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COMPACTNESS OF THE COMMUTATOR OF BILINEAR FOURIER MULTIPLIER OPERATOR

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Abstract. Let $b_1,\, b_2 \in \mathrm{CMO}(\mathbb{R}^n)$ and T_σ be the bilinear Fourier multiplier operator with associated multiplier σ satisfies the Sobolev regularity that $\sup_{\kappa\in\mathbb{Z}} \|\sigma_\kappa\|_{W^{s_1,\, s_2}(\mathbb{R}^{2n})} < \infty$ for some $s_1,\, s_2 \in (n/2,\, n]$. In this paper, it is proved that the commutator defined by

$$T_{\sigma,\vec{b}}(f_1, f_2)(x) = b_1(x)T_\sigma(f_1, f_2)(x)$$
$$-T_\sigma(b_1f_1, f_2)(x) + b_2(x)T_\sigma(f_1, f_2)(x) - T_\sigma(f_1, b_2f_2)(x)$$

is a compact operator from $L^{p_1}(\mathbb{R}^n)\times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ when $p_k \in (n/s_k,\, \infty)$ $(k = 1,\, 2)$, $p \in (1,\, \infty)$ with $1/p = 1/p_1 + 1/p_2$.

1. INTRODUCTION

As it is well known, the study of bilinear Fourier multiplier operator was origined by Coifman and Meyer. Let $\sigma \in L^\infty(\mathbb{R}^{2n})$. Define the bilinear Fourier multiplier operator T_σ by

$$T_\sigma(f_1, f_2)(x) = \int_{\mathbb{R}^{2n}} \exp(2\pi ix(\xi_1 + \xi_2))\sigma(\xi_1, \xi_2)\mathcal{F}f_1(\xi_1)\mathcal{F}f_2(\xi_2)\mathrm{d}\xi_1\mathrm{d}\xi_2 \tag{1.1}$$

for $f_1,\, f_2 \in \mathcal{S}(\mathbb{R}^n)$, where and in the following, for $f \in \mathcal{S}(\mathbb{R}^n)$, $\mathcal{F}f$ denotes the Fourier transform of f. Coifman and Meyer [5] proved that if $\sigma \in C^s(\mathbb{R}^{2n}\backslash\{0\})$ satisfies

$$|\partial^{\alpha_1}_{\xi_1}\partial^{\alpha_2}_{\xi_2}\sigma(\xi_1,\, \xi_2)| \le C_{\alpha_1,\,\alpha_2}(|\xi_1| + |\xi_2|)^{-(|\alpha_1|+|\alpha_2|)} \tag{1.2}$$

for all $|\alpha_1|+|\alpha_2| \le s$ with $s \ge 4n+1$, then T_σ is bounded from $L^{p_1}(\mathbb{R}^n)\times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for all $1 < p_1,\, p_2,\, p < \infty$ with $1/p = 1/p_1+1/p_2$. For the case of $s \ge 2n+1$,

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Kenig-Stein [14] and Grafakos-Torres [10] improved Coifman and Meyer's multiplier theorem to the indices $1/2 \le p \le 1$ by the multilinear Calderón-Zygmund operator theory. In the last several years, considerable attention has been paid to the behavior on function spaces for T_σ when the multiplier satisfies certain Sobolev regularity condition. An significant progress in this area was obtained by Tomita. Let $\Phi \in \mathcal{S}(\mathbb{R}^{2n})$ satisfy

$$\begin{cases} \operatorname{supp}\Phi \subset \Big\{(\xi_1,\, \xi_2) :\, 1/2 \le |\xi_1| + |\xi_2| \le 2\Big\}; \\ \displaystyle\sum_{\kappa\in\mathbb{Z}} \Phi(2^{-\kappa}\xi_1,\, 2^{-\kappa}\xi_2) = 1 \qquad \text{for all } (\xi_1,\, \xi_2) \in \mathbb{R}^{2n}\backslash\{0\}. \end{cases} \tag{1.3}$$

For $\kappa \in \mathbb{Z}$, set

$$\sigma_\kappa(\xi_1,\, \xi_2) = \Phi(\xi_1,\, \xi_2)\sigma(2^\kappa\xi_1,\, 2^\kappa\xi_2). \tag{1.4}$$

Tomita [16] proved that if

$$\sup_{\kappa\in\mathbb{Z}} \int_{\mathbb{R}^{2n}} (1 + |\xi_1|^2 + |\xi_2|^2)^s |\mathcal{F}\sigma_\kappa(\xi_1,\, \xi_2)|^2 \mathrm{d}\xi_1 \mathrm{d}\xi_2 < \infty \tag{1.5}$$

for some $s > n$, then T_σ is bounded from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ provided that $p_1,\, p_2,\, p \in (1,\, \infty)$ and $1/p = 1/p_1 + 1/p_2$. Grafakos and Si [9] considered the mapping properties from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for T_σ when σ satisfies (1.5) and $p \le 1$. Miyachi and Tomita [15] considered the problem to find minimal smoothness condition for bilinear Fourier multiplier. Let σ satisfies the Sobolev regularity that

$$\|\sigma_\kappa\|_{W^{s_1,\, s_2}(\mathbb{R}^{2n})} = \Big(\int_{\mathbb{R}^{2n}} \langle\xi_1\rangle^{2s_1} \langle\xi_2\rangle^{2s_2} |\mathcal{F}\sigma_\kappa(\xi_1,\, \xi_2)|^2 \mathrm{d}\xi_1 \mathrm{d}\xi_2 \Big)^{1/2},$$

where $\langle\xi_k\rangle := (1 + |\xi_k|^2)^{1/2}$. Miyachi and Tomita [15] proved that if

$$\sup_{\kappa\in\mathbb{Z}} \|\sigma_\kappa\|_{W^{s_1,\, s_2}(\mathbb{R}^{2n})} < \infty \tag{1.6}$$

for some $s_1,\, s_2 \in (n/2,\, n]$, then T_σ is is bounded from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for any $p_1,\, p_1 \in (1,\, \infty)$ and $p \ge 2/3$ with $1/p = 1/p_1 + 1/p_2$. Moreover, they also gives minimal smoothness condition for which T_σ is bounded from $H^{p_1}(\mathbb{R}^n) \times H^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$. For other works about the behavior of T_σ on various function spaces, we refer the papers [8, 7, 12] and the related references therein.

We now consider the commutator of the multiplier operator T_σ. Let T_σ be the multiplier operator definied by (1.1), $b_1,\, b_2 \in \mathrm{BMO}(\mathbb{R}^n)$ and $\vec{b} = (b_1,\, b_2)$. Define the commutator of $\vec{b}$ and T_σ by

$$T_{\sigma,\vec{b}}(f_1,\, f_2)(x) = \sum_{k=1}^{2} [b_k,\, T_\sigma]_k(f_1,\, f_2)(x), \tag{1.7}$$

with

$$[b_1, T_\sigma]_1(f_1, f_2)(x) = b_1(x)T_\sigma(f_1, f_2)(x) - T_\sigma(b_1 f_1, f_2)(x)$$

and

$$[b_2, T_\sigma]_2(f_1, f_2)(x) = b_2(x)T_\sigma(f_1, f_2)(x) - T_\sigma(f_1, b_2 f_2)(x).$$

Bui and Duong [3] established the weighted estimates with multiple weights for $T_{\sigma,\vec{b}}$ when σ satisfies (1.2) for $s \in (n, 2n]$. Hu and Yi [13] considered the behavior on $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ for $T_{\sigma,\vec{b}}$ when σ satisfies (1.6) for $s_1, s_2 \in (n/2, n]$, and showed that $T_{\sigma,\vec{b}}$ enjoys the same $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ mapping properties as that of the operator T_σ. In this paper, we will consider the compactness of $T_{\sigma,\vec{b}}$. Let $\mathrm{CMO}(\mathbb{R}^n)$ be the closure of $C_0^\infty(\mathbb{R}^n)$ in the $\mathrm{BMO}(\mathbb{R}^n)$ topology, which coincide with the space of functions of vanishing mean oscillation, see [2, 6]. Our main result in this paper can be stated as follows.

Theorem 1.1. *Let σ be a multiplier satisfying (1.6) for some $s_1, s_2 \in (n/2, n]$ and T_σ be the operator defined by (1.1). Let $t_k = n/s_k$, $p_k \in (t_k, \infty)$ $(k = 1, 2)$ and $p \in [1, \infty)$ with $1/p = 1/p_1 + 1/p_2$. Then for any $b_1, b_2 \in \mathrm{CMO}(\mathbb{R}^n)$, the commutator $T_{\sigma,\vec{b}}$ is a compact operators from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$.*

We remark that in this paper, we are very much motivated by the paper [17], and the recent work of Bényi and Torres [1]. Bényi and Torres [1] proved that if $b_1, b_2 \in \mathrm{CMO}(\mathbb{R}^n)$, and T is a bilinear Calderón-Zygmund operator, then for $p_1, p_2, \in (1, \infty)$, $p \in [1, \infty)$ with $1/p = 1/p_1 + 1/p_2$, the commutator $T_{\vec{b}}$ which is defined as (1.7), is a compact operator from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$. When the multiplier σ satisfies (1.6) for $s_1, s_2 \in (n/2, n]$, the operator T_σ is neither a bilinear Calderón-Zygmund operator, nor a bilinear singular integral operator whose kernel enjoys the bilinear L^r-Hörmander condition as in [3]. However, we can prove that T_σ can be approximated by a sequence of operator $\{T_{\sigma,N}\}_{N\in\mathbb{N}}$, and the kernels of $T_{\sigma,N}$ enjoy some variant of L^r-Hörmander condition, and certain L^r size condition. This will be useful in the proof of Theorem 1.1.

Throughout the article, C always denotes a positive constant that may vary from line to line but remains independent of the main variables. We use the symbol $A \lesssim B$ to denote that there exists a positive constant C such that $A \le CB$. For any set $E \subset \mathbb{R}^n$, χ_E denotes its characteristic function. We use $B(x, R)$ to denote a ball centered at x with radius R. For a ball $B \subset \mathbb{R}^n$ and $\lambda > 0$, we use λB to denote the ball concentric with B whose radius is λ times of B's.

2. Proof of Theorem 1.1.

Let $\sigma \in L^\infty(\mathbb{R}^{2n})$ and $\Phi \in \mathcal{S}(\mathbb{R}^{2n})$ satisfy (1.3). For $\kappa \in \mathbb{Z}$, define

$$\widetilde{\sigma}_\kappa(\xi_1, \xi_2) = \Phi(2^{-\kappa}\xi_1, 2^{-\kappa}\xi_2)\sigma(\xi_1, \xi_2).$$

Then

$$\widetilde{\sigma}_\kappa(\xi_1,\,\xi_2)=\sigma_\kappa(2^{-\kappa}\xi_1,\,2^{-\kappa}\xi_2)$$

and

$$\mathcal{F}^{-1}\widetilde{\sigma}_\kappa(\xi_1,\,\xi_2)=2^{2\kappa n}\mathcal{F}^{-1}\sigma_\kappa(2^{\kappa}\xi_1,\,2^{\kappa}\xi_2),$$

where $\mathcal{F}^{-1}f$ denotes the inverse Fourier transform of f.

Lemma 2.1. *Let* $q_1,\,q_2\in[2,\infty)$, *and* $s_1,\,s_2\geq 0$. *Then*

$$\Big(\int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n}|\mathcal{F}^{-1}\sigma_\kappa(\xi_1,\,\xi_2)|^{q_1}\langle\xi_1\rangle^{s_1}\mathrm{d}\xi_1\Big)^{q_2/q_1}\langle\xi_2\rangle^{s_2}\mathrm{d}\xi_2\Big)^{1/q_2}\lesssim\|\sigma_\kappa\|_{W^{s_1/q_1,s_2/q_2}(\mathbb{R}^{2n})}.$$

For the proof of Lemma 2.1, see Appendix A in [7].

Lemma 2.2. *Let* σ *be a bilinear multiplier satisfying (1.6) for some* $s_1,\,s_2\in(n/2,\,n]$, $r_1,\,r_2\in(1,\,2]$, $\gamma_1\in(n/r_1,\,s_1]$ *and* $\gamma_2\in(0,\,\min\{n/r_2,\,s_2\})$. *Then for every* $x\in\mathbb{R}^n$ *and* $R>0$,

$$\begin{aligned}&\int_{|x-y_1|\geq R}\int_{|x-y_2|<2R}|\mathcal{F}^{-1}\widetilde{\sigma}_\kappa(x-y_1,\,x-y_2)||f_1(y_1)f_2(y_2)|\mathrm{d}y_1\mathrm{d}y_2\\&\lesssim 2^{\kappa(n/r_1+n/r_2-\gamma_1-\gamma_2)}R^{n/r_1+n/r_2-\gamma_1-\gamma_2}\prod_{k=1}^{2}M_{r_k}f_k(x).\end{aligned}\tag{2.1}$$

Proof. Let $C(x,\,r)=B(x,\,2r)\backslash B(x,\,r)$. By the Hölder inequality and Lemma 2.1, we have

$$\begin{aligned}&\int_{C(x,\,r)}\int_{C(x,\,R)}|\mathcal{F}^{-1}\widetilde{\sigma}_\kappa(x-y_1,\,x-y_2)||f_1(y_1)f_2(y_2)|\mathrm{d}y_1\mathrm{d}y_2\\
\lesssim&\Big(\int_{C(x,\,r)}\Big(\int_{C(x,\,R)}|\mathcal{F}^{-1}\widetilde{\sigma}_\kappa(x-y_1,\,x-y_2)|^{r_2'}\langle 2^\kappa(x-y_2)\rangle^{r_2'\gamma_2}\mathrm{d}y_2\Big)^{\frac{r_1'}{r_2'}}\\
&\times\langle 2^\kappa(x-y_1)\rangle^{r_1'\gamma_1}\mathrm{d}y_1\Big)^{\frac{1}{r_1'}}(2^\kappa r)^{-\gamma_1}(2^\kappa R)^{-\gamma_2}\\
&\times\Big(\int_{C(x,\,r)}|f_1(y_1)|^{r_1}\mathrm{d}y_1\Big)^{\frac{1}{r_1}}\Big(\int_{C(x,\,R)}|f_2(y_1)|^{r_2}\mathrm{d}y_2\Big)^{\frac{1}{r_2}}\\
&\lesssim 2^{\kappa(n/r_1+n/r_2-\gamma_1-\gamma_2)}r^{n/r_1-\gamma_1}R^{n/r_2-\gamma_2}\prod_{k=1}^{2}M_{r_k}f_k(x),\end{aligned}$$

if $\gamma_k\in[0,\,s_k]$ with $k=1,\,2$. This in turn implies that

$$\begin{aligned}&\int_{r\leq|x-y_1|<2r}\int_{|x-y_2|<2R}|\mathcal{F}^{-1}\widetilde{\sigma}_\kappa(x-y_1,\,x-y_2)||f_1(y_1)f_2(y_2)|\mathrm{d}y_1\mathrm{d}y_2\\&\lesssim 2^{\kappa(n/r_1+n/r_2-\gamma_1-\gamma_2)}r^{n/r_1-\gamma_1}R^{n/r_2-\gamma_2}\prod_{k=1}^{2}M_{r_k}f_k(x),\end{aligned}$$

if $\gamma_2 < n/r_2$, and so

$$\begin{aligned}&\int_{|x-y_1|\ge r}\int_{|x-y_2|<2R}|\mathcal{F}^{-1}\widetilde{\sigma}_\kappa(x-y_1,\,x-y_2)||f_1(y_1)f_2(y_2)|\mathrm{d}y_1\mathrm{d}y_2\\&\lesssim 2^{\kappa(n/r_1+n/r_2-\gamma_1-\gamma_2)}r^{n/r_1-\gamma_1}R^{n/r_2-\gamma_2}\prod_{k=1}^{2}M_{r_k}f_k(x),\end{aligned}$$

if $\gamma_1 \in (n/r_1,\, s_1]$. Taking $r = R$ in the last inequality then gives (2.1).

Lemma 2.3. *Let σ be a bilinear multiplier satisfying (1.6) for some $s_1, s_2 \in (n/2, n]$, $r_1, r_2 \in (1, 2]$ with $r_2 s_2 > n$. Then for every $x \in \mathbb{R}^n$, $R > 0$ and $\gamma \in [0,\ \min\{s_1,\, 1+n/r_1\})$,*

$$\begin{aligned}&\int_{\mathbb{R}^n}\int_{|x-y_1|<R}|x-y_1||\mathcal{F}^{-1}\widetilde{\sigma}_\kappa(x-y_1,\,x-y_2)||f_1(y_1)f_2(y_2)|\mathrm{d}y_1\mathrm{d}y_2\\&\lesssim 2^{-\kappa(\gamma-n/r_1)}R^{1+n/r_1-\gamma}\textstyle\prod_{k=1}^{2}M_{r_k}f_k(x).\end{aligned}\tag{2.2}$$

Proof. Note that for $x \in \mathbb{R}^n$ and $\kappa \in \mathbb{Z}$,

$$\Big(\int_{\mathbb{R}^n}\frac{|f_2(y_2)|^{r_2}}{\langle 2^\kappa(x-y_2)\rangle^{s_2r_2}}\mathrm{d}y_2\Big)^{\frac{1}{r_2}}\lesssim 2^{-\kappa n/r_2}M_{r_2}f_2(x),$$

since $s_2 r_2 > n$. A trivial computation involving the Hölder inequality and Lemma 2.1 leads to that for $\gamma \in [0,\, s_1]$ and integer l

$$\begin{aligned}&\int_{\mathbb{R}^n}\int_{C(x,2^lR)}|\mathcal{F}^{-1}\widetilde{\sigma}_\kappa(x-y_1,x-y_2)||f_1(y_1)f_2(y_2)|\mathrm{d}y_1\mathrm{d}y_2\\&\lesssim M_{r_1}f_1(x)\Big(\int_{\mathbb{R}^n}\Big(\int_{C(x,2^lR)}|\mathcal{F}^{-1}\widetilde{\sigma}_\kappa\big(x-y_1,x-y_2\big)|^{r_1'}\langle 2^\kappa(x-y_1)\rangle^{r_1'\gamma}\mathrm{d}y_1\Big)^{\frac{r_2'}{r_1'}}\\&\quad\times\langle 2^\kappa(x-y_2)\rangle^{r_2's_2}\mathrm{d}y_2\Big)^{\frac{1}{r_2'}}\Big(\int_{\mathbb{R}^n}\frac{|f_2(y_2)|^{r_2}}{\langle 2^\kappa(x-y_2)\rangle^{s_2r_2}}\mathrm{d}y_2\Big)^{\frac{1}{r_2}}(2^lR)^{n/r_1}(2^\kappa 2^lR)^{-\gamma}\\&\lesssim\frac{2^{-\kappa(\gamma-n/r_1)}}{(2^lR)^{\gamma-n/r_1}}\prod_{k=1}^{2}M_{r_k}f_k(x)\\&\quad\Big(\int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n}|\mathcal{F}^{-1}\sigma_\kappa(z_1,z_2)|^{r_1'}\langle z_1\rangle^{r_1'\gamma}\mathrm{d}z_1\Big)^{\frac{r_2'}{r_1'}}\langle z_2\rangle^{r_2's_2}\mathrm{d}z_1\Big)^{\frac{1}{r_2'}}\\&\lesssim\frac{2^{-\kappa(\gamma-n/r_1)}}{(2^lR)^{\gamma-n/r_1}}\prod_{k=1}^{2}M_{r_k}f_k(x).\end{aligned}$$

If we choose γ such that $1 + n/r_1 > \gamma$, we then obtain that

$$\int_{\mathbb{R}^n}\int_{|x-y_1|<R}|x-y_1||\mathcal{F}^{-1}\widetilde{\sigma}_\kappa(x-y_1,\,x-y_2)||f_1(y_1)f_2(y_2)|\mathrm{d}y_1\mathrm{d}y_2$$

$$\leq \sum_{l=-\infty}^{-1} 2^l R \int_{\mathbb{R}^n} \int_{C(x,\,2^l R)} |\mathcal{F}^{-1}\widetilde{\sigma}_\kappa(x-y_1,\, x-y_2)||f_1(y_1)f_2(y_2)|\mathrm{d}y_1\mathrm{d}y_2$$
$$\lesssim 2^{-\kappa(\gamma-n/r_1)} R^{1+n/r_1-\gamma} \prod_{k=1}^{2} M_{r_k} f_k(x).$$

Lemma 2.4. *Let σ be a bilinear multiplier satisfying (1.6) for some $s_1,\, s_2 \in (n/2,\, n]$, $r_1,\, r_2 \in (1,\, 2]$ such that $r_2 s_2 > n$. Let $p_1 \in (r_1, \infty)$. Then for every $\gamma \in (0,\, s_1]$, $R > 0$ and $x \in \mathbb{R}^n$ with $|x| > 2R$,*

$$\begin{aligned}(2.3)\qquad &\int_{\mathbb{R}^n}\int_{|y_1|<R} |\mathcal{F}^{-1}\widetilde{\sigma}_\kappa(x-y_1,\, x-y_2)||f_1(y_1)f_2(y_2)|\mathrm{d}y_1\mathrm{d}y_2\\ &\lesssim 2^{-\kappa(\gamma-n/r_1)}|x|^{-\gamma}R^{n/r_1-n/p_1}\|f_1\|_{L^{p_1}(\mathbb{R}^n)}M_{r_2}f_2(x).\end{aligned}$$

Proof. As in the proof of Lemma 2.3, a trivial computation involving the Hölder inequality and Lemma 2.1 leads to that

$$\begin{aligned}
&\int_{\mathbb{R}^n}\int_{|y_1|<R} |\mathcal{F}^{-1}\widetilde{\sigma}_\kappa(x-y_1,\, x-y_2)||f_1(y_1)f_2(y_2)|\mathrm{d}y_1\mathrm{d}y_2\\
&\lesssim \Big(\int_{\mathbb{R}^n}\Big(\int_{|y_1|<R}|\mathcal{F}^{-1}\widetilde{\sigma}_\kappa\big(x-y_1,\, x-y_2\big)|^{r_1'}\mathrm{d}y_1\Big)^{\frac{r_2'}{r_1'}}\langle 2^\kappa(x-y_2)\rangle^{r_2's_2}\mathrm{d}y_2\Big)^{\frac{1}{r_2'}}\\
&\quad\times\Big(\int_{\mathbb{R}^n}\frac{|f_2(y_2)|^{r_2}}{\langle 2^\kappa(x-y_2)\rangle^{s_2r_2}}\mathrm{d}y_2\Big)^{\frac{1}{r_2}}\|f_1\chi_{\{|y_1|<R\}}\|_{L^{r_1}(\mathbb{R}^n)}\\
&\lesssim \Big(\int_{\mathbb{R}^n}\Big(\int_{|y_1|<R}\Big|\mathcal{F}^{-1}\sigma_\kappa\big(2^\kappa(x-y_1),\, 2^\kappa x-y_2\big)\Big|^{r_1'}\mathrm{d}y_1\Big)^{\frac{r_2'}{r_1'}}\langle 2^\kappa x-y_2\rangle^{r_2's_2}\mathrm{d}y_2\Big)^{\frac{1}{r_2'}}\\
&\quad\times 2^{\kappa n}M_{r_2}f_2(x)\|f_1\|_{L^{p_1}(\mathbb{R}^n)}R^{n/r_1-n/p_1}\\
&\lesssim 2^{-\kappa(\gamma-n/r_1)}|x|^{-\gamma}\Big(\int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n}|\mathcal{F}^{-1}\sigma_\kappa(z_1,\, z_2)|^{r_1'}\langle z_1\rangle^{r_1'\gamma}\mathrm{d}z_1\Big)^{\frac{r_2'}{r_1'}}\langle z_2\rangle^{r_2's_2}\mathrm{d}z_1\Big)^{\frac{1}{r_2'}}\\
&\quad\times M_{r_2}f_2(x)\|f_1\|_{L^{p_1}(\mathbb{R}^n)}R^{n/r_1-n/p_1}\\
&\lesssim 2^{-\kappa(\gamma-n/r_1)}|x|^{-\gamma}R^{n/r_1-n/p_1}M_{r_2}f_2(x)\|f_1\|_{L^{p_1}(\mathbb{R}^n)}.
\end{aligned}$$

Lemma 2.5. *Let σ be a bilinear multiplier satisfying (1.6) for some $s_1,\, s_2 \in (n/2,\, n]$, $r_k \in (n/s_k,\, 2]$ ($k = 1,\, 2$) and $s_1 + s_2 < n/r_1 + n/r_2 + 1$. Then there exists a constant $\varrho > 0$ such that for every $R > 0$, $x,\, t \in \mathbb{R}^n$ with $|t| < R/4$, bounded functions f_1 and f_2 with* $\operatorname{supp} f_k \subset \mathbb{R}^n\backslash 4B(x,\, R)$ *for some $k = 1,\, 2$*

$$\begin{aligned}(2.4)\qquad &\sum_{\kappa\in\mathbb{Z}}\int_{\mathbb{R}^{2n}}\big|W_{0,\kappa}(x, y_1, y_2;\, x+t)\big||f_1(y_1)f_2(y_2)|\mathrm{d}y_1\mathrm{d}y_2\\ &\lesssim (|t|R^{-1})^\varrho\prod_{k=1}^{2}\Big(M_{r_k}f_k(x)+M_{r_k}f_k(x+t)\Big),\end{aligned}$$

where and in the following

$$W_{0,\kappa}(x,\, y_1,\, y_2;\, x+t)=\mathcal{F}^{-1}\widetilde{\sigma}_\kappa(x-y_1, x-y_2)-\mathcal{F}^{-1}\widetilde{\sigma}_\kappa(x+t-y_1,\, x+t-y_2).$$

Proof. Let $S_0(B(x,\, R))=B(x,\, R)$ and $S_j(B(x,\, R))=2^jB(x,\, R)\backslash 2^{j-1}B(x,\, R)$. Repeating the proof of Lemma 3.3 in [12], we can obtain that for nonnegative integers j_1 and j_2,

$$\begin{aligned}&\Big(\int_{S_{j_1}(B(x,R))}\Big(\int_{S_{j_2}(B(x,R))}|W_{0,\kappa}(x,y_1,y_2;x+t)|^{r_2'}\mathrm{d}y_2\Big)^{\frac{r_1'}{r_2'}}\mathrm{d}y_1\Big)^{\frac{1}{r_1'}}\\&\qquad\lesssim t2^{-\kappa(s_1+s_2-n/r_1-n/r_2-1)}\prod_{k=1}^{2}(2^{j_k}R)^{-s_k}\end{aligned}$$

provided that $2^\kappa R<1$. On the other hand, as in the proof of Lemma 3.4 in [12], we can verify that for positive integer j_1, bounded function $f_1,\ f_2$ with $\mathrm{supp}f_1\subset\mathbb{R}^n\backslash 4B$,

$$\begin{aligned}&\int_{S_{j_1}(B(x,R))}\int_{\mathbb{R}^n}|W_{0,\kappa}(x,y_1,y_2;x+t)||f_1(y_1)f_2(y_2)|\mathrm{d}y_2\mathrm{d}y_1\\&\qquad\lesssim 2^{-\kappa(s_1-n/r_1)}(2^{j_1}R)^{n/r_1-s_1}\prod_{k=1}^{2}\Big(M_{r_k}f_k(x)+M_{r_k}f_k(x+t)\Big).\end{aligned}$$

A straightforward computation then shows that when $\mathrm{supp}\ f_1\subset\mathbb{R}^n\backslash 4B(x,\, R)$,

$$\begin{aligned}&\sum_{\kappa\in\mathbb{Z}}\int_{\mathbb{R}^{2n}}\big|W_{0,\kappa}(x,\,y_1,\,y_2;\,x+t)\big||f_1(y_1)f_2(y_2)|\mathrm{d}y_1\mathrm{d}y_2\\
&=\sum_{\kappa:2^\kappa R>A}\sum_{j_1=2}^{\infty}\int_{S_{j_1}(B(x,R))}\int_{\mathbb{R}^n}\big|W_{0,\kappa}(x,y_1,y_2;x+t)\big||f_1(y_1)f_2(y_2)|\mathrm{d}y_1\mathrm{d}y_2\\
&\quad+\sum_{\kappa:2^\kappa R\le A}\sum_{j_1=2}^{\infty}\sum_{j_2=0}^{\infty}\Big(\int_{S_{j_1}(B(x,R))}\Big(\int_{S_{j_2}(B(x,R))}|W_{0,\kappa}(x,y_1,y_2;x+t)|^{r_2'}\mathrm{d}y_2\Big)^{\frac{r_1'}{r_2'}}\mathrm{d}y_1\Big)^{\frac{1}{r_1'}}\\
&\quad\times\prod_{k=1}^{2}M_{r_k}f_k(x)2^{n(j_1/r_1+j_2r_2)}R^{n/r_1+n/r_2}\\
&\lesssim\Big(\sum_{\kappa:2^\kappa R>A}(2^\kappa R)^{n/r_1-s_1}+|t|R^{-1}\sum_{\kappa:2^\kappa R\le A}(2^\kappa R)^{n/r_1+n/r_2+1-s_1-s_2}\Big)\\
&\quad\times\prod_{k=1}^{2}\Big(M_{r_k}f_k(x)+M_{r_k}f_k(x+t)\Big)\\
&\lesssim(|t|R^{-1})^{(s_1-n/r_1)/(n/r_2+1-s_2)}\prod_{k=1}^{2}\Big(M_{r_k}f_k(x)+M_{r_k}f_k(x+t)\Big).\end{aligned}$$

if we choose $A=(|t|R^{-1})^{-1/(n/r_2+1-s_2)}$. A similar argument shows that (2.4) holds true when $\operatorname{supp} f_2\subset\mathbb{R}^n\backslash 4B(x,\,R)$.

Let K be a locally integrable function in $\mathbb{R}^{3n}$ away from the diagonal $\{(x,y_1,y_2):x=y_1=y_2\}$. We say that T is a bilinear singular integral operator with kernel K if T is bilinear, and for bounded functions $f_1,\ f_2$ with compact supports,

$$T(f_1,\,f_2)(x)=\int_{\mathbb{R}^{2n}}K(x;\,y_1,\,y_2)f_1(y_1)f_2(y_2)\mathrm{d}y_1\mathrm{d}y_2 \tag{2.5}$$

for everywhere $x\in\mathbb{R}^n\backslash\cap_{k=1}^2\operatorname{supp} f_k$. Associated with T, we define the maximal operator T^* by

$$T^*(f_1,\,f_2)(x)=\sup_{\epsilon>0}|T_\epsilon(f_1,\,f_2)(x)|,$$

where and in the following,

$$T_\epsilon(f_1,\,f_2)(x)=\int_{\max_{1\le k\le 2}|x-y_k|>\epsilon}K(x;\,y_1,\,y_2)\mathrm{d}y_1\mathrm{d}y_2.$$

For the relationship of T and T^*, we have the following conclusion.

Lemma 2.6. *Let $r_1,\,r_2\in(1,\,\infty)$, T be a bilinear singular integral operator with associated kernel K in the sense of (2.5). Suppose that*

(i) *T is bounded from $L^{r_1}(\mathbb{R}^n)\times L^{r_2}(\mathbb{R}^n)$ to $L^{r,\infty}(\mathbb{R}^n)$ with $1/r=1/r_1+1/r_2$;*

(ii)
$$\sup_{\epsilon>0}\int_{\substack{\min_{1\le k\le 2}|x-y_k|>\epsilon/2,\\ \max_{1\le k\le 2}|x-y_k|<2\epsilon}}|K(x;y_1,y_2)|f_1(y_1)f_2(y_2)|\mathrm{d}y_1\mathrm{d}y_2\lesssim M_{r_1}f_1(x)M_{r_2}f_2(x);$$

(iii) *for any ball B, $x,\,y\in B$ and bounded functions $f_1,\,f_2$ with $\operatorname{supp} f_k\subset\mathbb{R}^n\backslash 4B$ for some $k=1,\,2$,*

$$\begin{aligned}&\int_{\mathbb{R}^{2n}}|K(x;y_1,y_2)-K(y;y_1,y_2)||f_1(y_1)f_2(y_2)|\mathrm{d}y_1\mathrm{d}y_2\\ &\lesssim\prod_{k=1}^{2}\Big(M_{r_k}f_k(x)+M_{r_k}f_k(y)\Big);\end{aligned}$$

then for $\delta\in(0,\,\min\{1,\,r\})$ and everywhere $x\in\mathbb{R}^n$,

$$T^*(f_1,\,f_2)(x)\lesssim M_\delta\big(T(f_1,\,f_2)\big)(x)+\prod_{k=1}^{2}M_{r_k}f_k(x).$$

Proof. We will employ some ideas used in the proof of Theorem 1 in [11]. For each fixed $\epsilon>0$, $x,\,y\in\mathbb{R}^n$, let

$$\widetilde{T}_\epsilon(f_1,\,f_2)(y,\,x)=\int_{\{\mathbb{R}^{2n}:\ \min_{1\le k\le 2}|x-y_k|\ge\epsilon\}}K(y;\,y_1,\,y_2)f_1(y_1)f_2(y_2)\mathrm{d}y_1\mathrm{d}y_2.$$

For bounded functions $f_1,\ f_2$ with compact supports, let

$$f_k^1(y_k)=f_k(y_k)\chi_{B(x,\epsilon)}(y_k),\ f_k^2(y_k)=f_k(y_k)\chi_{\mathbb{R}^n\setminus B(x,\epsilon)}(y_k),\ k=1,\,2.$$

It is easy to verify that for $y\in B(x,\,\epsilon/2)$

$$\begin{aligned}&|\widetilde{T}_\epsilon(f_1,f_2)(x,x)|\\ \le\ &|\widetilde{T}_\epsilon(f_1,f_2)(x,x)-\widetilde{T}_\epsilon(f_1,f_2)(y,x)|+|\widetilde{T}_\epsilon(f_1,f_2)(y,x)|\\ \lesssim\ &\int_{\min_{1\le k\le 2}|x-y_k|>\epsilon}\big|K(x;y_1,y_2)-K(y;y_1,y_2)\big||f_1(y_1)f_2(y_2)|\mathrm{d}y_1\mathrm{d}y_2\\ &+\big|T(f_1,\,f_2)(y)-T(f_1^1,\,f_2^1)(y)\big|+\sum_{j=1}^{2}T_\epsilon^j(f_1,\,f_2)(y),\end{aligned}$$

where

$$\begin{aligned}T_\epsilon^1(f_1,\,f_2)(y)&=\int_{|y-y_1|>\epsilon/2}\int_{|y-y_2|<2\epsilon}|K(y;\,y_1,\,y_2)||f_1(y_1)f_2(y_2)|\mathrm{d}y_1\mathrm{d}y_2,\\ T_\epsilon^2(f_1,\,f_2)(y)&=\int_{|y-y_1|<2\epsilon}\int_{|y-y_2|>\epsilon/2}|K(y;\,y_1,\,y_2)||f_1(y_1)f_2(y_2)|\mathrm{d}y_1\mathrm{d}y_2.\end{aligned}$$

Thus, by assumptions (ii) and (iii), we know that for $y\in B(x,\,\epsilon/2)$,

$$\begin{aligned}|T_\epsilon(f_1,\,f_2)(x)|&\lesssim|\widetilde{T}_\epsilon(f_1,\,f_2)(x,\,x)|+\prod_{k=1}^{2}M_{r_k}(f_1,\,f_2)(x)\\ &\lesssim|T(f_1,\,f_2)(y)|+|T(f_1^1,\,f_2^1)(y)|+\prod_{k=1}^{2}M_{r_k}(f_1,\,f_2)(x).\end{aligned}$$

The fact that T is bounded from $L^{r_1}(\mathbb{R}^n)\times L^{r_2}(\mathbb{R}^n)$ to $L^{r,\infty}(\mathbb{R}^n)$, along with the argument in the proof of the Kolmogorov inequality, tells us that for $\delta\in(0,\ \min\{1,\,r\})$,

$$\begin{aligned}&\Big(\frac{1}{|B(x,\,\epsilon/2)|}\int_{B(x,\epsilon/2)}|T(f_1^1,\,f_2^1)(y)|^\delta\mathrm{d}y\Big)^{1/\delta}\\ \lesssim\ &\prod_{k=1}^{2}\Big(\frac{1}{|B(x,\,\epsilon)|}\int_{B(x,\epsilon)}|f_k(y_k)|^{r_k}\mathrm{d}y_k\Big)^{1/r_k}\\ \lesssim\ &\prod_{k=1}^{2}M_{r_k}f_k(x).\end{aligned}$$

On the other hand, we know from [4] that for $\delta\in(0,\,r)$,

$$\Big(\frac{1}{|B(x,\,\epsilon)|}\int_{B(x,\epsilon)}\big(M_{r_k}f_k(y)\big)^{\delta r_k/r}\mathrm{d}y\Big)^{r/(r_k\delta)}\lesssim M_{r_k}f_k(x).$$

Combining the estimates above yields

$$\begin{aligned}
&|T_\epsilon(f_1,f_2)(x)|\\
&\lesssim \Big(\frac{1}{|B(x,\epsilon/2)|}\int_{B(x,\epsilon/2)}|T(f_1,f_2)(y)|^\delta \mathrm{d}y\Big)^{1/\delta}\\
&\quad+\Big(\frac{1}{|B(x,\,\epsilon/2)|}\int_{B(x,\,\epsilon/2)}|T(f_1^1,\,f_2^1)(y)|^\delta \mathrm{d}y\Big)^{1/\delta}\\
&\quad+\prod_{k=1}^2\Big(\frac{1}{|B(x,\epsilon/2)|}\int_{B(x,\epsilon/2)}\big(M_{r_k}f_k(y)\big)^{\delta r_k/r}\mathrm{d}y\Big)^{r/r_k\delta}+\prod_{k=1}^2 M_{r_k}f_k(x)\\
&\lesssim M_\delta\big(T(f_1,\,f_2)\big)(x)+\prod_{k=1}^2 M_{r_k}f_k(x),
\end{aligned}$$

which gives us the desired conclusion directly.

Proof of Theorem 1.1. we will employ some ideas of Bényi and Torres [1]. For $N\in\mathbb{N}$, let

$$\sigma^N(\xi_1,\,\xi_2)=\sum_{|\kappa|\le N}\widetilde{\sigma}_\kappa(\xi_1,\,\xi_2)$$

and denote by $T_{\sigma,\,N}$ the multiplier operator associated with σ^N. It is obvious that $T_{\sigma,\,N}$ is a bilinear singular integral operator with kernel

$$K^N(x;\,y_1,\,y_2)=\mathcal{F}^{-1}\sigma^N(x-y_1,\,x-y_2)$$

in the sense of (2.5). For $b_1,\,b_2\in\mathrm{BMO}(\mathbb{R}^n)$, set

$$T_{\sigma,\,N;\,\vec{b}}(f_1,\,f_2)(x)=\sum_{k=1}^2[b_k,\,T_{\sigma,\,N}]_k(f_1,\,f_2)(x).$$

Let $p_k\in(t_k,\,\infty)$ $(k=1,\,2)$, $p\in[1,\,\infty)$ with $1/p=1/p_1+1/p_2$, and $b_1,\,b_2\in C_0^\infty(\mathbb{R}^n)$. Note that for any $f_1,\,f_2\in\mathcal{S}(\mathbb{R}^n)$ and almost every $x\in\mathbb{R}^n$,

$$\lim_{N\to\infty}T_{\sigma,\,N;\,\vec{b}}(f_1,\,f_2)(x)=T_{\sigma,\,\vec{b}}(f_1,\,f_2)(x).$$

Recall that $T_{\sigma,\,\vec{b}}$ is bounded from $L^{p_1}(\mathbb{R}^n)\times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$. If we can prove that

(a) for each fixed $\epsilon>0$, there exists an constant $A=A(\epsilon)$ which is independent of N, f_1 and f_2, such that

$$\Big(\int_{|x|>A}|T_{\sigma,\,N;\,\vec{b}}(f_1,\,f_2)|^p\mathrm{d}x\Big)^{1/p}\lesssim\epsilon\prod_{k=1}^2\|f_k\|_{L^{p_k}(\mathbb{R}^n)};\tag{2.6}$$

(b) for each fixed $\epsilon > 0$, there exists a constant $\rho = \rho_\epsilon$ which is independent of N, f_1 and f_2, such that for all t with $0 < |t| < \rho$,

$$\big\|T_{\sigma,\,N;\vec{b}}(f_1,\,f_2)(\cdot) - T_{\sigma,\,N;\vec{b}}(f_1,\,f_2)(\cdot+t)\big\|_{L^p(\mathbb{R}^n)} \lesssim \epsilon \prod_{k=1}^{2} \|f_k\|_{L^{p_k}(\mathbb{R}^n)}, \tag{2.7}$$

it then follows from the Fatou Lemma that the inequalities (2.6) and (2.7) still hold true if $T_{\sigma,\,N;\vec{b}}(f_1,\,f_2)$ is replaced by $T_{\sigma,\vec{b}}$. This, via Proposition 3 in [1] and the Fréchet-Kolmogorov theorem characterizing the pre-compactness of a set in L^p (see [18, p. 275]), implies the compactness of $T_{\sigma,\vec{b}}$ from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$.

In the following, we choose $r_k \in (t_k,\,p_k)$ $(k = 1,\,2)$ such that $s_1 + s_2 < n/r_1 + n/r_2 + 1$. We first prove the conclusion (a). For the sake of simplicity, we only consider $[b_1,\,T_\sigma]_1(f_1,\,f_2)$. Let $R > 0$ be large enough such that $\operatorname{supp} b_1 \subset B(0,\,R)$. Then for every x with $|x| > 2R$, we have by Lemma 2.4 that

$$\begin{aligned}
&\int_{\mathbb{R}^n}\int_{|y_1|<R} |K^N(x;\,y_1,\,y_2)||f_1(y_1)f_2(y_2)|\mathrm{d}y_1\mathrm{d}y_2\\
&\lesssim M_{r_2}f_2(x)\|f_1\|_{L^p(\mathbb{R}^n)}R^{n/r_1-n/p_1}|x|^{-s_1}\sum_{\kappa\in\mathbb{Z}:\,2^\kappa R>1} 2^{-\kappa(s_1-n/r_1)}\\
&\quad + M_{r_2}f_2(x)\|f_1\|_{L^p(\mathbb{R}^n)}R^{n/r_1-n/p_1}|x|^{-\theta}\sum_{\kappa\in\mathbb{Z}:\,2^\kappa R<1} 2^{-\kappa(\theta-n/r_1)}\\
&\lesssim \Big(R^{s_1-n/p_1}|x|^{-s_1} + R^{\theta-n/p_1}|x|^{-\theta}\Big)M_{r_2}f_2(x)\|f_1\|_{L^{p_1}(\mathbb{R}^n)},
\end{aligned}$$

if we choose $\gamma = s_1$ and $\gamma = \theta \in (n/p_1,\,n/r_1)$ in (2.3) respectively. Therefore, for $A > 2R$,

$$\begin{aligned}
&\Big(\int_{|x|>A} \big|[b_1,\,T_{\sigma,N}]_1(f_1,\,f_2)(x)\big|^p\mathrm{d}x\Big)^{1/p}\\
&\lesssim \|b_1\|_{L^\infty(\mathbb{R}^n)}\|f_1\|_{L^{p_1}(\mathbb{R}^n)}\|M_{r_2}f_2\|_{L^{p_2}(\mathbb{R}^n)}\Big\{R^{s_1-n/p_1}\Big(\int_{|x|>A}|x|^{-s_1p_1}\,\mathrm{d}x\Big)^{1/p_1}\\
&\quad + R^{\theta-n/p_1}\Big(\int_{|x|>A}|x|^{-\theta p_1}\,\mathrm{d}x\Big)^{1/p_1}\Big\}\\
&\lesssim \|b_1\|_{L^\infty(\mathbb{R}^n)}\|f_1\|_{L^{p_1}(\mathbb{R}^n)}\|f_2\|_{L^{p_2}(\mathbb{R}^n)}\Big(\frac{R}{A}\Big)^{\theta-n/p_1},
\end{aligned}$$

since $s_1 > \theta$. This in turn leads to conclusion (a) directly.

We turn our attention to conclusion (b). Again we only consider $[b_1,\,T_\sigma]_1$. As in [1], we write

$$[b_1,\,T_\sigma]_1(f_1,f_2)(x) - [b_1,T_\sigma]_1(f_1,f_2)(x+t) = \sum_{j=1}^{4}\mathrm{D}_j(x,t),$$

with

$$\begin{aligned}
\mathrm{D}_1(x,t) &= \big(b_1(x+t)-b_1(x)\big)\int_{\max_{1\le k\le 2}|x-y_k|\ge\delta_t} K^N(x;y_1,y_2)f_1(y_1)f_2(y_2)\mathrm{d}y_1\mathrm{d}y_2\\
\mathrm{D}_2(x,t) &= \int_{\max_{1\le k\le 2}|x-y_k|\ge\delta_t} \mathrm{E}^N(x,t;y_1,y_2)\big(b_1(y_1)-b_1(x+t)\big)f_1(y_1)f_2(y_2)\mathrm{d}y_1\mathrm{d}y_2,\\
\mathrm{D}_3(x,t) &= \int_{\max_{1\le k\le 2}|x-y_k|<\delta_t} K^N(x;y_1,y_2)\big(b_1(y_1)-b_1(x)\big)f_1(y_1)f_2(y_2)\mathrm{d}y_1\mathrm{d}y_2\\
\mathrm{D}_4(x,t) &= \int_{\max_{1\le k\le 2}|x-y_k|<\delta_t} K^N(x+t;y_1,y_2)\big(b_1(x+t)-b_1(y_1)\big)f_1(y_1)f_2(y_2)\mathrm{d}y_1\mathrm{d}y_2,
\end{aligned}$$

with $\delta_t > 4|t|$ a convenient choice to be determined later, and

$$\mathrm{E}^N(x,t;y_1,y_2)=K^N(x;y_1,y_2)-K^N(x+t;y_1,y_2).$$

It is obvious that

$$|\mathrm{D}_1(x,t)|\lesssim\|\nabla b_1\|_{L^\infty(\mathbb{R}^n)}|t|\sup_{\epsilon>0}\Big|\int_{\max_{1\le k\le 2}|x-y_k|\ge\epsilon}K^N(x;y_1,y_2)f_1(y_1)f_2(y_2)\mathrm{d}y_1\mathrm{d}y_2\Big|.$$

On the other hand, it follows from Lemma 2.2 that for any $R>0$,

$$\begin{aligned}
&\int_{|x-y_1|\ge R}\int_{|x-y_2|<2R}|K^N(x;,\,y_1,\,y_2||f_1(y_1)f_2(y_2)|\mathrm{d}y_2\mathrm{d}y_1\\
&\lesssim \sum_{\kappa:2^\kappa R>1}2^{\kappa(n/r_1+n/r_2-\gamma_1-\gamma_2)}R^{n/r_1+n/r_2-\gamma_1-\gamma_2}\prod_{k=1}^{2}M_{r_k}f_k(x)\\
&\quad+\sum_{\kappa:2^\kappa R\le 1}2^{\kappa(n/r_1+n/r_2-\widetilde{\gamma}_1-\widetilde{\gamma}_2)}R^{n/r_1+n/r_2-\gamma_1-\gamma_2}\prod_{k=1}^{2}M_{r_k}f_k(x)\\
&\lesssim\prod_{k=1}^{2}M_{r_k}f_k(x).
\end{aligned}$$

if we choose $\gamma_1,\,\gamma_2,\,\widetilde{\gamma}_1,\,\widetilde{\gamma}_2$ such that

$$n/r_1<\gamma_1,\,\widetilde{\gamma}_1<s_1,\;0<\gamma_2,\,\widetilde{\gamma}_2<n/r_2$$

and

$$\gamma_1+\gamma_2>n/r_1+n/r_2,\;\widetilde{\gamma}_1+\widetilde{\gamma}_2<n/r_1+n/r_2.$$

Similarly, we have that

$$\int_{|x-y_2|\ge R}\int_{|x-y_1|<2R}|K^N(x;,\,y_1,\,y_2||f_1(y_1)f_2(y_2)|\mathrm{d}y_1\mathrm{d}y_2\lesssim\prod_{k=1}^{2}M_{r_k}f_k(x)$$

Recall that T_σ is bounded from $L^{r_1}(\mathbb{R}^n)\times L^{r_2}(\mathbb{R}^n)$ to $L^r(\mathbb{R}^n)$ with $1/r=1/r_1+1/r_2$ (see [7, 15]). We have by Lemma 2.5 and Lemma 2.6 that

$$|\mathrm{D}_1(x,\,t)|\lesssim |t|\|\nabla b_1\|_{L^\infty(\mathbb{R}^n)}\Big(\prod_{k=1}^2 M_{r_k}f_k(x)+M_\delta(T(f_1,f_2))(x)\Big).$$

As for the term D_2, an application of Lemma 2.5 shows that for some constant $\varrho>0$,

$$\begin{aligned}|\mathrm{D}_2(x,t)| &\lesssim \|b_1\|_{L^\infty(\mathbb{R}^n)}\int_{\max_{1\le k\le 2}|x-y_k|\ge\delta_t}\big|\mathrm{E}^N(x,t;y_1,y_2)f_1(y_1)f_2(y_2)\big|\mathrm{d}y_1\mathrm{d}y_2\\ &\lesssim (|t|\delta_t^{-1})^{\varrho}\|b_1\|_{L^\infty(\mathbb{R}^n)}\prod_{k=1}^2\Big(M_{r_k}f_k(x)+M_{r_k}f_k(x+t)\Big).\end{aligned}$$

The estimates for D_3 and D_4 are fairly easy. In fact, by Lemma 2.3, we deduce that

$$\begin{aligned}|\mathrm{D}_3(x,t)| &\lesssim \|\nabla b_1\|_{L^\infty(\mathbb{R}^n)}\int_{\mathbb{R}^n}\int_{|x-y_1|<\delta_t}|x-y_1||K^N(x;y_1,y_2)||f_1(y_1)f_2(y_2)|\mathrm{d}y_1\mathrm{d}y_2\\ &\lesssim \|\nabla b_1\|_{L^\infty(\mathbb{R}^n)}\sum_{\kappa\in\mathbb{Z}:\,2^\kappa\delta_t>1}2^{-\kappa(s_1-n/r_1)}\delta_t^{1+n/r_1-s_1}\prod_{k=1}^2 M_{r_k}f_k(x)\\ &\quad+\|\nabla b_1\|_{L^\infty(\mathbb{R}^n)}\sum_{\kappa\in\mathbb{Z}:\,2^\kappa\delta_t>1}2^{\kappa n/r_1}\delta_t^{1+n/r_1}\prod_{k=1}^2 M_{r_k}f_k(x)\\ &\lesssim \delta_t\|\nabla b_1\|_{L^\infty(\mathbb{R}^n)}\prod_{k=1}^2 M_{r_k}f_k(x),\end{aligned}$$

if we choose $\gamma=s_1$ and $\gamma=0$ in the inequality (2.2) respectively (recall that $s_1<n/r_1+1$). Note that

$$\begin{aligned}&|\mathrm{D}_4(x,t)|\\ &\lesssim \int_{\mathbb{R}^n}\int_{|x+t-y_1|<\delta_t+|t|}\big|K^N(x+t;y_1,y_2)\big(b_1(x+t)-b_1(y_1)\big)f_1(y_1)f_2(y_2)\big|\mathrm{d}y_1\mathrm{d}y_2,\end{aligned}$$

an argument which is similar to what was used in the estimate for D_3 shows that

$$|\mathrm{D}_4(x,\,t)|\lesssim \delta_t\|\nabla b_1\|_{L^\infty(\mathbb{R}^n)}\prod_{k=1}^2 M_{r_k}f_k(x+t).$$

For each fixed $\epsilon>0$, set

$$\rho=\frac{A\epsilon}{2(1+\|\nabla b_1\|_{L^\infty(\mathbb{R}^n)})}\ \text{with}\ A=\min\Big\{1,\,\Big(\frac{\epsilon}{2(1+\|b_1\|_{L^\infty(\mathbb{R}^n)})}\Big)^{1/\varrho}\Big\},$$

and $\delta_t = |t|A^{-1}$ for each $t \in \mathbb{R}^n$. Our estimates for terms D_j ($j = 1, \dots, 4$) then leads to that when $0 < |t| < \rho$,

$$\begin{aligned}
&\big\|[b_1, T_\sigma]_1(f_1, f_2)(\cdot) - [b_1, T_\sigma]_1(f_1, f_2)(\cdot + t)\big\| \\
&\lesssim \big((|t| + \delta_t)\|\nabla b_1\|_{L^\infty(\mathbb{R}^n)} + \big(|t|\delta_t^{-1}\big)^{\varrho}\|b_1\|_{L^\infty(\mathbb{R}^n)}\big)\prod_{k=1}^{2}\|f_k\|_{L^{p_k}(\mathbb{R}^n)} \\
&\lesssim \epsilon\|f_k\|_{L^{p_k}(\mathbb{R}^n)}.
\end{aligned}$$

This establishes conclusion (b) and then completes the proof of Theorem 1.1.

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