TAIWANESE JOURNAL OF MATHEMATICS
Vol. 18, No. 1, pp. 257-264, February 2014
DOI: 10.11650/tjm.18.2014.2476
This paper is available online at http://journal.taiwanmathsoc.org.tw

MAPS ACTING ON SOME ZERO PRODUCTS

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Abstract. Let R be a prime ring with nontrivial idempotents. Assume $*$ is an involution of R. In this note we characterize the additive map $\delta\colon R \to R$ such that $\delta(x)y^* + x\delta(y)^* = 0$ whenever $xy^* = 0$ and $\phi\colon R \to R$ such that $\phi(x)\phi(y)^* = 0$ whenever $xy^* = 0$.

1. Introduction

Throughout, R denotes a prime ring with center Z, right (resp. left) Martindale quotient ring Q_r (resp. Q_ℓ), and symmetric Martindale quotient ring Q. The overrings Q, Q_ℓ and Q_r of R are also prime rings. The center C of Q is a field, which is called the extended centroid of R. We refer the reader to the book [1] for details.

By a derivation of R, we mean an additive map $d\colon R \to R$ such that $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. For $a \in R$, the map $\mathrm{ad}(a)\colon x \in R \longmapsto [a, x] \stackrel{\text{def.}}{=} ax - xa$ is a derivation of R, which is called the inner derivation induced by the element a. An additive map $g\colon R \to R$ is called a generalized derivation if there exists a derivation d of R such that $g(xy) = g(x)y + xd(y)$ for any $x, y \in R$. The simplest example of generalized derivation is a map of the form $g(x) = ax + xb$, for some $a, b \in R$.

In what follows, $*$ denotes an involution of R, that is, an anti-automorphism of period 2. An ideal I of R is called a $*$-ideal of R if $I = I^*$. It is well-known that any involution of R can be uniquely extended to an involution of Q (see [4]). A derivation d of R is called symmetric if $d(x^*) = d(x)^*$ for any $x \in R$ and is called anti-symmetric if $d(x^*) = -d(x)^*$ for any $x \in R$. Analogously, a homomorphism ϕ of R is called symmetric if $\phi(x^*) = \phi(x)^*$ for any $x \in R$. With some easy modifications, one can slightly extend the above definitions to (symmetric) derivations from an ideal I (with $I = I^*$) to R.

Received September 24, 2012, accepted July 30, 2013.
Communicated by Ngai-Ching Wong.
2010 *Mathematics Subject Classification*: 16N60, 16R60, 16W10, 16W25.
Key words and phrases: Prime ring, Additive map, Involution, Derivation, Functional identity, Zero products.
The research of the second author was supported by NSC and NCTS/TPE of Taiwan.
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For $a \in R$, let ℓ_a denote the left multiplication map by a. For a derivation d of R, it is clear that $d(x)y+xd(y)=0$ whenever $xy=0$. More generally, if an additive map ϕ is of the form $\ell_\alpha + d$, where $\alpha \in Z$ and d is a derivation, then $\phi(x)y + x\phi(y) = 0$ whenever $xy = 0$. In [3], Chebotar, Ke and Lee proved that the converse is true if R has an identity and possesses a nontrivial idempotent. Lee removed the assumption that R has an identity ([8, Corollary 1.2]).

In the vein, our goal is to characterize the additive map δ such that $\delta(x)y^* + x\delta(y)^* = 0$ whenever $xy^* = 0$. Precisely, in Section 3 we show the following.

Theorem 3.4. *Let R be a prime ring with an involution $*$. Assume R has nontrivial idempotents. If $\delta\colon R \to R$ is an additive map such that $\delta(x)y^*+x\delta(y)^* = 0$ whenever $xy^* = 0$. Then there exists a symmetric derivation $g\colon Q \to Q$ such that $\delta(xy) = \delta(x)y + xg(y)$ for any $x, y \in R$.*

Clearly, homomorphisms are also preserving zero products. If ϕ is a homomorphism of R, then $\phi(x)\phi(y) = 0$ whenever $xy = 0$. In [3], Chebotar, Ke and Lee considered the converse. They showed that if R has an identity and possesses a nontrivial idempotent, $\phi\colon R \to R$ is a bijective additive map such that $\phi(x)\phi(y) = 0$ whenever $xy = 0$, then $\phi(xy)\phi(z) = \phi(x)\phi(yz)$ for any $x, y, z \in R$. Moreover, if $1 \in R$, then $\phi(xy) = \lambda\phi(x)\phi(y)$ for any $x, y \in R$, where $\lambda = \phi(1)^{-1} \in C$ ([3, Theorem 3]).

Recently, Swain considered the result for involutions. He considered a bijective additive map $\phi\colon R \to R$ such that $\phi(x)\phi(y)^* = 0$ whenever $xy^* = 0$, and $\phi(x)^*\phi(y) = 0$ whenever $x^*y = 0$. He proved that if R contains nontrivial idempotents, then the map ϕ must be of the form $\phi(x) = tg(x)$, where $t \in Q$ with $tt^* \in C$ and $g\colon R \to Q$ is a symmetric monomorphism ([9, Theorem 6]). One can check that if $\phi(x) = ag(x)$, where $a \in Q$ and $g\colon R \to Q$ is a symmetric homomorphism, then $\phi(x)\phi(y)^* = 0$ whenever $xy^* = 0$, but we can not conclude that the map must be of this form if only one-sided condition is assumed. However, Swain considered a special case of this situation and showed that: If R is generated by all idempotents, then $\phi(xy) = \phi(x)g(y)$ for any $x, y \in R$, where $g\colon R \to Q$ is a symmetric homomorphism. In particular, if $1 \in R$, then $\phi(x) = tg(x)$, where $t = \phi(1)$ ([9, Theorem 4]). In Section 4, we extend Swain's theorem by removing the assumption that R is generated by all idempotents.

2. Preliminaries

In the following, we will always assume that R *is a prime ring with nontrivial idempotents*. Let E be the additive subgroup generated by idempotents of R, and $\overline{E}$ be the subring generated by E. We begin with a useful result for maps acting on zero products.

Theorem 2.1. ([5, Theorem 2.3]). *Let R be a prime ring with nontrivial idempotents. If $\Phi\colon R \times R \to R$ is a biadditive map such that $\Phi(x, y) = 0$ whenever $xy = 0$.*

Then $\Phi(xa, y) = \Phi(x, ay)$ *for any* $x, y \in R$ *and any* $a \in \overline{E}$*. In particular, there exists a nonzero ideal* I *of* R *such that* $\Phi(xa, y) = \Phi(x, ay)$ *for any* $x, y \in R$ *and any* $a \in I$*.*

We have the next lemma as a special case of [2, Lemma 4.5].

Lemma 2.2. ([2, Lemma 4.5]). *Let* R *be a prime ring. If* $f, g\colon R \to R$ *are additive maps such that* $f(x)y = xg(y)$ *for any* $x, y \in R$*. Then there exists* $q \in Q$ *such that* $f(x) = xq$ *and* $g(x) = qx$ *for any* $x \in R$*.*

3. Symmetric Derivations

In this section, we always assume that $\delta\colon R \to R$ is an additive map such that

$$\delta(x)y^* + x\delta(y)^* = 0 \text{ whenever } xy^* = 0. \tag{3.1}$$

We will characterize such map δ by a series of lemmas.

Lemma 3.1. *There exists a nonzero ideal* $I = I^*$ *of* R *such that*

$$\delta(xa)y + xa\delta(y^*)^* = \delta(x)ay + x\delta(y^*a^*)^* \tag{3.2}$$

for any $x, y \in R$ *and any* $a \in I$*.*

Proof. Define $\Phi(x, y) = \delta(x)y + x\delta(y^*)^*$ for $x, y \in R$. Then for $xy = 0$ we have $x(y^*)^* = 0$, hence $\Phi(x, y) = \delta(x)(y^*)^* + x\delta(y^*)^* = 0$ by (3.1). In view of Theorem 2.1, there exists a nonzero ideal I of R such that $\Phi(xa, y) = \Phi(x, ay)$ for any $x, y \in R$ and any $a \in I$. This means, $\delta(xa)y + xa\delta(y^*)^* = \delta(x)ay + x\delta(y^*a^*)^*$. We may replace I by $I \cap I^*$ and just assume $I^* = I$. ∎

In the following I denotes the specific ideal of R in Lemma 3.1.

Lemma 3.2. *There exists a symmetric derivation* $g\colon I \to Q$ *such that* $\delta(xa) = \delta(x)a + xg(a)$ *for all* $x \in R$ *and* $a \in I$*.*

Proof. By Lemma 3.1 we have

$$\big(\delta(xa) - \delta(x)a\big)y = x\big(\delta(y^*a^*)^* - a\delta(y^*)^*\big) \tag{3.3}$$

for all $x, y \in R$ and $a \in I$. Applying Lemma 2.2 to (3.3), there exists an additive map $g\colon I \to Q$ such that

$$\delta(xa) - \delta(x)a = xg(a) \tag{3.4}$$

and

$$\delta(y^*a^*)^* - a\delta(y^*)^* = g(a)y. \tag{3.5}$$

Combining (3.4) and (3.5),

$$\delta(xa) = \delta(x)a + xg(a) = \delta(x)a + xg(a^*)^*. \tag{3.6}$$

So $g(a^*) = g(a)^*$ for all $a \in I$. Moreover, using (3.6) to expand $\delta(xab)$ in two ways, we have

$$\begin{aligned}&\delta(x(ab)) = \delta(x)ab + xg(ab)\\ =&\delta((xa)b) = \delta(xa)b + xag(b) = \delta(x)ab + xg(a)b + xag(b)\end{aligned}$$

for all $x \in R$ and $a, b \in I$. Hence $g(ab) = g(a)b + ag(b)$ for all $a, b \in I$, as asserted. ■

Lemma 3.3. *g can be uniquely extended to a symmetric derivation on Q.*

Proof. Note that from (3.4) and (3.5) we know $Rg(I)$ and $g(I)R$ are both contained in R. Hence, if we set $J = I^2$, we have $J^* = J$ and $g(J) \subseteq g(I)I + Ig(I) \subseteq R$. This means, g restricted on J is a derivation from J into R. Hence g can be uniquely extended to a derivation on Q (see [6]). For any $q \in Q$, choose W to be a nonzero ideal of R such that $W \subseteq I$ and $qW + Wq \subseteq R$. Since $g(a)^* = g(a^*)$ for all $a \in I$, we see

$$\begin{aligned}&g(wq)^* = (g(w)q + wg(q))^* = q^*g(w)^* + g(q)^*w^*\\ =&g((wq)^*) = g(q^*w^*) = g(q^*)w^* + q^*g(w^*) = g(q^*)w^* + q^*g(w)^*,\end{aligned}$$

for all $w \in W^2$. So $g(q^*) = g(q)^*$ for any $q \in Q$. ■

Now we are ready to characterize completely the map δ satisfying (3.1).

Theorem 3.4. *Let R be a prime ring with an involution $*$. Assume R has nontrivial idempotents. If $\delta\colon R \to R$ is an additive map such that $\delta(x)y^* + x\delta(y)^* = 0$ whenever $xy^* = 0$. Then there exists a symmetric derivation $g\colon Q \to Q$ such that $\delta(xy) = \delta(x)y + xg(y)$ for any $x, y \in R$.*

Proof. From Lemmas 3.2 and 3.3 we know there is a symmetric derivation $g\colon Q \to Q$ and a nonzero ideal I of R with $I^* = I$, such that $\delta(xa) = \delta(x)a + xg(a)$ for any $x \in R$ and $a \in I$. Take $x, y \in R$ and $a, b \in I$, from (3.2) we can compute $\delta(xya)b + xya\delta(b^*)^*$ in two ways:

$$\begin{aligned}&\delta((xy)a)b + (xy)a\delta(b^*)^* = \delta(xy)ab + xy\delta(b^*a^*)^*\\ =&\delta(x(ya))b + x(ya)\delta(b^*)^* = \delta(x)yab + x\delta(b^*a^*y^*)^*\\ =&\delta(x)yab + x\big(\delta(b^*)a^*y^* + b^*g(a^*y^*)\big)^*\\ =&\delta(x)yab + x\big(\delta(b^*)a^*y^* + b^*g(a^*)y^* + b^*a^*g(y^*)\big)^*\\ =&\delta(x)yab + x\big(\delta(b^*a^*)y^* + b^*a^*g(y)^*\big)^*\\ =&\delta(x)yab + xy\delta(b^*a^*)^* + xg(y)ab.\end{aligned}$$

So $\big(\delta(xy) - \delta(x)y - xg(y)\big)I^2 = 0$, and this implies that $\delta(xy) = \delta(x)y + xg(y)$ for any $x, y \in R$. This completes the proof of our theorem. ■

Recall that a derivation d of R is called anti-symmetric if $d(x^*) = -d(x)^*$ for any $x \in R$. Analogous to Theorem 3.4, we have

Theorem 3.5. *Let R be a prime ring with an involution $*$. Assume R has nontrivial idempotents. If $\delta\colon R \to R$ is an additive map such that $\delta(x)y^* - x\delta(y)^* = 0$ whenever $xy^* = 0$. Then there exists a anti-symmetric derivation $g\colon Q \to Q$ such that $\delta(xy) = \delta(x)y + xg(y)$ for any $x, y \in R$.*

4. Homomorphism Type with Involutions

The aim of this section is to generalize Swain's result in [9, Theorem 4] by removing the condition $\overline{E} = R$. Throughout this section, we always assume that $\phi\colon R \to R$ is a bijective additive map such that

$$\phi(x)\phi(y)^* = 0 \text{ whenever } xy^* = 0. \tag{4.1}$$

Lemma 4.1. *There exists a nonzero ideal $I = I^*$ of R such that*

$$\phi(xa)\phi(y^*)^* = \phi(x)\phi(y^*a^*)^* \tag{4.2}$$

for any $x, y \in R$ and any $a \in I$.

Proof. Define $\tilde{\Phi}(x, y) = \phi(x)\phi(y^*)^*$ for $x, y \in R$. Then for $xy = 0 = x(y^*)^*$, we have $\tilde{\Phi}(x, y) = \phi(x)\phi(y^*)^* = 0$ by (4.1). In view of Theorem 2.1, there exists a nonzero ideal I of R such that $\tilde{\Phi}(xa, y) = \tilde{\Phi}(x, ay)$ for any $x, y \in R$ and any $a \in I$. This means, $\phi(xa)\phi(y^*)^* = \phi(x)\phi(y^*a^*)^*$. We may replace I by $I \cap I^*$ and just assume $I^* = I$. ■

In the following I denotes the specific ideal of R in Lemma 4.1.

Lemma 4.2. *If $r\phi(J)^* = 0$ or $\phi(J)r = 0$ for some $r \in R$ and some nonzero ideal J of R. Then $r = 0$.*

Proof. Assume $r\phi(J)^* = 0$. By replacing J by $J \cap J^*$, we may assume $J^* = J$. Since ϕ is bijective, there exists $r' \in R$ such that $\phi(r') = r$. Now $0 = r\phi(R^*(I \cap J)^*)^* = \phi(r')\phi(R^*(I \cap J)^*)^* = \phi(r'(I \cap J))\phi(R)^* = \phi(r'(I \cap J))R$, so $r'(I \cap J) = 0$, and hence $r' = 0$, implying $r = 0$. The other case can be shown analogously. ■

Lemma 4.3. *There exists a symmetric monomorphism $g\colon I \to Q$ such that $\phi(xa) = \phi(x)g(a)$ for any $x \in R$ and $a \in I$.*

Proof. Set $X = \phi(x)$ and $Y = \phi(y^*)^*$ in (4.2) for $x, y \in R$. Since ϕ is surjective, we obtain that

$$\phi(\phi^{-1}(X)a)Y = X\phi(\phi^{-1}(Y^*)a^*)^*, \tag{4.3}$$

for any $X, Y \in R$, and any $a \in I$. Applying Lemma 2.2 to (4.3), there exists an additive map $g\colon I \to Q$ such that

$$\phi(\phi^{-1}(X)a) = Xg(a) \tag{4.4}$$

and

$$\phi(\phi^{-1}(Y^*)a^*)^* = g(a)Y, \tag{4.5}$$

for all $X, Y \in R$ and $a \in I$. Setting $X = \phi(x)$ in (4.4), we get

$$\phi(xa) = \phi(x)g(a) \tag{4.6}$$

for any $x \in R$ and $a \in I$. Similarly, (4.5) yields that

$$\phi(xa^*) = \phi(x)g(a)^*. \tag{4.7}$$

Replacing a by a^* in (4.6), we see $\phi(xa^*) = \phi(x)g(a^*)$. Comparing with (4.7) we get $\phi(x)\big(g(a^*) - g(a)^*\big) = 0$ for all $x \in R$. Hence $g(a^*) = g(a)^*$ for all $a \in I$. For any $x \in R$ and $a, b \in I$ we have

$$\begin{aligned} &\phi(x(ab)) = \phi(x)g(ab) \\ =&\phi((xa)b) = \phi(xa)g(b) = \phi(x)g(a)g(b). \end{aligned}$$

So $g(ab) = g(a)g(b)$ for any $a, b \in I$. Moreover, if $g(a) = 0$ for some $a \in I$, $\phi(x)g(a) = \phi(xa) = 0$ for any $x \in R$. So $Ra = 0$ since ϕ is injective, and $a = 0$ follows. This means, g is a symmetric monomorphism on I. ∎

Lemma 4.4. *If $q \cdot g(J) = 0$ for some $q \in Q_\ell$ and some nonzero ideal J of R, then $q = 0$. Analogously, if $g(J) \cdot q' = 0$ for some $q' \in Q_r$ and some nonzero ideal J of R, then $q' = 0$.*

Proof. Assume $q \cdot g(J) = 0$. There exists a nonzero ideal M of R such that $Mq \subseteq R$. So for any $m \in M$, $0 = mq \cdot g(J \cap I) = \phi(r)g(J \cap I) = \phi\big(r(J \cap I)\big)$ for some $r \in R$ with $\phi(r) = mq$, hence $r(J \cap I) = 0$, implying $r = 0$. That is, $Mq = 0$, so $q = 0$ follows. The other case can be shown analogously. ∎

Recall that $*$ can be extended to Q and an ideal I is called a $*$-ideal if $I = I^*$. Before stating the main result, we define a new notion.

Definition. Let R be a prime ring with an involution $*$. Assume $g\colon R \to Q_\ell$ is a homomorphism. If there exists a nonzero $*$-ideal I of R such that $g(I) \subseteq Q$ and $g(a)^* = g(a^*)$ for all $a \in I$, then g is called *partially* symmetric on R.

Now we prove the main result of this section.

Theorem 4.5. *Let R be a prime ring with an involution $*$. Assume R has nontrivial idempotents. If $\phi\colon R \to R$ is a bijective additive map such that $\phi(x)\phi(y)^* = 0$ whenever $xy^* = 0$. Then there exists a monomorphism $g\colon R \to Q_\ell$ partially symmetric on R such that $\phi(xy) = \phi(x)g(y)$ for any $x, y \in R$.*

Proof. Continuing with Lemma 4.3, we extend $g\colon I \to Q$ to a map from R to Q_ℓ by the following:

For $r \in R$, define $g_r\colon R\phi(R) \to R$ by the rule

$$g_r(\sum_i x_i\phi(y_i)) = \sum_i x_i\phi(y_i r),$$

where $x_i, y_i \in R$. Note that $R\phi(R)$ is a nonzero ideal of R. It is clear that $g_a = g(a)$ for every $a \in I$.

Claim the map g_r is *well-defined* for $r \in R$: If $\sum_i x_i\phi(y_i) = 0$, then

$$\begin{aligned} 0 = \sum_i x_i\phi(y_i)g(rI) &= \sum_i x_i\phi(y_i rI) \\ &= \sum_i x_i\phi(y_i r)g(I). \end{aligned}$$

So by Lemma 4.4 we know $\sum_i x_i\phi(y_i r) = 0$.

Since the map is a left R-module map, g_r can be regarded as an element in Q_ℓ. Hence we extend $g\colon I \to Q$ to $g\colon R \to Q_\ell$, and the extension is unique. Moreover, by definition we have $\phi(x)g(y) = \phi(xy)$ for any $x, y \in R$.

For $x, y, z \in R$, we expand $\phi(xyz)$ in two ways:

$$\begin{aligned} &\phi(x)g(yz) = \phi(xyz) \\ =&\phi(xy)g(z) = \phi(x)g(y)g(z). \end{aligned}$$

Since $\phi(R) = R$, $g(yz) = g(y)g(z)$ for any $y, z \in R$.

If $g(y) = 0$ for $y \in R$, then $\phi(R)g(y) = \phi(Ry) = 0$, implying $Ry = 0$, and $y = 0$ follows. Hence $g\colon R \to Q_\ell$ is a partially symmetric monomorphism. This completes the proof of the theorem. ∎

In the case when R is a simple ring, we see that $I = R$ in the proof of Theorem 4.5. Therefore we have the following theorem.

Theorem 4.6. *Let R be a simple ring with an involution $*$. Assume R has nontrivial idempotents. If $\phi\colon R \to R$ is a bijective additive map such that $\phi(x)\phi(y)^* = 0$ whenever $xy^* = 0$. Then there is a symmetric monomorphism $g\colon R \to Q$ such that $\phi(xy) = \phi(x)g(y)$ for any $x, y \in R$. Moreover, if $1 \in R$, then $\phi(y) = \phi(1)g(y)$ for all $y \in R$.*

We remark that the above theorem can also be obtained by [9, Theorem 4] and [7, Lemma 2].

Acknowledgments

We would like to thank the referee for his suggestions and careful reading of the manuscript. The authors are supported by NTU under Grant 99R40044 and by NSC under Grant NSC100-2811-M-002-059, respectively.

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