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A NOTE ON THE APPROXIMATION BY THE q-HYBRID SUMMATION INTEGRAL TYPE OPERATORS

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Abstract. In this study, it is introduced a $q-$Stancu type of hybrid summation integral type operators. It is investigated their approximation properties. It is given a weighted approximation theorem and obtained rates of convergence of these operators for continuous functions.

1. Introduction

The well known hybrid summation integral type operators are defined as

$$M_n(f,x) = (n-1)\sum_{k=1}^{\infty} s_{n,k}(x) \int_0^{\infty} p_{n,k-1}(t) f(t)\, dt + e^{-nx} f(0), \tag{1.1}$$

where

$$s_{n,k}(x) = e^{-nx}\frac{(nx)^k}{k!}$$

and

$$p_{n,k}(x) = \binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}$$

are respectively Szász and Baskakov basis functions. This operators were studied in [1, 2]. In [3], based on $q-$integer and $q-$binomial coefficients, firstly, Lupaş introduced a $q-$analogue of the Bernstein operators. After then several interesting generalization about $q-$calculus were given in [4, 5, 6, 7, 8, 9, 10]. Our aim is to obtain generalization to $q-$calculus of hybrid summation integral type operators. We use without further explanation the basic notations and formulas, from the theory of $q-$calculus as set out in [11, 12, 13, 14, 15].

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2. q-HYBRID OPERATORS

Let $k \in \mathbb{N}$, $n \in \mathbb{N} \backslash \{0\}$, $A > 0$ and f be a real valued continuous function defined on the interval $[0, \infty)$. We introduce $q-$hybrid summation integral Stancu type linear positive operators for $0 < q \leq 1$ as

$$M_{n,q}^{(\alpha,\beta)}(f,x) = [n-1]_q \sum_{k=1}^{\infty} s_{n,k}^q(x) \int_0^{\infty/A} p_{n,k-1}^q(t) f\left(\frac{[n]_q t + \alpha}{[n]_q + \beta}\right) d_q t + e^{-[n]_q x} f\left(\frac{\alpha}{[n]_q + \beta}\right), \tag{2.1}$$

where

$$s_{n,k}^q(x) = \left([n]_q x\right)^k \frac{e^{-[n]_q x}}{[k]_q!},$$

and

$$p_{n,k}^q(x) = \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q q^{k(k+1)} \frac{x^k}{(1+x)_q^{n+k}}.$$

If we write $q = 1$ in (2.1), then the operators $M_{n,q}^{(\alpha,\beta)}$ are reduced to hybrid summation integral type operators given in (1.1).

Now we are ready to give the following lemma for the Korovkin test functions.

Lemma 1. *Let* $e_m(t) = t^m$, $m = 0, 1, 2$. *we get*

$$(i)\ M_{n,q}^{(\alpha,\beta)}(e_0, x) = 1,$$

$$(ii)\ M_{n,q}^{(\alpha,\beta)}(e_1, x) = \frac{[n]_q^2}{q\left([n]_q + \beta\right)[n-2]_q} x + \frac{\alpha}{[n]_q + \beta},$$

$$(iii)\ M_{n,q}^{(\alpha,\beta)}(e_2, x) = \frac{[n]_q^4}{q^4\left([n]_q + \beta\right)^2 [n-3]_q [n-2]_q} x^2 + \left\{ \frac{[2]_q [n]_q^3}{q^3\left([n]_q + \beta\right)^2 [n-3]_q [n-2]_q} + \frac{2[n]_q^2 \alpha}{q [n-2]_q \left([n]_q + \beta\right)^2} \right\} x + \frac{\alpha^2}{\left([n]_q + \beta\right)^2}.$$

Proof. Using $q-$Gamma and $q-$Beta functions in [14, 15], we obtain the estimate,

$$(2.2)\quad \begin{aligned} &\int_0^{\infty/A} p^q_{n,k-1}(t)t^m d_qt \\ &= \left[\begin{array}{c} n+k-2 \\ k-1 \end{array}\right]_q \int_0^{\infty/A} q^{k(k-1)} \frac{t^{k-1+m}}{(1+t)_q^{n+k-1}} d_qt \\ &= \frac{[n+k-2]_q! B_q(k+m, n-m-1) q^{k(k-1)}}{[k-1]_q! [n-1]_q! K(A, k+m)} \\ &= \frac{[m+k-1]_q! [n-m-2]_q! q^{\{-(k+m)(k+m-1)+2k(k-1)\}/2}}{[n-1]_q! [k-1]_q!}. \end{aligned}$$

Then, using (2.2) for $m = 0$, we get

$$\begin{aligned} M_{n,q}^{(\alpha,\beta)}(e_0, x) &= [n-1]_q \sum_{k=1}^{\infty} s^q_{n,k}(x) \int_0^{\infty/A} p^q_{n,k-1}(t) d_qt + e^{-[n]_q x} \\ &= e^{-[n]_q x} \sum_{k=0}^{\infty} \frac{\left([n]_q x\right)^k}{[k]_q!} q^{k(k-1)/2} \\ &= e^{-[n]_q x} E_q^{[n]_q x} \\ &= 1, \end{aligned}$$

and the proof of (i) is finished. With a direct computation, we obtain (ii) as follows:

$$\begin{aligned} &M_{n,q}^{(\alpha,\beta)}(e_1, x) \\ &= [n-1]_q \sum_{k=1}^{\infty} s^q_{n,k}(x) \int_0^{\infty/A} p^q_{n,k-1}(t) \left(\frac{[n]_q t + \alpha}{[n]_q + \beta} \right) d_qt + e^{-[n]_q x} \frac{\alpha}{[n]_q + \beta} \\ &= \frac{[n]_q}{\left([n]_q + \beta\right) [n-2]_q} \sum_{k=1}^{\infty} \frac{\left([n]_q x\right)^k}{[k-1]_q!} q^{k(k-3)/2} e^{-[n]_q x} \\ &\quad + \frac{\alpha}{[n]_q + \beta} \sum_{k=1}^{\infty} \frac{\left([n]_q x\right)^k}{[k]_q!} q^{k(k-1)/2} e^{-[n]_q x} + e^{-[n]_q x} \frac{\alpha}{[n]_q + \beta} \\ &= \frac{[n]_q^2 x}{q\left([n]_q + \beta\right) [n-2]_q} e^{-[n]_q x} E_q^{[n]_q x} + \frac{\alpha}{[n]_q + \beta} e^{-[n]_q x} E_q^{[n]_q x} \\ &= \frac{[n]_q^2 x}{q\left([n]_q + \beta\right) [n-2]_q} + \frac{\alpha}{[n]_q + \beta}. \end{aligned}$$

Using the equality

$$[n]_q = [s]_q + q^s[n-s]_q,\ 0 \le s \le n, \tag{2.3}$$

we get

$$\begin{aligned}
&M_{n,q}^{(\alpha,\beta)}(e_2,x)\\
&= [n-1]_q \sum_{k=1}^{\infty} s_{n,k}^q(x) \int_0^{\infty/A} p_{n,k-1}^q(t) \left(\frac{[n]_q t+\alpha}{[n]_q+\beta}\right)^2 d_qt + e^{-[n]_q x}\left(\frac{\alpha}{[n]_q+\beta}\right)^2\\
&= \frac{[n-1]_q [n]_q^2}{\left([n]_q+\beta\right)_q^2} \sum_{k=1}^{\infty} s_{n,k}^q(x) \int_0^{\infty/A} p_{n,k-1}^q(t) t^2 d_qt\\
&\quad + \frac{2\alpha [n-1]_q [n]_q}{\left([n]_q+\beta\right)^2} \sum_{k=1}^{\infty} s_{n,k}^q(x) \int_0^{\infty/A} p_{n,k-1}^q(t) t d_qt\\
&\quad + \frac{\alpha^2 [n-1]_q}{\left([n]_q+\beta\right)^2} \sum_{k=1}^{\infty} s_{n,k}^q(x) \int_0^{\infty/A} p_{n,k-1}^q(t) d_qt + e^{-[n]_q x}\left(\frac{\alpha}{[n]_q+\beta}\right)^2\\
&= \frac{[n]_q^4}{q^4\left([n]_q+\beta\right)^2 [n-3]_q [n-2]_q} x^2\\
&\quad + \left\{ \frac{[2]_q [n]_q^3}{q^3\left([n]_q+\beta\right)^2 [n-3]_q [n-2]_q} + \frac{2[n]_q^2\alpha}{\left([n]_q+\beta\right)^2 [n-2]_q} \right\} x\\
&\quad + \frac{\alpha^2}{\left([n]_q+\beta\right)^2},
\end{aligned}$$

and so we have proof of *(iii)*. ■

To obtain our main results we need computing second moment.

Lemma 2. *Let $q \in (0,1)$ and $n > 3$. Then we have the following inequality*

$$M_{n,q}^{(\alpha,\beta)}((t-x)^2,x) \le \left(\frac{2(1-q^3)}{q^4} + \frac{512(\alpha+\beta+1)^2[n]_q}{q^4[n-3]_q[n-2]_q}\right) x(x+1) + \frac{\alpha^2}{\left([n]_q+\beta\right)^2}.$$

Proof. *From linearity of $M_{n,q}^{(\alpha,\beta)}$ operators and Lemma 1, we write the second moment as*

$$
\begin{aligned}
& M_{n,q}^{(\alpha,\beta)}((t-x)^2,x) \\
&= M_{n,q}^{(\alpha,\beta)}(t^2;x) - 2xM_{n,q}^{(\alpha,\beta)}(t;x) + x^2 M_{n,q}^{(\alpha,\beta)}(1;x) \\
&= \left\{ \frac{[n]_q^4}{q^4\left([n]_q+\beta\right)^2 [n-3]_q\,[n-2]_q} - \frac{2[n]_q^2}{q\left([n]_q+\beta\right)[n-2]_q} + 1 \right\} x^2 \\
&\quad + \left\{ \frac{[2]_q[n]_q^3}{q^3\left([n]_q+\beta\right)^2 [n-3]_q\,[n-2]_q} \right. \\
&\quad \left. + \frac{2[n]_q^2\alpha}{q\left([n]_q+\beta\right)^2 [n-2]_q} - \frac{2\alpha}{[n]_q+\beta} \right\} x + \frac{\alpha^2}{\left([n]_q+\beta\right)^2}, \\
&\le \left\{ \frac{\left|[n]_q^4\,(1+q^4) - 2q^3\,[n-3]_q^4\right| + 2q^4\beta\,[n]_q\,[n-3]_q\,[n-2]_q}{q^4\left([n]_q+\beta\right)^2 [n-3]_q\,[n-2]_q} \right. \\
&\quad \left. + \frac{q^4\beta^2\,[n-3]_q\,[n-2]_q + q\,[2]_q\,[n]_q^3 + 2q^3\alpha\,[n]_q^2\,[n-3]_q}{q^4\left([n]_q+\beta\right)^2 [n-3]_q\,[n-2]_q} \right\} x(x+1) \\
&\quad + \frac{\alpha^2}{\left([n]_q+\beta\right)^2}.
\end{aligned}
$$

From (2.3), we have

$$
\begin{aligned}
& M_{n,q}^{(\alpha,\beta)}((t-x)^2,x) \\
&\le \left\{ \frac{[n-3]_q^4\left|q^{12}+q^{16}-2q^3\right|}{q^4\left([n]_q+\beta\right)^2 [n-3]_q\,[n-2]_q} \right. \\
&\quad + \frac{(1+q^4)\left\{4q^9\,[n-3]_q^3\,[3]_q + 6q^6\,[n-3]_q^3\,[3]_q^2 + 4q^3\,[n-3]_q\,[3]_q^3 + [3]_q^4\right\}}{q^4\left([n]_q+\beta\right)^2 [n-3]_q\,[n-2]_q} \\
&\quad \left. + \frac{2\,[n]_q^3\,(\beta+\alpha+1)^2}{q^4\left([n]_q+\beta\right)^2 [n-3]_q\,[n-2]_q} \right\} x(x+1) + \frac{\alpha^2}{\left([n]_q+\beta\right)^2} \\
&\le \left(\frac{2(1-q^3)}{q^4} + \frac{512(\alpha+\beta+1)^2[n]_q}{q^4[n-3]_q[n-2]_q} \right) x(x+1) + \frac{\alpha^2}{\left([n]_q+\beta\right)^2}.
\end{aligned}
$$

And the proof of the Lemma 2 is now finished. ∎

Now we consider, $B[0,\infty)$ denotes the set of all bounded functions from $[0,\infty)$ to $\mathbb{R}$. $B[0,\infty)$ is a normed space with the norm $\|f\|_B = \sup\{|f(x)| : x \in [0,\infty)\}$.

$C_B[0,\infty)$ denotes the subspace of all continuous functions in $B[0,\infty)$. We denote first modulus of continuity on finite interval $[0,b]$, $b>0$

$$\omega_{[0,b]}(f,\delta)=\sup_{0<h\leq\delta,x\in[0,b]}|f(x+h)-f(x)|\,. \tag{2.4}$$

The Peetre's $K-$functional is defined by

$$K_2(f,\delta)=\inf\left\{\|f-g\|_B+\delta\left\|g''\right\|_B : g\in W_\infty^2\right\},\ \delta>0$$

where $W_\infty^2=\{g\in C_B[0,\infty):g',g''\in C_B[0,\infty)\}$. By , p. 177, Theorem 2.4 in [16], there exists a positive constant C such that

$$K_2(f,\delta)\leq C\omega_2(f,\sqrt{\delta}) \tag{2.5}$$

where

$$\omega_2(f,\sqrt{\delta})=\sup_{0<h\leq\sqrt{\delta}}\sup_{x\in[0,\infty)}|f(x+2h)-2f(x+h)-f(x)|\,.$$

The weighted Korovkin- type theorems was proved by Gadzhiev [17]. We give the Gadzhiev's results in weighted spaces. Let $\rho(x)=1+x^2$. $B_\rho[0,\infty)$ denotes the set of all functions f, from $[0,\infty)$ to $\mathbb{R}$, satisfying growth condition $|f(x)|\leq N_f\rho(x)$, where N_f is a constant depending only on f. $B_\rho[0,\infty)$ is a normed space with the norm $\|f\|_\rho=\sup\left\{\dfrac{|f(x)|}{\rho(x)}:x\in\mathbb{R}\right\}$. $C_\rho[0,\infty)$ denotes the subspace of all continuous functions in $B_\rho[0,\infty)$ and $C_\rho^*[0,\infty)$ denotes the subspace of all functions $f\in C_\rho[0,\infty)$ for which $\lim\limits_{|x|\to\infty}\dfrac{|f(x)|}{\rho(x)}$ exists finitely.

Thus we are ready to give direct results. The following lemma is routine and its proof is omitted.

Lemma 3. *Let*

$$\overline{M}_{n,q}^{(\alpha,\beta)}(f,x)=M_{n,q}^{(\alpha,\beta)}(f,x)-f\left(\frac{[n]_q^2\,x}{q\left([n]_q+\beta\right)[n-2]_q}+\frac{\alpha}{[n]_q+\beta}\right)+f\left(x\right). \tag{2.6}$$

Then the following assertions are hold for the operators (2.6):

$$\begin{aligned}(i)\quad \overline{M}_{n,q}^{(\alpha,\beta)}(1,x)&=1,\\ (ii)\quad \overline{M}_{n,q}^{(\alpha,\beta)}(t,x)&=x,\\ (iii)\quad \overline{M}_{n,q}^{(\alpha,\beta)}(t-x,x)&=0.\end{aligned}$$

Lemma 4. *Let* $q \in (0,1)$ *and* $n > 3$. *Then for every* $x \in [0,\infty)$ *and* $f'' \in C_B[0,\infty)$, *we have the inequality*

$$\left|\overline{M}_{n,q}^{(\alpha,\beta)}(f,x) - f(x)\right| \le \delta_{n,q}^{(\alpha,\beta)}(x)\left\|f''\right\|_B,$$

where $\delta_{n,q}^{(\alpha,\beta)}(x) = \left(\frac{1-q^3}{q^4} + \frac{514(\alpha+\beta+1)^2[n]_q}{q^4[n-3]_q[n-2]_q}\right)x(x+1) + \frac{\alpha^2}{\left([n]_q+\beta\right)^2}$.

Proof. Using Taylor's expansion

$$f(t) = f(x) + (t-x)f'(x) + \int_x^t (t-u)f''(u)du$$

and from Lemma 3, we obtain

$$\overline{M}_{n,q}^{(\alpha,\beta)}(f,x) - f(x) = \overline{M}_{n,q}^{(\alpha,\beta)}\left(\int_x^t (t-u)f''(u)du; x\right).$$

Then, using the Lemma1 and the inequality

$$\left|\int_x^t (t-u)f''(u)du\right| \le \|f''\|_B \frac{(t-x)^2}{2},$$

we get

$$\begin{aligned}
&\left|\overline{M}_{n,q}^{(\alpha,\beta)}(f,x) - f(x)\right| \\
&\le \left|M_{n,q}^{(\alpha,\beta)}\left(\int_x^t (t-u)f''(u)du, x\right)\right. \\
&\quad \left. - \int_x^{\frac{[n]_q^2 x}{q\left([n]_q+\beta\right)[n-2]_q} + \frac{\alpha}{[n]_q+\beta}} \left(\frac{[n]_q^2 x}{q\left([n]_q+\beta\right)[n-2]_q} + \frac{\alpha}{[n]_q+\beta} - u\right)f''(u)du\right| \\
&\le \frac{\|f''\|_B}{2} M_{n,q}^{(\alpha,\beta)}((t-x)^2, x) + \frac{\|f''\|_B}{2}\left(\left(\frac{[n]_q^2 x}{q\left([n]_q+\beta\right)[n-2]_q} - 1\right)x\right. \\
&\quad \left. + \frac{\alpha}{[n]_q+\beta}\right)^2
\end{aligned}$$

$$\leq \frac{\|f''\|_B}{2}\left\{\left(\frac{2\left(1-q^3\right)}{q^4}+\frac{512(\alpha+\beta+1)^2[n]_q}{q^4[n-3]_q[n-2]_q}\right)x(x+1)+\frac{\alpha^2}{\left([n]_q+\beta\right)^2}\right\}$$
$$+\frac{\|f''\|_B}{2}\left\{\frac{[n]_q^4\,x^4}{q^2\left([n]_q+\beta\right)^2[n-2]_q^2}-2\frac{[n]_q^2\,x^2}{q\left([n]_q+\beta\right)[n-2]_q}+x^2\right.$$
$$\left.+2\frac{\alpha}{[n]_q+\beta}\left(\frac{[n]_q^2\,x}{q\left([n]_q+\beta\right)[n-2]_q}-1\right)x+\left(\frac{\alpha}{[n]_q+\beta}\right)^2\right\}$$
$$\leq\left\{\left(\frac{1-q^3}{q^4}+\frac{514(\alpha+\beta+1)^2[n]_q}{q^4[n-3]_q[n-2]_q}\right)x(x+1)+\frac{\alpha^2}{\left([n]_q+\beta\right)^2}\right\}\|f''\|_B.$$

And the proof of the Lemma 4 is now completed. ■

Theorem 1. *Let $(q_n)\subset(0,1)$ a sequence such that $q_n\to 1$ as $n\to\infty$. Then for every $n>3$, $x\in[0,\infty)$ and $f\in C_B[0,\infty)$, we have the inequality*

$$\left|M_{n,q_n}^{(\alpha,\beta)}(f,x)-f(x)\right|\leq 2M\omega_2\left(f,\sqrt{\delta_{n,q_n}^{(\alpha,\beta)}(x)}\right)+w\left(f,\eta_{n,q_n}^{(\alpha,\beta)}(x)\right),$$

where $\eta_{n,q_n}^{(\alpha,\beta)}(x)=\left(\frac{[n]_{q_n}^2}{q_n\left([n]_{q_n}+\beta\right)[n-2]_{q_n}}-1\right)x+\frac{\alpha}{[n]_{q_n}+\beta}$.

Proof. Using (2.6) for any $g\in W_\infty^2$, we obtain the equality

$$\left|M_{n,q_n}^{(\alpha,\beta)}(f,x)-f(x)\right|\leq\left|\overline{M}_{n,q_n}^{(\alpha,\beta)}(f-g,x)-(f-g)(x)+\overline{M}_{n,q_n}^{(\alpha,\beta)}(g,x)-g(x)\right|$$
$$+\left|f\left(\frac{[n]_{q_n}^2\,x}{q\left([n]_{q_n}+\beta\right)[n-2]_{q_n}}+\frac{\alpha}{[n]_{q_n}+\beta}\right)-f(x)\right|.$$

From Lemma 4, we get

$$\left|M_{n,q_n}^{(\alpha,\beta)}(f,x)-f(x)\right|\leq 2\,||f-g||_B+\delta_{n,q_n}^{(\alpha,\beta)}(x)\,\|g''\|_B$$
$$+\left|f\left(\frac{[n]_{q_n}^2\,x}{q_n\left([n]_{q_n}+\beta\right)[n-2]_{q_n}}+\frac{\alpha}{[n]_{q_n}+\beta}\right)-f(x)\right|.$$

By using equality (2.4) we have

$$\left|M_{n,q_n}^{(\alpha,\beta)}(f,x)-f(x)\right|\leq 2\left\|f-g\right\|_B+\delta_{n,q_n}^{(\alpha,\beta)}(x)\left\|g''\right\|_B+w\left(f,\eta_{n,q_n}^{(\alpha,\beta)}(x)\right).$$

Taking infimum over $g\in W_\infty^2$ on the right hand side of the above inequality and using the inequality (2.5), we get the desired result. ■

Theorem 2. *Let* $(q_n)\subset(0,1)$ *a sequence such that* $q_n\to1$ *as* $n\to\infty$. *Then* $f\in C_\rho^*[0,\infty)$, *we have*

$$\lim_{n\to\infty}\|M_{n,q_n}^{(\alpha,\beta)}(f)-f\|_\rho=0.$$

Proof. From Lemma 1, it is obvious that $\|M_{n,q_n}^{(\alpha,\beta)}(e_0)-e_0\|_\rho=0$. Since $\left|\frac{[n]_{q_n}^2}{q_n\left([n]_{q_n}+\beta\right)[n-2]_{q_n}}x+\frac{\alpha}{[n]_{q_n}+\beta}-x\right|\leq(x+1)o(1)$ and $\frac{x+1}{1+x^2}$ is positive and bounded from above for each $x\geq0$, we obtain

$$\|M_{n,q_n}^{(\alpha,\beta)}(e_1)-e_1\|_\rho\leq\frac{x+1}{1+x^2}o(1).$$

And then $\lim\limits_{n\to\infty}\|M_{n,q_n}^{(\alpha,\beta)}(e_1)-e_1\|_\rho=0.$

Similarly for every $n>3$, we write

$$\begin{aligned}&\|M_{n,q_n}^{(\alpha,\beta)}(e_2)-e_2\|_\rho\\&=\sup_{x\in[0,\infty)}\left\{\frac{\left|\frac{[n]_{q_n}^4}{q_n\left([n]_{q_n}+\beta\right)^2[n-3]_{q_n}[n-2]_{q_n}}x^2\right.}{1+x^2}\right.\\&\left.+\frac{\left.\left\{\frac{[2]_{q_n}[n]_{q_n}^3}{q_n\left([n]_{q_n}+\beta\right)^2[n-3]_{q_n}[n-2]_{q_n}}+\frac{2[n]_{q_n}^2\alpha}{[n-2]_{q_n}\left([n]_{q_n}+\beta\right)^2}\right\}x+\frac{\alpha^2}{\left([n]_{q_n}+\beta\right)^2}-x^2\right|}{1+x^2}\right\}\\&\leq\sup_{x\in[0,\infty)}\frac{1+x+x^2}{1+x^2}o(1),\end{aligned}$$

we get $\lim\limits_{n\to\infty}\|M_{n,q_n}^{(\alpha,\beta)}(e_2)-e_2\|_\rho=0$. Thus, from A.D. Gadzhiev's Theorem in [17], we obtain desired result of Theorem 2. ■

Lemma 5. *Let* $f\in C_\rho[0,\infty),(q_n)\subset(0,1)$ *a sequence such that* $q_n\to1$ *as* $n\to\infty$ *and* $\omega_{[0,b+1]}(f,\delta)$ *be its modulus of continuity on the finite interval* $[0,b+1]$,

$b > 0$. *Then for every* $n > 3$, *there exists a constant* $C > 0$ *such that the inequality holds*

$$\|M_{n,q_n}^{(\alpha,\beta)}(f,x) - f(x)\|_{C[0,b]} \leq C\left\{(b+1)^2\xi_{n,q_n}^{(\alpha,\beta)}(b) + \omega_{[0,b+1]}\left(f; \sqrt{\xi_{n,q_n}^{(\alpha,\beta)}(b)}\right)\right\},$$

where

$$\xi_{n,q_n}^{(\alpha,\beta)}(b) = \left(\frac{2(1-q_n^3)}{q_n^4} + \frac{512(\alpha+\beta+1)^2[n]_{q_n}}{q_n^4[n-3]_{q_n}[n-2]_{q_n}}\right)b(b+1) + \frac{\alpha^2}{\left([n]_{q_n}+\beta\right)^2}.$$

Proof. Let $x \in [0,b]$ and $t > b+1$. Since $t - x > 1$, we have

$$\begin{aligned} |f(t) - f(x)| &\leq N_f(2 + (t-x+x)^2 + x^2) \\ &\leq 3N_f(1+b)^2(t-x)^2. \end{aligned} \tag{2.7}$$

Let $x \in [0,b]$, $t < b+1$ and $\delta > 0$. Then, we have

$$|f(t) - f(x)| \leq \left(1 + \frac{|t-x|}{\delta}\right)\omega_{[0,b+1]}(f,\delta). \tag{2.8}$$

Due to(2.7) and (2.8), we can write

$$|f(t) - f(x)| \leq 3N_f(1+b)^2(t-x)^2 + \left(1 + \frac{|t-x|}{\delta}\right)\omega_{[0,b+1]}(f,\delta).$$

Then, using Cauchy- Schwarz' s inequality and Lemma 2, we get

$$\begin{aligned} &\left|M_{n,q_n}^{(\alpha,\beta)}(f,x) - f(x)\right| \\ &\leq 3N_f(1+b)^2M_{n,q_n}^{(\alpha,\beta)}\left((t-x)^2,x\right) + \omega_{[0,b+1]}(f;\delta)\left[1 + \frac{1}{\delta}\left(M_{n,q_n}^{(\alpha,\beta)}\left((t-x)^2,x\right)\right)^{1/2}\right] \\ &\leq 3N_f(1+b)^2\xi_{n,q_n}^{(\alpha,\beta)}(x) + \omega_{[0,b+1]}(f;\delta)\left[1 + \frac{1}{\delta}\left(\xi_{n,q_n}^{(\alpha,\beta)}(x)\right)^{1/2}\right], \end{aligned}$$

where

$$\xi_{n,q_n}^{(\alpha,\beta)}(x) = \left(\frac{2(1-q_n^3)}{q_n^4} + \frac{512(\alpha+\beta+1)^2[n]_{q_n}}{q_n^4[n-3]_{q_n}[n-2]_{q_n}}\right)x(x+1) + \frac{\alpha^2}{\left([n]_{q_n}+\beta\right)^2}.$$

Choosing,

$$\delta^2 := \xi_{n,q_n}^{(\alpha,\beta)}(b) = \left(\frac{2\left(1-q_n^3\right)}{q_n^4} + \frac{512(\alpha+\beta+1)^2[n]_{q_n}}{q_n^4[n-3]_{q_n}[n-2]_{q_n}}\right)b(b+1) + \frac{\alpha^2}{\left([n]_{q_n}+\beta\right)^2}$$

and $C = \min\{3N_f, 2\}$. We reach the proof of Lemma 5. ∎

Theorem 3. *Let* $\gamma > 0$, $(q_n) \subset (0,1)$ *a sequence such that* $q_n \to 1$ *as* $n \to \infty$ *and* $f \in C_\rho^*[0,\infty)$. *Then, we have*

$$\lim_{n\to\infty} \sup_{x\geq 0} \frac{\left|M_{n,q_n}^{(\alpha,\beta)}(f,x) - f(x)\right|}{1+x^{2+\gamma}} = 0.$$

Proof. For $\gamma > 0$, $f \in C_\rho^*[0,\infty)$ and $b \geq 1$ the following inequality is satisfied

$$\begin{aligned}
&\sup_{x\geq 0} \frac{\left|M_{n,q_n}^{(\alpha,\beta)}(f,x) - f(x)\right|}{1+x^{2+\gamma}} \\
&\leq \sup_{0\leq x<b} \frac{\left|M_{n,q_n}^{(\alpha,\beta)}(f,x) - f(x)\right|}{1+x^{2+\gamma}} + \sup_{b\leq x} \frac{\left|M_{n,q_n}^{(\alpha,\beta)}(f,x) - f(x)\right|}{1+x^{2+\gamma}} \\
&\leq \left\|M_{n,q_n}^{(\alpha,\beta)}(f,x) - f(x)\right\|_{C[0,b]} + \sup_{b\leq x} \frac{\left|M_{n,q_n}^{(\alpha,\beta)}(f,x) - f(x)\right|}{1+x^{2}} \\
&\leq \left\|M_{n,q_n}^{(\alpha,\beta)}(f,x) - f(x)\right\|_{C[0,b]} + \left\|M_{n,q_n}^{(\alpha,\beta)}(f) - f\right\|_{\rho}.
\end{aligned}$$

Using Lemma 5 and Theorem 2, we complete the proof of Theorem 3. ∎

Remark 1. A q-analogue of the Szász-Beta type operators, which was studied by Gupta et al in [18] can be studied similarly.

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