) = o\left(T\left(r, e^{(P-h)-h_0}\right)\right). \end{cases} \tag{3.9}$$

Thus, applying Lemma 2.2 to (3.8), by (3.9), we obtain

$$g_{3m}(z) \equiv 0 \quad (m = 1,\ 2,\ 3,\ 4).$$

Clearly, this is a contradiction.

Case 2. Suppose that $1 \leq s < k$. Then, (3.1) is rewritten as

$$[t(z)e^{P(z)} - d(z)]e^{-h(z)} = \sum_{j=0}^{n} (-1)^{n-j} C_n^j H(z+j\eta) e^{h(z+j\eta)-h(z)} - H(z)e^{P(z)}. \tag{3.10}$$

By $t(z) = a(z) - b(z)$, (3.2), (3.3), we have $\sigma(te^P - d) = s < k$. Thus, since $\deg h = k$, $\deg(h(z + j\eta) - h(z)) = k - 1\,(j = 1,\, 2,\, \cdots, n)$, $\deg P = s < k$ and $\sigma(H) < k$, we can see that the order of growth of the left side of (3.10) is equal to k, but the order of growth of the right side of (3.10) is less than k. This is a contradiction.

Thus, we have proved that P is only a constant (including $P(z) \equiv 0$), that is

$$\frac{\Delta_\eta^n f(z) - b(z)}{f(z) - b(z)} = A, \tag{3.11}$$

where A is a nonzero constant.

Second step. Applying Lemma 2.7 to (3.11), we can obtain

$$A = \frac{\Delta_\eta^n a(z) - b(z)}{a(z) - b(z)} = (e^{c_1\eta} - 1)^n, \quad \text{and} \quad f(z) = a(z) + ce^{c_1 z},$$

where c, c_1 are two nonzero constants.

Thus, Theorem 1.1 is proved.

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