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THE TIME OPTIMAL CONTROL OF PARABOLIC INTEGRODIFFERENTIAL EQUATION IN REACTOR DYNAMICS

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Abstract. In this paper, we first study the lack of global null controllability and the local null controllability respectively. Based on the null controllability, we establish the existence of the time optimal control for nonlinear parabolic integrodifferential equations with internal distributive control.

1. Introduction

Let $T > 0$ and Ω be a bounded domain in $\mathbb{R}^N$, with the smooth boundary $\partial\Omega$. We consider the following controlled parabolic integrodifferential equation:

$$(1.1) \qquad \begin{cases} y_t - \Delta y = y \displaystyle\int_0^t y(x,s)ds + by + \chi_\omega u, & \text{in } Q_\infty, \\ y(x,t) = 0, & \text{on } \Sigma_\infty, \\ y(x,0) = y_0(x), & \text{in } \Omega, \end{cases}$$

where y and u are respectively the state variable and the control variable. b is a constant. Put $Q_\infty = \Omega \times (0,\infty)$ and $\Sigma_\infty = \partial\Omega \times (0,\infty)$. Assume ω to be two given nonempty open subset of Ω. Denote by χ_ω the characteristic function of the set ω.

The equation (1.1) with $u \equiv 0$ arises in the analysis of space time dependent nuclear reactor dynamics. If the effect of a linear temperature feedback is taken into consideration and the reactor model is considered as an infinite rod, then the one group

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neutron flux $y(x,t)$ and the temperature $v(x,t)$ in the reactor are given by the following coupled equation (see [4, 7] and [9]):

$$(1.2)\qquad \begin{cases} y_t - (a(x)y_x)_x = (c_1 v + c_2 - 1)\Sigma_f y, & -\infty < x < \infty,\ t > 0,\\ \tilde{\rho} c_2 v_t = c_3 \Sigma_g y, \end{cases}$$

where a is the diffusion coefficient and $\Sigma_f, \Sigma_g, \tilde{\rho}, c_1, c_2, c_3$ are the physical quantities. By integrating the second equation in (1.2) in the interval $(0,t)$ and substituting it into the first equation, we get the following nonlinear integrodifferential diffusion equation:

$$y_t - (a(x)y_x)_x = \beta y \int_0^t y(x,s)ds + by \ -\infty < x < \infty,\ t > 0,$$

where β, b are the constants associated with the initial temperature and various physical parameters. Throughout the paper, we will take $a \equiv 1$ and $\beta = 1$ for the sake of simplicity. All the results can be extended without difficulty to the other diffusion coefficient which is uniformly elliptic and β arbitrary.

Let the control function u taken from a given set

$$\mathcal{U}_\rho = \{u \in L^\infty(\Omega \times (0,\infty)); \|u\|_{L^\infty(\Omega\times(0,\infty))} \le \rho\},$$

where ρ is a positive constant. In this paper, we shall study the following time optimal control problem:

$$\text{(P)}\quad \min \mathcal{T} := \min\{\, T; y(\cdot,T) = 0,\ \text{a.e. in } \Omega, u \in \mathcal{U}_\rho \text{ and } y \text{ is the solution to (1.1) corresponding to } u\,\}.$$

A function $u \in \mathcal{U}_\rho$ is called admissible if the corresponding solution y to (1.1) satisfying $y(\cdot,T) = 0,$ a.e. in Ω for some $T > 0$. And $T^* := \min\{T; y(\cdot,T) = 0,\ \text{a.e. in } \Omega, u \in \mathcal{U}_\rho\}$ is called the minimal time for (P) and a control $u^* \in \mathcal{U}_\rho$ such that $y^*(\cdot,T^*) = 0,$ a.e. in Ω is called a time optimal control.

The time optimal control problem was studied first for the finite-dimensional case [6]. Thereafter, the problem was developed to infinite-dimensional case (see [1, 2] and [8]). However, the method is suitable only for the case where the control is distributed in the whole domain Ω. In [13], the authors obtained the existence of a time optimal control for phase-field systems with control distributed in a subdomain $\omega \subset \Omega$. Their method is based on a modified Carleman inequality. Recently, in [14], by the local null controllability and some special type of feedback stabilization, the time optimal control was obtained for the general nonlinear parabolic equation. It follows from [14] that the key to get the existence of a time optimal control on a local domain ω is to show the existence of an admissible control which is related to a type of controllability of the equation with some suitable control functions.

Our main aim in this paper is to show the existence of the time optimal control for (1.1). Following some of the ideas developed by Wang [14], the proof is based on the controllability control for (1.1). We prove the null controllability of system (1.1) does not hold for large initial data. In other words, no matter what control function is chosen, making use of a localized estimate in $\Omega\backslash\bar{\omega}$, we can see that the blow-up phenomena will still happen. On the other hand, we shall show that the system (1.1) is null controllability for small initial data. It is worth mentioning that the local null controllability we obtain improved the results in [10] and [11], since we do not need the technical condition on the kernel at 0 and T. With the help of null controllability, we obtain the existence of the time optimal control.

2. Exact Null Controllability

In this section, we prove the lack of global null controllability and the local null controllability respectively. Let $T > 0$. Denote $Q_T = \Omega\times(0,T)$ and $\Sigma_T = \partial\Omega\times(0,T)$. Consider the following system:

$$
\begin{cases} y_t - \Delta y = y\displaystyle\int_0^t y(x,s)ds + by + \chi_\omega u, & \text{in } Q_T, \\ y(x,t) = 0, & \text{on } \Sigma_T, \\ y(x,0) = y_0(x), & \text{in } \Omega. \end{cases} \tag{2.1}
$$

As in [12], for any $y_0 \in L^\infty(\Omega)\cap H_0^1(\Omega)$ and $u \in L^\infty(\Omega\times(0,T))$, we can give the definition of the weak solution to (2.1) in $C([0,T];L^2(\Omega))\cap L^2(0,T;H_0^1(\Omega))\cap L^\infty(Q_T)$. The existence and uniqueness of the local weak solution can be obtained by exactly the same methods as used in [12]. So, we omit the proof here.

2.1. Lack of global null controllability

We first give the proof of the lack of global null controllability for the system (2.1). We prove a localized estimate in $\Omega\backslash\bar{\omega}$ which shows that the control cannot compensate the blow-up phenomena occurring.

Theorem 2.1. *Let $T > 0$. Then, there exist initial conditions $y_0 \in L^\infty(\Omega)\cap H_0^1(\Omega)$ such that for any control function $u \in L^\infty(\Omega\times(0,T))$ the associated solution y to (2.1) is not identically equal zero at time T .*

Proof. In order to obtain this result, we employ the eigenfunction method. Let $\omega' \subset \Omega$ be a subdomain of Ω such that $\omega \subset \omega'$ and $E = \Omega\setminus\overline{\omega'}$. Let φ_1 be the first eigenfunction of the following eigenvalue problem:

$$
\begin{cases} -\Delta\varphi_1 = \lambda_1\varphi_1, & \text{in } E, \\ \varphi_1(x) = 0, & \text{on } \partial E, \end{cases}
$$

where λ_1 is the first eigenvalue. Then it is known that φ_1 is a non-negative smooth function on $\overline{E}$ and φ_1 is positive in E. In particular, we shall normalize φ_1 in sup-norm, that is, $\sup\limits_{x\in\overline{E}}\varphi_1(x)=1$. Denote

$$\rho=\begin{cases}\varphi_1, & \text{in } E,\\ 0, & otherwise.\end{cases}$$

Multiplying (1.1) by ρ, we have

$$\begin{aligned}\rho y_t-\rho\Delta y&=\rho y\int_0^t y(x,s)ds+\rho by\\ &\geq\frac{1}{2}\partial_t(\int_0^t\rho y(x,s)ds)^2+\rho by.\end{aligned}$$

Integrating this by parts over $(0,t)\times\Omega$, we get

$$\int_\Omega\rho y(x,t)dx-\int_0^t\int_\Omega\Delta\rho ydxdt\geq\int_\Omega\rho y_0dx+\frac{1}{2}\int_\Omega(\int_0^t\rho ydt)^2dx+\int_0^t\int_\Omega\rho bydxdt,$$

that is,

$$\int_E\varphi_1y(x,t)dx-\int_0^t\int_E\Delta\varphi_1ydxdt\geq\int_E\varphi_1y_0dx+\frac{1}{2}\int_E(\int_0^t\varphi_1ydt)^2dx+b\int_0^t\int_E\varphi_1ydxdt.$$

According to Hölder's inequality, we obtain

$$\begin{aligned}&\int_E\varphi_1y(x,t)dx\\ &\geq\int_E\varphi_1y_0dx+(b-\lambda_1)\int_0^t\int_E\varphi_1ydxdt+\frac{1}{2}\int_E(\int_0^t\varphi_1ydt)^2dx\\ &\geq\int_E\varphi_1y_0dx+(b-\lambda_1)\int_0^t\int_E\varphi_1ydxdt+\frac{1}{2|\Omega|}(\int_0^t\int_E\varphi_1ydxdt)^2.\end{aligned}\tag{2.2}$$

Define the function

$$Z(t)=\int_0^t\int_E\varphi_1ydxdt,$$

then, $Z\in C^1([0,1))$ with $Z(0)=0$. In view of (2.2), we have

$$Z'\geq\int_E\varphi_1y_0dx+(b-\lambda_1)Z+\frac{1}{2|\Omega|}Z^2,$$

where $'$ denotes once differentiation with respect to time t. Thus, we easily see that if we take an arbitrary initial data $y_0 \geq 0$ such that

$$\int_E \varphi_1 y_0 dx > \frac{(b-\lambda_1)^2|\Omega|}{2},$$

then Z blows up in finite time T^* (dependent of y_0). Obviously, as $y_0 \to \infty$, the blow-up time of Z tends to zero and so is y. This completes the proof of Theorem 2.1. ■

2.2. Local null controllability

Let us consider the well-known linear parabolic system:

$$\begin{cases} y_t - \Delta y = ay + \chi_\omega u, & \text{in } Q_T, \\ y(x,t) = 0, & \text{on } \Sigma_T, \\ y(x,0) = y_0(x), & \text{in } \Omega, \end{cases} \tag{2.3}$$

where $a \in L^\infty(Q_T)$. The following result holds:

Lemma 2.1. ([3]). *For any $T > 0$, any $a \in L^\infty(Q_T)$ and any $y_0 \in L^2(\Omega)$, there exist controls $u \in L^\infty(\omega \times (0,T))$ such that the corresponding solution of (2.3) satisfies $y(x,T) = 0$, a.e. in Ω. Furthermore, u can be chosen such that the following estimate holds:*

$$\begin{aligned} &\|u\|_{L^\infty(\omega\times(0,T))} \\ &\leq \exp\left[C\left(1+\frac{1}{T}+T+(T^{1/2}+T)\|a\|_{L^\infty(Q_T)}+\|a\|^{2/3}_{L^\infty(Q_T)}\right)\right]\|y_0\|_{L^2(\Omega)}. \end{aligned} \tag{2.4}$$

From the classical L^∞ estimates on the solutions of (2.3) [5], we can get the following result.

Corollary 2.1. *For any $T > 0$, any $a \in L^\infty(Q_T)$ and any $y_0 \in L^\infty(\Omega) \cap H_0^1(\Omega)$, there exist controls $u \in L^\infty(\Omega \times (0,T))$ such that (2.3) has a solution $y \in C([0,T]; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega)) \cap L^\infty(Q_T)$ satisfying $y(x,T) = 0$, a.e. in Ω. Furthermore, u can be chosen such that the following estimate holds:*

$$\begin{aligned} &\|u\|_{L^\infty(\Omega\times(0,T))} \\ &\leq \exp\left[C\left(1+\frac{1}{T}+T+(T^{1/2}+T)\|a\|_{L^\infty(Q_T)}+\|a\|^{2/3}_{L^\infty(Q_T)}\right)\right]\|y_0\|_{L^\infty(\Omega)}. \end{aligned} \tag{2.5}$$

With the null controllability for the linear parabolic system at hand, we are in a position to prove the local null controllability of (1.1). The proof is based on the Kakutani fixed point theorem.

Theorem 2.2. *Let $T > 0$. Then there exists a constant $c > 0$ such that for each $y_0 \in L^\infty(\Omega) \cap H_0^1(\Omega)$ satisfying*

$$\|y_0\|_{L^\infty(\Omega)} \leq \exp\left[-c\left(1 + T + \frac{1}{T}\right)\right] min\{1, \rho\}, \tag{2.6}$$

there is a control function $u \in \mathcal{U}_\rho$ such that the system (1.1) has a solution $y \in C([0,T]; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega)) \cap L^\infty(Q_T)$ satisfying $y(x,T) = 0$, a.e. in Ω.

Proof. Define

$$K_T = \{z \in L^\infty(Q_T); \|z\|_{L^\infty(Q_T)} \leq 1\}.$$

For each $z \in K_T$, we consider the controllability of the following linearized system:

$$\begin{cases} y_t - \Delta y = y\displaystyle\int_0^t z(x,s)ds + by + \chi_\omega u, & \text{in } Q_T, \\ y(x,t) = 0, & \text{on } \Sigma_T, \\ y(x,0) = y_0(x), & \text{in } \Omega. \end{cases} \tag{2.7}$$

Now, by Corollary 2.1, the system (2.7) is null controllable at time T. Moreover, it follows from (2.5) that

$$\|u\|_{L^\infty(\Omega\times(0,T))} \leq \exp\left[C_0\left(1 + T + \frac{1}{T}\right)\right] \|y_0\|_{L^\infty(\Omega)}. \tag{2.8}$$

where the constant C_0 is independent of T.

For each $z \in K_T$, we define a map $\Phi : K_T(\subset L^2(Q_T)) \to 2^{L^2(Q_T)}$ by

$$\begin{aligned}\Phi(z) = \{y \in C([0,T]; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega)) \cap L^\infty(Q_T);\ \exists\, u \in L^\infty(\Omega \times (0,T)) \\ \text{satisfying (2.8) such that } y \text{ is the solution to the system} \\ \text{(2.3) corresponding to } z, u \text{ and } y(x,T) = 0, \text{ a.e. in } \Omega\}.\end{aligned}$$

It is readily seen that $\Phi(z)$ is nonempty, closed and convex in $L^2(Q_T)$. Then we prove $\Phi(K_T) \subset K_T$. By (2.5) and the classical L^∞ estimates on the solutions of (2.7), we have

$$\begin{aligned}\|y\|_{L^\infty(Q_T)} &\leq e^{T^2+bT}\|y_0\|_{L^\infty(\Omega)} + Te^{T^2+bT}\|u\|_{L^\infty(\Omega\times(0,T))} \\ &\leq \exp\left[C_1C_0\left(1 + T + \frac{1}{T}\right)\right] \|y_0\|_{L^\infty(\Omega)},\end{aligned}$$

where the constant $C_1 > 1$ is independent of T. Let $c = C_1C_0$. It follow from (2.6) that

$$\|y\|_{L^\infty(Q_T)} \leq 1.$$

This implies that $\Phi(K_T) \subset K_T$.

Moreover, from parabolic regularity, $\Phi(K_T)$ is a relatively compact subset of $L^2(Q_T)$ and Φ is upper semicontinuous in $L^2(Q_T)$ exactly as in [3].

Then applying the Kakutani fixed point theorem (see, for instance, [1]), we infer that there is at least one $y \in K_T$ such that $y \in \Phi(y)$. Moreover, by (2.5) and (2.8), it is easy to check that $u \in \mathcal{U}_\rho$ (which can be extended by 0 to the whole time interval). Hence, our assertion is proved. ∎

3. Time Optimal Control

Based on the null controllability of the system (1.1), we now study the existence of the time optimal control.

Theorem 3.1. *Let c be the constant in Theorem 2.2. Then, for each $y_0 \in L^\infty(\Omega) \cap H_0^1(\Omega)$ satisfying*

(3.1) $$\|y_0\|_{L^\infty(\Omega)} \leq \exp(-3c) min\{1, \rho\},$$

there exists at least one time optimal control for problem (P).

Proof. Note that the function $\exp[-c(1 + T + \frac{1}{T})]$ attains its maximum $\exp(-3c)$ at $T = 1$. It follows from Theorem 2.2 that for any $y_0 \in L^\infty(\Omega) \cap H_0^1(\Omega)$ satisfying (3.1), there is a control function $u \in \mathcal{U}_\rho$ such that the solution of the system (1.1)(where $T = 1$) satisfying $y(x, 1) = 0$, a.e. in Ω. This implies that the set $\mathcal{T}$ is nonempty.

Let $T^* = \min \mathcal{T}$. Then there exist sequences $\{T_n\}_{n=1}^\infty$ with $T_n \to T^*, T_n \geq T^*$ and $\{u_n\}_{n=1}^\infty \subset \mathcal{U}_\rho$ such that the solutions y_n to (1.1), where $u = u_n$, satisfies $y_n(\cdot, T_n) = 0$, a.e. in Ω for all $n \in \mathbb{N}^+$. Denote

$$\tilde{u}_n(x, t) = \begin{cases} u_n(x, t),\ x \in \Omega,\ 0 \leq t \leq T_n, \\ 0,\ x \in \Omega,\ t > T_n. \end{cases}$$

Let $\tilde{y}_n$ be the solutions of (1.1) with $u = \tilde{u}_n$. Then, $\tilde{y}_n(\cdot, T_n) = 0$, a.e. in Ω. Given $T > T^*$, we can take $n^* \in \mathbb{N}^+$ such that $T \geq T_n$ for all $n > n^*$. Since $\|\tilde{u}_n\|_{L^\infty(Q_T)} \leq \rho$, there exist a subsequence of $\{\tilde{u}_n\}_{n=1}^\infty$, denoted also by itself, and a $\tilde{u}^*$ such that

(3.2) $$\tilde{u}_n \to \tilde{u}^* \text{ weakly star in } L^\infty(Q_T),$$

which implies $\|\tilde{u}^*\|_{L^\infty(Q_T)} \leq \rho$.

By the standard energy estimate and the boundedness of $\{\tilde{y}_n\}_{n=1}^\infty$ in $L^\infty(Q_T)$ (from the proof in Theorem 2.2 $\tilde{y}_n \in K_T$), there is $\tilde{y}^*$ such that (selecting a subsequence if necessary)

$$
\tilde{y}_n \to \tilde{y}^* \text{ weakly star in } L^\infty(Q_T), \tag{3.3}
$$
$$
\text{weakly in } W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)),
$$
$$
\text{strongly in } C([0,T];L^2(\Omega)).
$$

It follows from (3.2) and (3.3) that

$$
\begin{cases} \tilde{y}_t^* - \Delta \tilde{y}^* = \tilde{y}^* \displaystyle\int_0^t \tilde{y}^*(x,s)ds + b\tilde{y}^* + \chi_\omega \tilde{u}^*, & \text{in } Q_T, \\ \tilde{y}^*(x,t) = 0, & \text{on } \Sigma_T, \\ \tilde{y}^*(x,0) = y_0(x), & \text{in } \Omega. \end{cases}
$$

Since $\tilde{y}_n(\cdot, T_n) = 0$, a.e. in Ω, we have

$$
\begin{aligned} \|\tilde{y}^*(\cdot,T^*) - \tilde{y}_n(\cdot,T_n)\|_{L^2(\Omega)} \le & \|\tilde{y}^*(\cdot,T^*) - \tilde{y}_n(\cdot,T^*)\|_{L^2(\Omega)} \\ & + \|\tilde{y}_n(\cdot,T_n) - \tilde{y}_n(\cdot,T^*)\|_{L^2(\Omega)} \to 0, \end{aligned}
$$

as $n \to \infty$, which implies $\tilde{y}^*(\cdot, T^*) = 0$. Finally, denote

$$
u^*(x,t) = \begin{cases} \tilde{u}^*(x,t), \ x \in \Omega, \ 0 \le t \le T^*, \\ 0, \ x \in \Omega, \ t > T^*. \end{cases}
$$

and y^* be the solution to (1.1) with $u = u^*$. Then, $y^*(\cdot, T^*) = 0$, a.e. in Ω, and hence T^* is the optimal time of the problem (P). This completes the proof. ■

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