TAIWANESE JOURNAL OF MATHEMATICS
Vol. 17, No. 6, pp. 1901-1919, December 2013
DOI: 10.11650/tjm.17.2013.3263
This paper is available online at http://journal.taiwanmathsoc.org.tw

X-POSETS OF CERTAIN COXETER GROUPS

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Abstract. Let X be a subgroup of a Coxeter group W. In [5], the authors developed the notion of X-posets, which are defined on certain equivalence classes of the (right) cosets of X in W. These posets can be thought of as a generalization of the well-known Bruhat order of W. This article provides a catalogue of all the X-posets for various small Coxeter groups.

1. Introduction

Suppose that W is a Coxeter group and let X be a subgroup of W. Then, in [5], the authors introduced and began the investigation of X-posets, which are defined on certain equivalence classes of the (right) cosets of X in W. The special case when X is the trivial subgroup yields the well known, and important, Bruhat order [6]. While taking X to be a standard parabolic subgroup of W delivers us the (generalized) Bruhat order defined on the cosets in W of that standard parabolic subgroup [3]. The study of X-posets is in its infancy and will benefit greatly from a well organized collection of examples. The aim here is to provide a catalogue of all the X-posets for various small Coxeter groups. Specifically we look at the Coxeter groups of type A_2, A_3, A_4, A_5, B_2, B_3 and D_4.

We now describe X-posets in more detail as well as establishing our notation and briefly recapping some basic facts about Coxeter groups. Assume that W is a finite Coxeter group with X a subgroup of W. Then, by definition, W has a presentation

$$W = \langle R \mid (rs)^{m_{rs}} = 1, r, s \in R\rangle$$

where $m_{rs} \in \mathbb{N}$, $m_{rr} = 1$ and for $r, s \in R$, $r \neq s$, $m_{rs} = m_{sr} \geq 2$. Let V be a real vector space with basis $\Pi = \{\alpha_r | r \in R\}$, upon which we define the symmetric bilinear form $\langle \cdot, \cdot \rangle$ by

Received September 5, 2010, accepted April 30, 2013.
Communicated by Ruibin Zhang.
2010 *Mathematics Subject Classification*: 20F55.
Key words and phrases: Coxeter group, Cosets, Bruhat order, Partially ordered set.
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$$\langle \alpha_r, \alpha_s \rangle = -\cos\left(\frac{\pi}{m_{rs}}\right) \quad \text{where } r, s \in R.$$

For $r, s \in R$ we also define

$$r \cdot \alpha_s = \alpha_s - 2\langle \alpha_r, \alpha_s \rangle \alpha_r.$$

This extends to give an action of W which is faithful and respects $\langle \cdot, \cdot \rangle$ (see [6]). The root system Φ (of W) is the following subset of V

$$\Phi = \{w \cdot \alpha_r \mid r \in R, w \in W\},$$

with $\Phi^+ = \{\sum_{r \in R} \lambda_r \alpha_r \in \Phi \mid \lambda_r \geq 0 \text{ for all } r \in R\}$ and $\Phi^- = -\Phi^+$ being, respectively, the positive and negative roots of Φ. As is well-known $\Phi = \Phi^+ \dot{\cup} \Phi^-$. The elements in R are called the fundamental reflections of W and $\mathrm{Ref}(W)$, the set of reflection of W, consists of all W-conjugates of the fundamental reflections.

For Y a subset of W define

$$N(Y) = \{\alpha \in \Phi^+ | w \cdot \alpha \in \Phi^- \text{ for some } w \in Y\}$$

and $l(Y) = |N(Y)|$. We call $l(Y)$ the Coxeter length of Y. This is a generalization of the usual length function in Coxeter groups, first defined in [7].

For right cosets Xg and Xh of X we write $Xg \sim Xh$ whenever $Xgt = Xh$ for some $t \in \mathrm{Ref}(W)$ and Xg and Xh have the same Coxeter length. Let $\approx$ be the equivalence relation generated by $\sim$ on the set of right cosets of X in W and let $\mathfrak{X}$ be the set of $\approx$ equivalence classes. (We remark that our choice of right, as opposed to left, cosets is due to the fact that W acts on the right of Φ – see [5] for more on this.) Now let $\mathbf{x}, \mathbf{x}' \in \mathfrak{X}$. We write $\mathbf{x} \rightsquigarrow \mathbf{x}'$ if there is a right coset Xg in $\mathbf{x}$ and a right coset Xh in $\mathbf{x}'$ such that $Xgt = Xh$ for some $t \in \mathrm{Ref}(W)$ and $l(Xg) < l(Xh)$. The partial order $\preccurlyeq$ on $\mathfrak{X}$ is defined by $\mathbf{x} \prec \mathbf{x}'$ if and only if there exist $\mathbf{x}_1, \ldots, \mathbf{x}_m \in \mathfrak{X}$ such that $\mathbf{x} \rightsquigarrow \mathbf{x}_1 \rightsquigarrow \ldots \rightsquigarrow \mathbf{x}_m \rightsquigarrow \mathbf{x}'$ and we call $\mathfrak{X}$ the X-poset (of W).

A standard parabolic subgroup of W is a subgroup generated by S where $S \subseteq R$ and is usually denoted by W_S. The (generalized) Bruhat order defined on the cosets of W_S will be denoted by $\mathcal{B}(W_S)$. Any conjugate of a standard parabolic subgroup is called a parabolic subgroup of W. For $X \leq W$ let $\langle X \rangle_{\mathrm{sp}}$ denote the standard parabolic closure of X which is the intersection of all standard parabolic subgroups of W containing X. The following three results from [5] have a bearing on our calculations here.

Theorem 1.1. ([5], Corollary 1.4). *Suppose that $X \leq W$ where W is finite. If $N(X) = N(\langle X \rangle_{\mathrm{sp}})$, then the X-poset $\mathfrak{X}$ is poset isomorphic to $\mathcal{B}(\langle X \rangle_{\mathrm{sp}})$.*

Theorem 1.2. ([5], Proposition 4.1). *Suppose that $X \leq W$ where W is finite. If X is not contained in any proper parabolic subgroup of W, then $|\mathfrak{X}| = 1$. In particular, $\mathfrak{X}$ is poset isomorphic to $\mathcal{B}(W_R)$.*

Theorem 1.3. ([5], Theorem 3.8). *Let $X \leq Y \leq W$ where X and Y are finite and Y is a reflection subgroup of W. Let $\mathfrak{X}$ and $\mathfrak{Y}$ denote, respectively, the X-poset and the Y-poset. If $N(X) = N(Y)$ then $\mathfrak{X}$ is poset isomorphic to $\mathfrak{Y}$. If W is finite and $\mathfrak{X}$ is poset isomorphic to $\mathfrak{Y}$ then $N(X) = N(Y)$.*

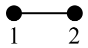
Accompanying each Coxeter group of rank n is its Coxeter graph with the nodes labelled $\{1, \ldots, n\}$ (in one-to-one correspondence with elements R). Let $R = \{r_1, \ldots, r_n\}$. To compress our tabular information in Section 2 when giving elements of W we suppress the symbol "r"; so, for example, for W of type A_4 instead of $r_1r_2r_4r_3$ we shall write [1243]. We use $\mathbb{Z}_m$, 2^m, Dih(m), Alt(m), $\mathrm{Sym}(m)$ to denote, respectively, the cyclic group of order m, the elementary abelian group of order 2^m, the dihedral group of order m, the alternating group of degree m and the symmetric group of degree m.

Our next section describes the structure of various X-posets – this information was obtained with the assistance of Magma [2]. Our third section draws some lessons from these examples.

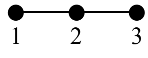
2. X-Posets for Small Coxeter Groups

In compiling the data below on X-posets where $X \leq W$, we take the view that the (generalized) Bruhat order on cosets of a standard parabolic subgroup is "known". Thus, because of Theorem 1.2, we only need concern ourselves with subgroups contained in parabolic subgroups of W. Also Theorem 1.1 tells us that we can ignore any X for which $N(X) = N(\langle X \rangle_{\mathrm{sp}})$.

For $m \in \mathbb{N}$, $\mathcal{C}_m$ will denote the totally ordered set with m elements – $\mathcal{C}_m$ is sometimes called the m-chain poset. In the posets presented in the figures below we have only joined $\mathbf{x}_1$ and $\mathbf{x}_2$ (where $\mathbf{x}_1, \mathbf{x}_2 \in \mathfrak{X}$) if $\mathbf{x}_1 \prec \mathbf{x}_2$ and $l(\mathbf{x}_2) = l(\mathbf{x}_1) + 1$. For $\mathbf{x} \in \mathfrak{X}$ the length of $\mathbf{x}$, $l(\mathbf{x})$ is defined to be $l(Xg)$ where Xg is any coset in $\mathbf{x}$ – clearly $l(\mathbf{x})$ is well defined. Also, in these figures we have indicated on the right-hand side the lengths of the poset elements. Here we cover the Coxeter groups of type A_2, A_3, A_4, A_5, B_2, B_3 and D_4. An example of an X-poset for type F_4 is given in [4].

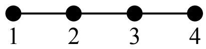
(2.1) W of type A_2, 

. For $X \leq W$, $\mathfrak{X} \cong \mathcal{B}(\langle X \rangle_{\mathrm{sp}})$.

(2.2) W of type A_3, 

. For $X \leq W$, either $\mathfrak{X} \cong \mathcal{B}(\langle X \rangle_{\mathrm{sp}})$ or one of the following holds:-

X	$\mathfrak{X}$
$\langle [32123] \rangle \cong \mathbb{Z}_2$	$\mathcal{C}_2$
$\langle [2132] \rangle \cong \mathbb{Z}_2$	$\mathcal{C}_3$

(2.3) W of type A_4,  . For $X \leq W$, either $\mathfrak{X} \cong \mathcal{B}(\langle X \rangle_{\mathrm{sp}})$ or one of the following holds:-

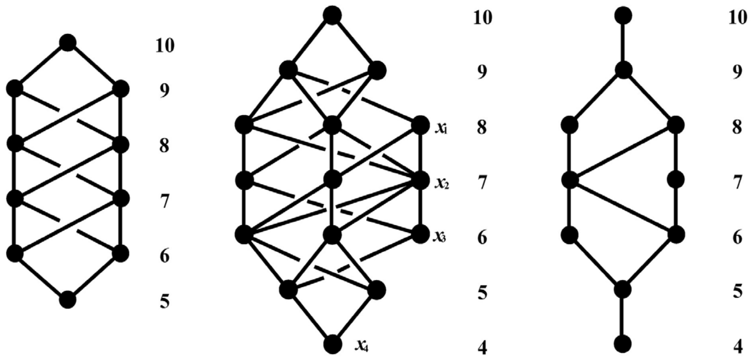
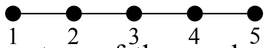


Fig. 1. $A_4(i)$(left), $A_4(ii)$(centre) and $A_4(iii)$(right).

Next we look at the case when W is of type A_5 – so we have the Coxeter diagram:  . Unlike for W of type A_4 (in (2.3)), here we shall take advantage of the graph automorphism of order 2 (which interchanges vertices 1 and 5, vertices 2 and 4, and fixes vertex 3) and only consider subgroups of W conjugate to subgroups of W_{1234} $(= W_{\{1,2,3,4\}})$, W_{1235} $(= W_{\{1,2,3,5\}})$ and W_{1245} $(= W_{\{1,2,4,5\}})$.

In (2.4) we first list the subgroups X which are contained in W_{1234} and whose poset is not of the form $\mathcal{B}(\langle X \rangle_{\mathrm{sp}})$ (note that these are given in the same order as in (2.3)). Then we consider similar subgroups of W_{1235} (which is of type $A_3 \times A_1$). Let X_{123}, respectively X_5, denote the projection of X in W_{123}, respectively W_5. If the X_{123}-poset in W_{123} is $\mathcal{B}(\langle X_{123} \rangle_{\mathrm{sp}})$, and the X_5-poset in W_5 is $\mathcal{B}(\langle X_5 \rangle_{\mathrm{sp}})$, then $\mathfrak{X} \cong \mathcal{B}(\langle X \rangle_{\mathrm{sp}})$. Hence, consulting (2.2), we see that in this subcase we only need examine $X = \langle [321235] \rangle$ and $X = \langle [21325] \rangle$. Similar considerations apply to the subgroups of W_{1245} with (2.1) showing that we need not consider any subgroups of W_{1245}.

A number of X-posets we encounter when W is of type A_5 have quite a large number of elements – too large to draw a comprehensible lattice. To describe these larger posets we use the following scheme. For $i \in \mathbb{N}$ and X-poset $\mathfrak{X}$ we set $\mathfrak{X}_i = \{\mathbf{x} \in \mathfrak{X} \mid l(\mathbf{x}) = i\}$. If $|\mathfrak{X}| = t$, then we will label the elements of $\mathfrak{X}$ by $1, 2, \ldots, t$. In $A_5(vii)$, $A_5(viii)$, $A_5(ix)$ and $A_5(x)$, for each non-empty $\mathfrak{X}_i$ and $i < |\Phi^+|$ we give the element $\mathbf{x}$ of $\mathfrak{X}_i$ followed by the set of all elements $\mathbf{y}$ in $\mathfrak{X}_{i+1}$, with the property that $\mathbf{x} \prec \mathbf{y}$. So, for example, in $A_5(vii)$ we see that $\mathfrak{X}_7 = \{8, 9, 10, 11, 12, 13\}$ and that the element 8 is less than 14 and 16, the element 9 is less than 14, 15, 17, and so on. One further point, we may reduce our computational labours by using Theorem 1.3. Thus we observe in (2.4) that for $X \leq Y \leq W$ with $N(X) = N(Y)$ and Y a reflection subgroup we have $\mathfrak{X} \cong \mathcal{Y}$ for the pairs (X, Y):-

$(\langle[12324321]\rangle, \langle[4321234], [43421234]\rangle)$, $(\langle[12134321]\rangle, \langle[1234321], [12134321]\rangle)$, $(\langle[1214]\rangle, \langle[4], [212]\rangle)$, $(\langle[1343]\rangle, \langle[1], [343]\rangle)$, $(\langle[134321]\rangle, \langle[121], [343]\rangle)$, $(\langle[234321]\rangle, \langle[1], [23432]\rangle)$, $(\langle[124321]\rangle, \langle[12321], [4]\rangle)$, $(\langle[2321]\rangle, \langle[1], [232]\rangle)$, $(\langle[1321]\rangle, \langle[121], [3]\rangle)$, $(\langle[3432]\rangle, \langle[2], [343]\rangle)$, $(\langle[2432]\rangle, \langle[232], [4]\rangle)$.

X	$\mathfrak{X}$
$\langle[12324321]\rangle \cong \mathbb{Z}_2$	$\mathcal{C}_3$
$\langle[21321432]\rangle \cong \mathbb{Z}_2$	$\mathcal{C}_3$
$\langle[12134321]\rangle \cong \mathbb{Z}_2$	$\mathcal{C}_3$
$\langle[4321234]\rangle \cong \mathbb{Z}_2$	$\mathcal{C}_4$
$\langle[132143]\rangle \cong \mathbb{Z}_2$	$\mathcal{C}_5$
$\langle[213432]\rangle \cong \mathbb{Z}_2$	$\mathcal{C}_5$
$\langle[1214]\rangle \cong \mathbb{Z}_2$	$\lvert\mathfrak{X}\rvert = 10$; see $A_4(iii)$
$\langle[1343]\rangle \cong \mathbb{Z}_2$	$\lvert\mathfrak{X}\rvert = 10$; see $A_4(iii)$
$\langle[32123]\rangle \cong \mathbb{Z}_2$	$\lvert\mathfrak{X}\rvert = 10$; see $A_4(i)$
$\langle[23432]\rangle \cong \mathbb{Z}_2$	$\lvert\mathfrak{X}\rvert = 10$; see $A_4(i)$
$\langle[2132]\rangle \cong \mathbb{Z}_2$	$\lvert\mathfrak{X}\rvert = 15$; see $A_4(ii)$
$\langle[3243]\rangle \cong \mathbb{Z}_2$	$\lvert\mathfrak{X}\rvert = 15$; see $A_4(ii)$
$\langle[234321]\rangle \cong \mathbb{Z}_3$	$\mathcal{C}_2$
$\langle[124321]\rangle \cong \mathbb{Z}_3$	$\mathcal{C}_2$
$\langle[134321]\rangle \cong \mathbb{Z}_3$	$\mathcal{C}_2$
$\langle[1321]\rangle \cong \mathbb{Z}_3$	$\mathcal{C}_5$
$\langle[3432]\rangle \cong \mathbb{Z}_3$	$\mathcal{C}_5$
$\langle[2321]\rangle \cong \mathbb{Z}_3$	$\mathcal{C}_5$
$\langle[2432]\rangle \cong \mathbb{Z}_3$	$\mathcal{C}_5$
$\langle[4321234], [43421234]\rangle \cong 2^2$	$\mathcal{C}_3$
$\langle[32123], [342123]\rangle \cong 2^2$	$\mathcal{C}_3$
$\langle[1234321], [12134321]\rangle \cong 2^2$	$\mathcal{C}_3$
$\langle[23432], [213432]\rangle \cong 2^2$	$\mathcal{C}_3$
$\langle[4], [212]\rangle \cong 2^2$	$\lvert\mathfrak{X}\rvert = 10$; see $A_4(iii)$
$\langle[1], [343]\rangle \cong 2^2$	$\lvert\mathfrak{X}\rvert = 10$; see $A_4(iii)$
$\langle[121], [343]\rangle \cong \mathrm{Sym}(3)$	$\mathcal{C}_2$
$\langle[1], [23432]\rangle \cong \mathrm{Sym}(3)$	$\mathcal{C}_2$
$\langle[12321], [4]\rangle \cong \mathrm{Sym}(3)$	$\mathcal{C}_2$
$\langle[1], [232]\rangle \cong \mathrm{Sym}(3)$	$\mathcal{C}_5$
$\langle[121], [3]\rangle \cong \mathrm{Sym}(3)$	$\mathcal{C}_5$
$\langle[2], [343]\rangle \cong \mathrm{Sym}(3)$	$\mathcal{C}_5$
$\langle[232], [4]\rangle \cong \mathrm{Sym}(3)$	$\mathcal{C}_5$

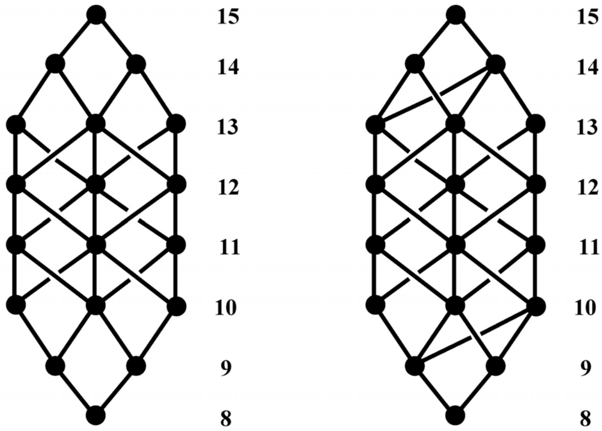
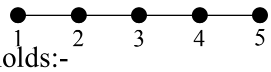


Fig. 2. $A_5(i)$(left) and $A_5(ii)$.

(2.4) W of type A_5, 

. For $X \leq W$, either $\mathfrak{X} \cong \mathcal{B}(\langle X \rangle_{\mathrm{sp}})$ or one of the following holds:-

	X	$\mathfrak{X}$
X a subgroup of W_{1234}	$\langle[12324321]\rangle \cong \mathbb{Z}_2$	$\lvert\mathfrak{X}\rvert = 18$; see $A_5(i)$
	$\langle[21321432]\rangle \cong \mathbb{Z}_2$	$\lvert\mathfrak{X}\rvert = 18$; see $A_5(ii)$
	$\langle[12134321]\rangle \cong \mathbb{Z}_2$	$\lvert\mathfrak{X}\rvert = 18$; see $A_5(iii)$
	$\langle[4321234]\rangle \cong \mathbb{Z}_2$	$\lvert\mathfrak{X}\rvert = 24$; see $A_5(iv)$
	$\langle[132143]\rangle \cong \mathbb{Z}_2$	$\lvert\mathfrak{X}\rvert = 30$; see $A_5(v)$
	$\langle[213432]\rangle \cong \mathbb{Z}_2$	$\lvert\mathfrak{X}\rvert = 30$; see $A_5(vi)$
	$\langle[1214]\rangle \cong \mathbb{Z}_2$	$\lvert\mathfrak{X}\rvert = 60$; see $A_5(vii)$
	$\langle[1343]\rangle \cong \mathbb{Z}_2$	$\lvert\mathfrak{X}\rvert = 60$; see $A_5(viii)$
	$\langle[32123]\rangle \cong \mathbb{Z}_2$	$\lvert\mathfrak{X}\rvert = 60$; see $A_5(ix)$
	$\langle[23432]\rangle \cong \mathbb{Z}_2$	$\lvert\mathfrak{X}\rvert = 60$; see $A_5(x)$
	$\langle[2132]\rangle \cong \mathbb{Z}_2$	$\lvert\mathfrak{X}\rvert = 90$; see $A_5(xi)$
	$\langle[3243]\rangle \cong \mathbb{Z}_2$	$\lvert\mathfrak{X}\rvert = 90$; see $A_5(xii)$
	$\langle[234321]\rangle \cong \mathbb{Z}_3$	$\lvert\mathfrak{X}\rvert = 12$; see $A_5(xiii)$
	$\langle[124321]\rangle \cong \mathbb{Z}_3$	$\lvert\mathfrak{X}\rvert = 12$; see $A_5(xiii)$
	$\langle[134321]\rangle \cong \mathbb{Z}_3$	$\lvert\mathfrak{X}\rvert = 12$; see $A_5(xiii)$
	$\langle[1321]\rangle \cong \mathbb{Z}_3$	$\lvert\mathfrak{X}\rvert = 30$; see $A_5(xiv)$
	$\langle[3432]\rangle \cong \mathbb{Z}_3$	$\lvert\mathfrak{X}\rvert = 30$; see $A_5(xv)$
	$\langle[2321]\rangle \cong \mathbb{Z}_3$	$\lvert\mathfrak{X}\rvert = 30$; see $A_5(xiv)$
	$\langle[2432]\rangle \cong \mathbb{Z}_3$	$\lvert\mathfrak{X}\rvert = 30$; see $A_5(xv)$
	$\langle[4321234], [43421234]\rangle \cong 2^2$	$\lvert\mathfrak{X}\rvert = 18$; see $A_5(i)$
	$\langle[32123], [342123]\rangle \cong 2^2$	$\lvert\mathfrak{X}\rvert = 18$; see $A_5(i)$

	$\langle[1234321],[12134321]\rangle \cong 2^2$	$\lvert\mathfrak{X}\rvert = 18$; see $A_5(iii)$
	$\langle[23432],[213432]\rangle \cong 2^2$	$\lvert\mathfrak{X}\rvert = 18$; see $A_5(iii)$
	$\langle[4],[212]\rangle \cong 2^2$	$\lvert\mathfrak{X}\rvert = 60$; see $A_5(vii)$
	$\langle[1],[343]\rangle \cong 2^2$	$\lvert\mathfrak{X}\rvert = 60$; see $A_5(viii)$
	$\langle[121],[343]\rangle \cong \mathrm{Sym}(3)$	$\lvert\mathfrak{X}\rvert = 12$; see $A_5(xiii)$
	$\langle[1],[23432]\rangle \cong \mathrm{Sym}(3)$	$\lvert\mathfrak{X}\rvert = 12$; see $A_5(xiii)$
	$\langle[12321],[4]\rangle \cong \mathrm{Sym}(3)$	$\lvert\mathfrak{X}\rvert = 12$; see $A_5(xiii)$
	$\langle[1],[232]\rangle \cong \mathrm{Sym}(3)$	$\lvert\mathfrak{X}\rvert = 30$; see $A_5(xiv)$
	$\langle[121],[3]\rangle \cong \mathrm{Sym}(3)$	$\lvert\mathfrak{X}\rvert = 30$; see $A_5(xiv)$
	$\langle[2],[343]\rangle \cong \mathrm{Sym}(3)$	$\lvert\mathfrak{X}\rvert = 30$; see $A_5(xv)$
	$\langle[232],[4]\rangle \cong \mathrm{Sym}(3)$	$\lvert\mathfrak{X}\rvert = 30$; see $A_5(xv)$
X a subgroup of W_{1235}	$\langle[321235]\rangle \cong \mathbb{Z}_2$	$\lvert\mathfrak{X}\rvert = 30$; see $A_5(xvi)$
	$\langle[21325]\rangle \cong \mathbb{Z}_2$	$\lvert\mathfrak{X}\rvert = 45$; see $A_5(xvii)$

X conjugate to a subgroup of W_{1234} and X not contained in any proper standard parabolic subgroup of W.

	X	$\mathfrak{X}$
$X \cong \mathbb{Z}_2$	$\langle[12132432154321]\rangle$	$\mathcal{C}_2$
	$\langle[121321454321]\rangle$ $\langle[213243215432]\rangle$ $\langle[123243254321]\rangle$	$\mathcal{C}_4$
	$\langle[2132145432]\rangle$ $\langle[1232454321]\rangle$ $\langle[1324321543]\rangle$	$\mathcal{C}_6$
	$\langle[21345432]\rangle$ $\langle[13214543]\rangle$ $\langle[12432154]\rangle$	$\mathcal{C}_8$
	$\langle[1213454321]\rangle$ $\langle[1234354321]\rangle$	$\lvert\mathfrak{X}\rvert = 8$; see $A_5(xviii)$
	$\langle[123454321]\rangle$	$\lvert\mathfrak{X}\rvert = 8$; see $A_5(xix)$
$X \cong \mathbb{Z}_3$	$\langle[12134543]\rangle$ $\langle[12321454]\rangle$ $\langle[12345432]\rangle$ $\langle[12343215]\rangle$	$\mathcal{C}_4$
$X \cong \mathbb{Z}_4$	18 possible X's	$\mathcal{C}_2$
$X \cong 2^2$	X conjugate to $\langle[13],[2132]\rangle$ (6 possibilities) $\langle[23432],[12132432154321]\rangle$ $\langle[1234321],[213243215432]\rangle$	$\mathcal{C}_2$
	$\langle[232],[121321454321]\rangle$ $\langle[12321],[2132145432]\rangle$ $\langle[343],[123243254321]\rangle$ $\langle[1234321],[1324321543]\rangle$	$\mathcal{C}_4$
	$\langle[3],[1232454321]\rangle$ $\langle[12321],[13214543]\rangle$	$\mathcal{C}_6$

	$\langle [121], [21345432] \rangle$ $\langle [2], [1213454321] \rangle$ $\langle [4], [1234354321] \rangle$ $\langle [1234321], [12432154] \rangle$	$\lvert\mathfrak{X}\rvert = 8$; see $A_5(xviii)$
$X \cong \mathbb{Z}_6$	$\langle [2432154] \rangle$ $\langle [1214354] \rangle$ $\langle [1243254] \rangle$ $\langle [1432154] \rangle$	$\mathcal{C}_3$
	$\langle [3214543] \rangle$ $\langle [1324543] \rangle$	$\mathcal{C}_4$
$X \cong$ Dih(8)	18 possible X's	C_2
$X \cong$ Alt(4)	3 possible X's	C_2
$X \cong$ Sym(4)	6 possible X's	C_2

X conjugate to a subgroup of W_{1245} which is not a subgroup of W_{1234} and X not contained in any proper standard parabolic subgroup of W.

	X	$\mathfrak{X}$
$X \cong \mathbb{Z}_3$	$\langle [13243254] \rangle$ $\langle [21324325] \rangle$ $\langle [21324354] \rangle$ $\langle [23214325] \rangle$ $\langle [32432154] \rangle$	$\mathcal{C}_4$
	$\langle [321435] \rangle$ $\langle [132435] \rangle$	$\mathcal{C}_7$

X conjugate to a subgroup of W_{1235} which is not a subgroup of W_{1234} nor W_{1245} and X not contained in any proper standard parabolic subgroup of W.

	X	$\mathfrak{X}$
$X \cong \mathbb{Z}_2$	$\langle [2132143215432] \rangle$ $\langle [1232143254321] \rangle$	$\mathcal{C}_3$
	$\langle [13214321543] \rangle$ $\langle [23214325432] \rangle$	$\mathcal{C}_5$
	$\langle [121432154] \rangle$ $\langle [321432543] \rangle$ $\langle [213435432] \rangle$	$\mathcal{C}_7$
	$\langle [2143254] \rangle$	$\mathcal{C}_9$
	$\langle [12134354321] \rangle$	$\lvert\mathfrak{X}\rvert = 6$; see $A_5(xx)$

$X \cong Z_4$	8 possible X's	$\mathcal{C}_5$
$X \cong 2^2$	$\langle [32432543], [321432543] \rangle$	$\mathcal{C}_3$
	$\langle [21321432], [321432543] \rangle$	
	$\langle [132143], [23214325432] \rangle$	
	$\langle [2132145432], [23214325432] \rangle$	
	$\langle [324543], [13214321543] \rangle$	
	$\langle [1324321543], [13214321543] \rangle$	
	$\langle [213243215432], [2132143215432] \rangle$	
	$\langle [13214543], [321432543] \rangle$	
	$\langle [3243], [1232143254321] \rangle$	
	$\langle [121432154], [14354] \rangle$	$\mathcal{C}_5$
	$\langle [13214321543], [321432543] \rangle$	
	$\langle [121432154], [2143254] \rangle$	
	$\langle [213435432], [21325] \rangle$	
	$\langle [2143254], [14354] \rangle$	
	$\langle [2143254], [21325] \rangle$	
	$\langle [213435432], [2143254] \rangle$	
	$\langle [23214325432], [321432543] \rangle$	

ctd.	X	$\mathfrak{X}$
$X \cong 2^2$	$\langle [213432], [243254] \rangle$	$\lvert\mathfrak{X}\rvert = 6$; see $A_5(xx)$
	$\langle [12134321], [12432154] \rangle$	
	$\langle [21345432], [23435432] \rangle$	
	$\langle [1213454321], [1234354321] \rangle$	
	$\langle [12134321], [121432154] \rangle$	
	$\langle [1213454321], [12134354321] \rangle$	
	$\langle [24], [12134354321] \rangle$	
	$\langle [121454], [2143254] \rangle$	
	$\langle [1214], [213435432] \rangle$	
	$\langle [23435432], [213435432] \rangle$	
	$\langle [1234354321], [12134354321] \rangle$	
	$\langle [2454], [121432154] \rangle$	
	$\langle [213432], [2143254] \rangle$	$\mathcal{C}_7$
	$\langle [21345432], [213435432] \rangle$	
	$\langle [12432154], [121432154] \rangle$	
	$\langle [243254], [2143254] \rangle$	
$X \cong 2^3$	$\langle [121], [4], [2345432] \rangle$	$\lvert\mathfrak{X}\rvert = 6$; see $A_5(xx)$
	$\langle [2], [4], [123454321] \rangle$	
	$\langle [2], [454], [1234321] \rangle$	
	$\langle [121], [454], [23432] \rangle$	

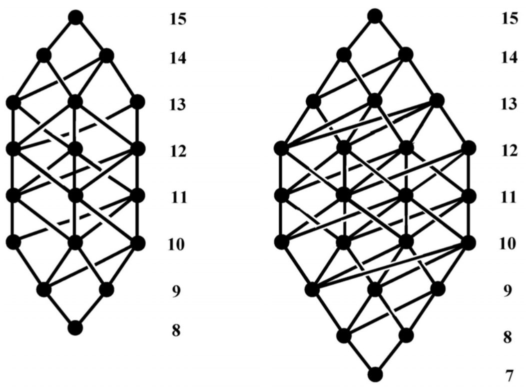


Fig. 3. $A_5(iii)$(left) and $A_5(iv)$(right).

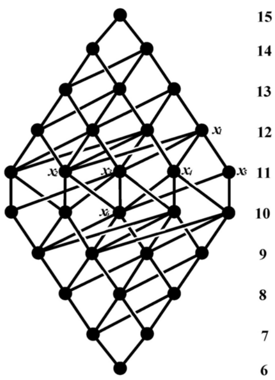


Fig. 4. $A_5(v)$.

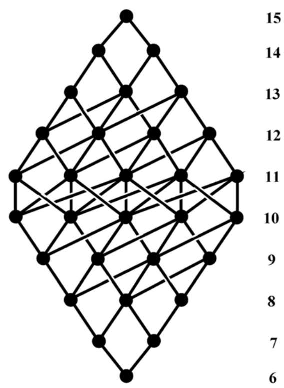


Fig. 5. $A_5(vi)$.

Table 1. $A_5(vii)$

$\mathfrak{X}_i$	$j\{k \in \mathfrak{X}_{i+1} \mid j \prec k\}(j \in \mathfrak{X}_i)$
$\mathfrak{X}_4$	$1\{2,3\}$
$\mathfrak{X}_5$	$2\{4,5,6\}$; $3\{6,7\}$
$\mathfrak{X}_6$	$4\{9,10,12\}$; $5\{8,9,11\}$; $6\{10,11,12,13\}$; $7\{12,13\}$
$\mathfrak{X}_7$	$8\{14,16\}$; $9\{14,15,17\}$; $10\{17,18\}$; $11\{16,17,19,21\}$; $12\{18,19,20\}$; $13\{20,21\}$
$\mathfrak{X}_8$	$14\{23,25\}$; $15\{22,24,28\}$; $16\{23,25,30\}$; $17\{23,24,26\}$; $18\{26,27\}$; $19\{25,26,28,29\}$; $20\{27,28,29\}$; $21\{29,30\}$
$\mathfrak{X}_9$	$22\{31,32,35\}$; $23\{32,33\}$; $24\{32,34,36\}$; $25\{33,35,39\}$; $26\{33,34,36,37\}$; $27\{36,37\}$; $28\{35,36,38\}$; $29\{37,38,39\}$; $30\{39\}$
$\mathfrak{X}_{10}$	$31\{40,44\}$; $32\{40,41,42\}$; $33\{41,42,46\}$; $34\{41,43\}$; $35\{42,44,47\}$; $36\{42,43,45\}$; $37\{45,46\}$; $38\{44,45,47\}$; $39\{46,47\}$
$\mathfrak{X}_{11}$	$40\{48,50\}$; $41\{48,49\}$; $42\{49,50,53\}$; $43\{49,51\}$; $44\{50,52\}$; $45\{50,51,53\}$; $46\{53\}$; $47\{52,53\}$
$\mathfrak{X}_{12}$	$48\{54,55\}$; $49\{54,55,57\}$; $50\{55,56\}$; $51\{55,57\}$; $52\{56\}$; $53\{56,57\}$
$\mathfrak{X}_{13}$	$54\{58\}$; $55\{58,59\}$; $56\{59\}$; $57\{59\}$
$\mathfrak{X}_{14}$	$58\{60\}$; $59\{60\}$

Table 2. $A_5(viii)$

$\mathfrak{X}_i$	$j\{k \in \mathfrak{X}_{i+1} \mid j \prec k\}(j \in \mathfrak{X}_i)$
$\mathfrak{X}_4$	$1\{2,3\}$
$\mathfrak{X}_5$	$2\{4,5,6\}$; $3\{6,7\}$
$\mathfrak{X}_6$	$4\{9,10\}$; $5\{8,9,11\}$; $6\{10,11,12\}$; $7\{12,13\}$
$\mathfrak{X}_7$	$8\{14,16,19\}$; $9\{14,15,17\}$; $10\{17,18\}$; $11\{16,17,19,20\}$; $12\{18,19,20,21\}$; $13\{20,21\}$
$\mathfrak{X}_8$	$14\{22,23,26\}$; $15\{22,24\}$; $16\{23,25\}$; $17\{23,24,26,27\}$; $18\{26,27,30\}$; $19\{25,26,28\}$; $20\{27,28,29\}$; $21\{29,30\}$
$\mathfrak{X}_9$	$22\{31,32,34\}$; $23\{32,33\}$; $24\{32,34,37\}$; $25\{33,35\}$; $26\{33,34,36\}$; $27\{36,37,39\}$; $28\{35,36,38\}$; $29\{37,38,39\}$; $30\{39\}$
$\mathfrak{X}_{10}$	$31\{40,43\}$; $32\{40,41\}$; $33\{41,42\}$; $34\{41,43,45\}$; $35\{42,44\}$; $36\{42,43,45,47\}$; $37\{45,46\}$; $38\{44,45,47\}$; $39\{46,47\}$
$\mathfrak{X}_{11}$	$40\{48,49\}$; $41\{48,49,50\}$; $42\{49,50,52\}$; $43\{49,51\}$; $44\{50,52\}$; $45\{50,51,53\}$; $46\{53\}$; $47\{52,53\}$
$\mathfrak{X}_{12}$	$48\{54\}$; $49\{54,55\}$; $50\{55,56\}$; $51\{55,57\}$; $52\{56\}$; $53\{56,57\}$
$\mathfrak{X}_{13}$	$54\{58\}$; $55\{58,59\}$; $56\{59\}$; $57\{59\}$
$\mathfrak{X}_{14}$	$58\{60\}$; $59\{60\}$

Table 3. $A_5(ix)$

$\mathfrak{X}_i$	$j\{k \in \mathfrak{X}_{i+1} \mid j \prec k\}(j \in \mathfrak{X}_i)$
$\mathfrak{X}_5$	$1\{2,3,4\}$
$\mathfrak{X}_6$	$2\{5,6,7,9\}$; $3\{5,6,8\}$; $4\{7,8,9\}$
$\mathfrak{X}_7$	$5\{10,12,14\}$; $6\{10,13,15\}$; $7\{11,12,13,16\}$; $8\{12,13,14,15\}$; $9\{11,14,15,16\}$
$\mathfrak{X}_8$	$10\{20,21,22\}$; $11\{17,18,19,23\}$; $12\{19,20,24\}$; $13\{18,20,22,25\}$; $14\{19,21,22,24\}$; $15\{18,21,25\}$; $16\{17,22,23,24,25\}$
$\mathfrak{X}_9$	$17\{26,27,30\}$; $18\{26,29,32\}$; $19\{27,28,29,33\}$; $20\{28,29,34,35\}$; $21\{29,31,34\}$; $22\{28,31,35\}$; $23\{28,30,32,33\}$; $24\{27,33,34,35\}$; $25\{26,31,32,34\}$
$\mathfrak{X}_{10}$	$26\{37,39\}$; $27\{36,37,40\}$; $28\{36,38,41,42\}$; $29\{37,38,43\}$; $30\{36,39,40\}$; $31\{38,41,44\}$; $32\{38,39,43\}$; $33\{40,41,42,43\}$; $34\{37,41,43,44\}$; $35\{36,42,44\}$
$\mathfrak{X}_{11}$	$36\{45,46,48\}$; $37\{45,47\}$; $38\{45,49,50\}$; $39\{45,47\}$; $40\{46,47,48\}$; $41\{48,49,51\}$; $42\{46,50,51\}$; $43\{47,49,50\}$; $44\{45,48,50,51\}$
$\mathfrak{X}_{12}$	$45\{52,53\}$; $46\{52,54,55\}$; $47\{52,53\}$; $48\{53,54,55\}$; $49\{53,55,56\}$; $50\{52,55,56\}$; $51\{54,56\}$
$\mathfrak{X}_{13}$	$52\{57,58\}$; $53\{57,58\}$; $54\{57,59\}$; $55\{58,59\}$; $56\{57,59\}$
$\mathfrak{X}_{14}$	$57\{60\}$; $58\{60\}$; $59\{60\}$

Table 4. $A_5(x)$

$\mathfrak{X}_i$	$j\{k \in \mathfrak{X}_{i+1} \mid j \prec k\}(j \in \mathfrak{X}_i)$
$\mathfrak{X}_5$	$1\{2,3,4\}$
$\mathfrak{X}_6$	$2\{5,7,9\}$; $3\{5,6,8\}$; $4\{6,7,8,9\}$
$\mathfrak{X}_7$	$5\{11,12,13\}$; $6\{11,14,15\}$; $7\{10,11,12,16\}$; $8\{12,13,14,15\}$; $9\{10,13,16\}$
$\mathfrak{X}_8$	$10\{17,19,22\}$; $11\{20,21,23\}$; $12\{19,20,23,24\}$; $13\{19,21,24\}$; $14\{18,20,25\}$; $15\{18,21,25\}$; $16\{17,21,22,24\}$
$\mathfrak{X}_9$	$17\{26,28,30\}$; $18\{27,29,31\}$; $19\{26,30,32,34\}$; $20\{27,31,32,33\}$; $21\{30,31,32,35\}$; $22\{28,32,34\}$; $23\{27,30,33,35\}$; $24\{26,31,34,35\}$; $25\{19,32,33\}$
$\mathfrak{X}_{10}$	$26\{36,38\}$; $27\{39,40\}$; $28\{38,41,43\}$; $29\{37,39,42\}$; $30\{36,41,43\}$; $31\{37,40,42\}$; $32\{37,41,44\}$; $33\{39,41,42,44\}$; $34\{37,38,43,44\}$; $35\{36,40,42,43,44\}$

$\mathfrak{X}_{11}$	$36\{47,48\}$; $37\{45,46,49\}$; $38\{47,48\}$; $39\{46,50\}$; $40\{46,50\}$; $41\{45,47,51\}$; $42\{45,49,40\}$; $43\{45,48,51\}$; $44\{46,47,49,51\}$
$\mathfrak{X}_{12}$	$45\{52,53,54\}$; $46\{52,55\}$; $47\{53,56\}$; $48\{53,56\}$; $49\{53,54,55\}$; $50\{52,55\}$; $51\{52,54,56\}$
$\mathfrak{X}_{13}$	$52\{57,58\}$; $53\{57,59\}$; $54\{58,59\}$; $55\{57,58\}$; $56\{57,59\}$
$\mathfrak{X}_{14}$	$57\{60\}$; $58\{60\}$; $59\{60\}$

Table 5. $A_5(xi)$

$\mathfrak{X}_i$	$j\{k \in \mathfrak{X}_{i+1} \mid j \prec k\}(j \in \mathfrak{X}_i)$
$\mathfrak{X}_4$	$1\{2,3,4\}$
$\mathfrak{X}_5$	$2\{5,6,7,8\}$; $3\{6,8,9,10\}$; $4\{7,9,10\}$
$\mathfrak{X}_6$	$5\{11,12,13\}$; $6\{11,13,14,15,16\}$; $7\{12,14,15,17,19\}$; $8\{13,16,17,19\}$; $9\{14,17,18\}$; $10\{15,18,19\}$
$\mathfrak{X}_7$	$11\{20,21,22,23\}$; $12\{21,22,24,26\}$; $13\{20,23,24,26,27,29\}$; $14\{21,24,25,28\}$; $15\{22,25,26,29,31\}$; $16\{23,27,28,31\}$; $17\{24,28,29,30\}$; $18\{25,29,30\}$; $19\{26,30,31\}$
$\mathfrak{X}_8$	$20\{32,33,35,36,38\}$; $21\{33,34,37\}$; $22\{34,35,38,40\}$; $23\{32,36,37,40,42\}$; $24\{33,37,38,39,41\}$; $25\{34,38,39,42,44\}$; $26\{35,39,40,43,45\}$; $27\{36,41,45\}$; $28\{37,41,42,44\}$; $29\{38,42,43\}$; $30\{39,43,44\}$; $31\{40,44,45\}$
$\mathfrak{X}_9$	$32\{46,47,50,52\}$; $33\{47,48,49,50\}$; $34\{48,49,52,54\}$; $35\{49,50,53,55\}$; $36\{46,51,55,58\}$; $37\{47,51,52,54,57\}$; $38\{48,52,53,58\}$; $39\{49,53,54,57,59\}$; $40\{50,54,55,56\}$; $41\{51,58,59\}$; $42\{52,56,57,58\}$; $43\{53,56,57\}$; $44\{54,56,59\}$; $45\{55,59\}$
$\mathfrak{X}_{10}$	$46\{60,64,70\}$; $47\{60,61,63,67\}$; $48\{61,62,70\}$; $49\{62,63,67,71\}$; $50\{63,64,66\}$; $51\{60,69,70,71\}$; $52\{61,66,67,70\}$; $53\{62,66,67,68,69\}$; $54\{63,65,66,71\}$; $55\{64,68,71\}$; $56\{65,66,68\}$; $57\{65,67,69\}$; $58\{68,69,70\}$; $59\{68,71\}$
$\mathfrak{X}_{11}$	$60\{78,79,80\}$; $61\{74,76,79\}$; $62\{74,76,77,78\}$; $63\{73,74,80\}$; $64\{77,80\}$; $65\{72,73,75\}$; $66\{72,73,74,77\}$; $67\{73,76,78\}$; $68\{75,77\}$; $69\{72,75,78\}$; $70\{72,77,78,79\}$; $71\{75,77,80\}$
$\mathfrak{X}_{12}$	$72\{81,82\}$; $73\{81,83,84\}$; $74\{81,83\}$; $75\{82,84\}$; $76\{83,86\}$; $77\{82,84,85\}$; $78\{84,86\}$; $79\{81,85,86\}$; $80\{84,85\}$
$\mathfrak{X}_{13}$	$81\{87,88\}$; $82\{88\}$; $83\{87,89\}$; $84\{88,89\}$; $85\{88,89\}$; $86\{87,89\}$
$\mathfrak{X}_{14}$	$87\{90\}$; $88\{90\}$; $89\{90\}$

Table 6. $A_5(xii)$

$\mathfrak{X}_i$	$j\{k \in \mathfrak{X}_{i+1} \mid j \prec k\}(j \in \mathfrak{X}_i)$
$\mathfrak{X}_4$	$1\{2,3,4\}$
$\mathfrak{X}_5$	$2\{5,6,7,8,10\}$; $3\{6,8,9\}$; $4\{7,9,10\}$
$\mathfrak{X}_6$	$5\{11,12,13,15\}$; $6\{11,13,14,16,18\}$; $7\{12,14,15,17,19\}$; $8\{13,16,17\}$; $9\{14,17,18\}$; $10\{15,18,19\}$
$\mathfrak{X}_7$	$11\{20,21,23,25\}$; $12\{21,22,24,26\}$; $13\{20,23,24,27,29\}$; $14\{21,24,25,28,30\}$; $15\{22,25,26,29,31\}$; $16\{23,27,28\}$; $17\{24,28,29\}$; $18\{25,29,30\}$; $19\{26,30,31\}$
$\mathfrak{X}_8$	$20\{32,33,36,38\}$; $21\{33,34,37,39\}$; $22\{34,35,38,40\}$; $23\{32,36,37,43\}$; $24\{33,37,38,41,44\}$; $25\{34,38,39,42,45\}$; $26\{35,39,40,43\}$; $27\{36,41,42\}$; $28\{37,41,42,43\}$; $29\{38,42,43,44\}$; $30\{39,43,44,45\}$; $31\{40,44,45\}$
$\mathfrak{X}_9$	$32\{46,47,53\}$; $33\{47,48,51,54\}$; $34\{48,49,52,55\}$; $35\{49,50,53\}$; $36\{46,51,52,58\}$; $37\{47,51,52,53,56\}$; $38\{48,52,53,54,57,59\}$; $39\{49,53,54,55,58\}$; $40\{50,54,55,56\}$; $41\{51,59\}$; $42\{52,58,59\}$; $43\{53,56,58\}$; $44\{54,56,57\}$; $45\{55,57\}$

$\mathfrak{X}_{10}$	$46\{60,61,70\}$; $47\{60,61,62,65\}$; $48\{61,62,63,69,71\}$; $49\{62,63,64,70\}$; $50\{63,64,65\}$; $51\{60,68,71\}$; $52\{61,66,70,71\}$; $53\{62,65,67,68,70\}$; $54\{63,65,66,69\}$; $55\{64,67,69\}$; $56\{65,66,67\}$; $57\{67,69\}$; $58\{66,68,70\}$; $59\{68,71\}$
$\mathfrak{X}_{11}$	$60\{76,80\}$; $61\{74,79,80\}$; $62\{73,75,76,79\}$; $63\{73,74,78\}$; $64\{75,78\}$; $65\{72,73,74,75\}$; $66\{72,74,77\}$; $67\{75,77\}$; $68\{72,76\}$; $69\{75,77,78\}$; $70\{74,76,77,79\}$; $71\{72,76,80\}$
$\mathfrak{X}_{12}$	$72\{81,82\}$; $73\{81,83,84\}$; $74\{81,83,86\}$; $75\{82,84,86\}$; $76\{81,82,85\}$; $77\{82,86\}$; $78\{84,86\}$; $79\{83,85,86\}$; $80\{81,85\}$
$\mathfrak{X}_{13}$	$81\{87,88\}$; $82\{88\}$; $83\{87,89\}$; $84\{88,89\}$; $85\{87,88\}$; $86\{88,89\}$
$\mathfrak{X}_{14}$	$87\{90\}$; $88\{90\}$; $89\{90\}$

Table 7. $A_5(xvii)$

$\mathfrak{X}_i$	$j\{k \in \mathfrak{X}_{i+1} \mid j \prec k\}(j \in \mathfrak{X}_i)$
$\mathfrak{X}_5$	$1\{2,3\}$
$\mathfrak{X}_6$	$2\{4,5,7\}$; $3\{4,5,6\}$;
$\mathfrak{X}_7$	$4\{8,9,10,11\}$; $5\{10,11,12\}$; $6\{8,12\}$; $7\{9,10\}$
$\mathfrak{X}_8$	$8\{13,16,18\}$; $9\{15,16,17\}$; $10\{13,14,15,17,19\}$; $11\{17,18,19\}$; $12\{13,14,18\}$
$\mathfrak{X}_9$	$13\{20,21,23,25\}$; $14\{20,26\}$; $15\{20,22,23,24\}$; $16\{21,23\}$; $17\{21,22,24,26\}$; $18\{21,25,26\}$; $19\{24,25\}$
$\mathfrak{X}_{10}$	$20\{27,28,31\}$; $21\{28,29,30,32\}$; $22\{28,29,33\}$; $23\{27,29,30\}$; $24\{30,31,33\}$; $25\{30,31\}$; $26\{28,31,32\}$
$\mathfrak{X}_{11}$	$27\{34,37\}$; $28\{34,35,37\}$; $29\{34,35,36\}$; $30\{36,37,38\}$; $31\{37,38\}$; $32\{35,38\}$; $33\{36,37\}$
$\mathfrak{X}_{12}$	$34\{39,42\}$; $35\{41,42\}$; $36\{39,41\}$; $37\{39,40,41\}$; $38\{40,41\}$
$\mathfrak{X}_{13}$	$39\{43,44\}$; $40\{43\}$; $41\{43,44\}$; $42\{44\}$
$\mathfrak{X}_{14}$	$43\{45\}$; $44\{45\}$

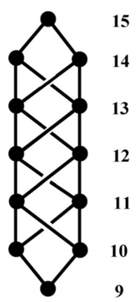


Fig. 6. $A_5(xiii)$.

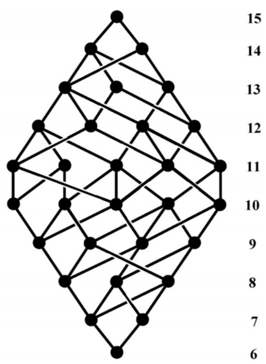


Fig. 7. $A_5(xiv)$.

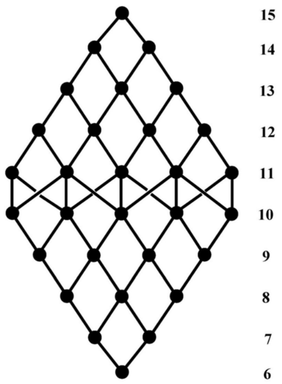


Fig. 8. $A_5(xv)$.

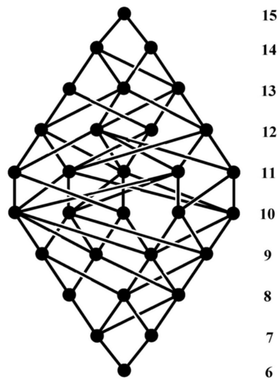


Fig. 9. $A_5(xvi)$.

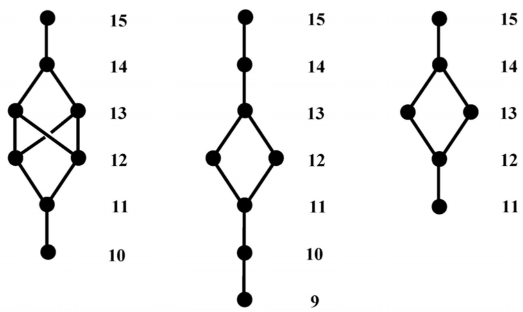
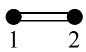
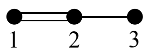


Fig. 10. $A_5(xviii)$(left), $A_5(xix)$(centre) and $A_5(xx)$(right).

(2.5) W of type B_2, 

. For $X \leq W$ either $\mathfrak{X} \cong \mathcal{B}(\langle X\rangle_{\rm sp})$ or one of the following holds:-

X	$\mathfrak{X}$
$\langle[212]\rangle \cong \mathbb{Z}_2$	$\mathcal{C}_2$
$\langle[121]\rangle \cong \mathbb{Z}_2$	$\mathcal{C}_2$

(2.6) W of type B_3, 

. For $X \leq W$ either $\mathfrak{X} \cong \mathcal{B}(\langle X\rangle_{\rm sp})$ or one of the following holds:-

X	$\mathfrak{X}$
$\langle[21232123]\rangle \cong \mathbb{Z}_2$	$\mathcal{C}_2$
$\langle[12123212]\rangle \cong \mathbb{Z}_2$	$\mathcal{C}_2$
$\langle[12132123]\rangle \cong \mathbb{Z}_2$	$\mathcal{C}_2$

$\langle[2123212]\rangle \cong \mathbb{Z}_2$	$\mathcal{C}_3$
$\langle[132123]\rangle \cong \mathbb{Z}_2$	$\mathcal{C}_4$
$\langle[121321]\rangle \cong \mathbb{Z}_2$	$\mathcal{C}_4$
$\langle[232123]\rangle \cong \mathbb{Z}_2$	$\mathcal{C}_4$
$\langle[32123]\rangle \cong \mathbb{Z}_2$	$\lvert\mathfrak{X}\rvert = 6$; see $B_3(v)$
$\langle[2132]\rangle \cong \mathbb{Z}_2$	$\lvert\mathfrak{X}\rvert = 8$; see $B_3(vi)$
$\langle[12321]\rangle \cong \mathbb{Z}_2$	$\lvert\mathfrak{X}\rvert = 8$; see $B_3(iii)$
$\langle[232]\rangle \cong \mathbb{Z}_2$	$\lvert\mathfrak{X}\rvert = 8$; see $B_3(iv)$
$\langle[212]\rangle \cong \mathbb{Z}_2$	$\lvert\mathfrak{X}\rvert = 12$; see $B_3(i)$
$\langle[121]\rangle \cong \mathbb{Z}_2$	$\lvert\mathfrak{X}\rvert = 12$; see $B_3(ii)$
$\langle[1213]\rangle \cong \mathbb{Z}_3$	$\mathcal{C}_4$
$\langle[2123]\rangle \cong \mathbb{Z}_4$	$\mathcal{C}_2$
$\langle[1232]\rangle \cong \mathbb{Z}_4$	$\mathcal{C}_2$
$\langle[2123], [21232123]\rangle \cong 2^2$	$\mathcal{C}_2$
$\langle[1232], [132123]\rangle \cong 2^2$	$\mathcal{C}_2$
$\langle[212], [21232123]\rangle \cong 2^2$	$\mathcal{C}_2$
$\langle[1], [2123212]\rangle \cong 2^2$	$\mathcal{C}_2$
$\langle[212], [12321]\rangle \cong 2^2$	$\mathcal{C}_2$
$\langle[32123], [121]\rangle \cong 2^2$	$\mathcal{C}_2$
$\langle[212], [232]\rangle \cong 2^2$	$\mathcal{C}_4$
$\langle[32123], [2]\rangle \cong 2^2$	$\mathcal{C}_4$
$\langle[1], [132123]\rangle \cong 2^2$	$\mathcal{C}_4$
$\langle[121], [3]\rangle \cong \mathrm{Sym}(3)$	$\mathcal{C}_4$
$X = W_{12}^g \neq W_{12}$ (twice) $\cong \mathrm{Dih}(8)$	$\mathcal{C}_2$
$X = W_{12}^h \neq W_{12}$ (twice) $\cong \mathrm{Dih}(8)$	$\mathcal{C}_4$

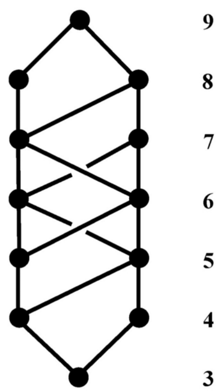


Fig. 11. $B_3(i)$.

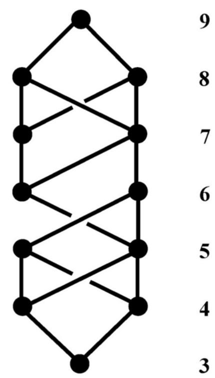


Fig. 12. $B_3(ii)$.

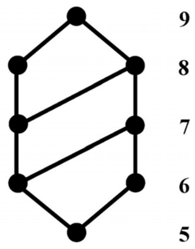


Fig. 13. $B_3(iii)$.

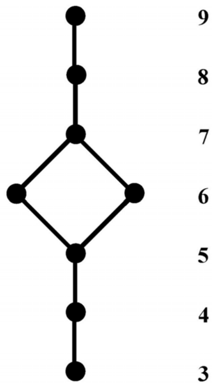


Fig. 14. $B_3(iv)$.

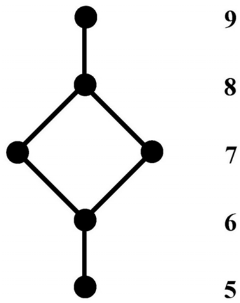


Fig. 15. $B_3(v)$.

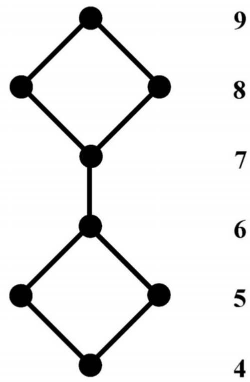
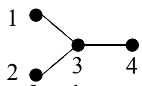


Fig. 16. $B_3(vi)$.

(2.7) W of type D_4, 

. On account of the graph automorphisms we shall only consider conjugates of subgroups of $W_{234} = \langle [2], [3], [4] \rangle$ and $W_{124} = \langle [1], [2], [4] \rangle$. For $X \leq W$, either $\mathfrak{X} \cong \mathcal{B}(\langle X \rangle_{\rm sp})$ or one of the following holds:-

X	$\mathfrak{X}$
$\langle [12312343123] \rangle$, $\langle [12312431234] \rangle$, $\langle [13123431234] \rangle$, $\langle [23123431234] \rangle$	$\mathcal{C}_2$
$\langle [1312343123] \rangle$	$\mathcal{C}_3$
$\langle [123124312] \rangle$, $\langle [132431234] \rangle$, $\langle [231431234] \rangle$, $\langle [312343123] \rangle$	$\mathcal{C}_4$

$X \cong \mathbb{Z}_2$	$\langle[12324312]\rangle$	$\mathcal{C}_5$
	$\langle[1312431]\rangle$, $\langle[2312432]\rangle$, $\langle[3431234]\rangle$, $\langle[1234312]\rangle$	$\mathcal{C}_6$
	$\langle[132431]\rangle$	$\mathcal{C}_7$
	$\langle[31243]\rangle$	$\mathcal{C}_8$
	$\langle[43234]\rangle$	$\lvert\mathfrak{X}\rvert = 16$; see $D_4(i)$
	$\langle[3243]\rangle$	$\lvert\mathfrak{X}\rvert = 24$; see $D_4(ii)$
$X \cong \mathbb{Z}_3$	$\langle[124312]\rangle$ $\langle[123432]\rangle$ $\langle[134312]\rangle$	$\mathcal{C}_4$
$X \cong \mathbb{Z}_4$	$\langle[12431]\rangle$ $\langle[32431]\rangle$ $\langle[14312]\rangle$	$\mathcal{C}_3$
	$\langle[1324312]\rangle$	$\mathcal{C}_5$
$X \cong 2^2$	X conjugate to $Y_1 = \langle[12],[124]\rangle$, $X \neq Y_1, X_1$ or X_2 (9 conjugates)	$\mathcal{C}_2$
	X conjugate to $Y_2 = \langle[14],[24]\rangle$, $X \neq Y_2, X_3$ (10 conjugates)	$\mathcal{C}_2$
	$\langle[132431],[3243]\rangle$ $\langle[13431],[132431]\rangle$ $\langle[1234312],[12324312]\rangle$ $\langle[312343123],[1312343123]\rangle$	$\mathcal{C}_3$
	$\langle[132431],[24]\rangle$	$\mathcal{C}_5$
	$X_1 = \langle[3123],[31243]\rangle$ $X_2 = \langle[431234],[3431234]\rangle$ $X_3 = \langle[3243],[3143]\rangle$	$\mathcal{C}_6$
$X \cong \mathrm{Sym}(3)$	$\langle[4],[123124]\rangle$ $\langle[23432],[123432]\rangle$ $\langle[13431],[123431]\rangle$	$\mathcal{C}_4$
$X \cong \mathrm{Dih}(8)$	$\langle[1234312],[24]\rangle$ $\langle[312343123],[3243]\rangle$ $\langle[13431],[24]\rangle$	$\mathcal{C}_3$
	$\langle[4],[132431]\rangle$	$\mathcal{C}_5$

3. Some Observations

3.1. Möbius Functions

It was first shown by Deodhar [3] that the Möbius function of the Bruhat order of any Coxeter group takes values -1, 1. Generalized quotients, introduced by Björner and Wachs [1] and having the Bruhat order of a Coxeter group as a special case, have

Möbius functions which only take values -1, 0, 1. This is not true for X-posets as may be seen by looking at $A_5(v)$ where W is of type A_5 and $X = \langle[132143]\rangle \cong \mathbb{Z}_2$. The elements $\mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$, $\mathbf{x}_5$ are all the elements of length 11 which are greater than $\mathbf{x}_6$ (of length 10) and less than $\mathbf{x}_1$ (of length 12). So (where μ denotes the Möbius function of $\mathfrak{X}$)

$$\mu(\mathbf{x}_6, \mathbf{x}_1) + \mu(\mathbf{x}_2, \mathbf{x}_1) + \mu(\mathbf{x}_3, \mathbf{x}_1) + \mu(\mathbf{x}_4, \mathbf{x}_1) + \mu(\mathbf{x}_5, \mathbf{x}_1) + \mu(\mathbf{x}_1, \mathbf{x}_1) = 0$$

whence $\mu(\mathbf{x}_6, \mathbf{x}_1) - 1 - 1 - 1 - 1 + 1 = 0$, and hence $\mu(\mathbf{x}_6, \mathbf{x}_1) = 3$. In particular, $\mathfrak{X}$ cannot be a generalized quotient.

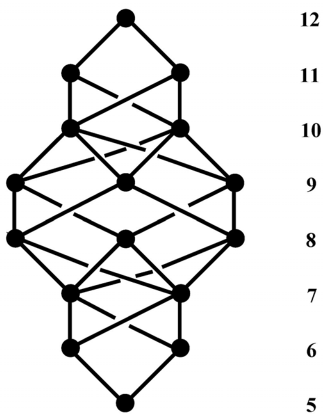


Fig. 17. $D_4(i)$.

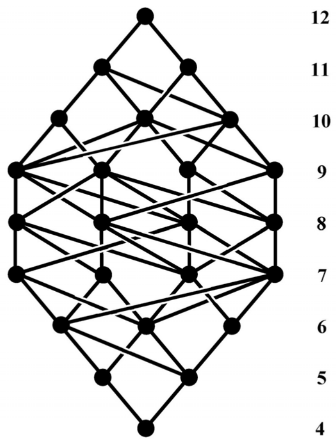


Fig. 18. $D_4(ii)$.

3.2. Odd and even elements in intervals

In any interval of the Bruhat order of a Coxeter group there are the same number of odd (length) elements as there are even (length) elements [6]. This property is not shared by X-posets. For example in $A_4(ii)$ the interval between (and including) $\mathbf{x}_1$ and $\mathbf{x}_3$ has two even elements and only one odd element (as a more substantial example take the interval between $\mathbf{x}_1$ and $\mathbf{x}_4$).

3.3. X-posets in standard parabolic subgroups

Suppose that X is a subgroup of Y where Y is a standard parabolic subgroup of W. Let $\mathfrak{X}_Y$ be the X-poset in Y. Then there are a number of connections between $\mathfrak{X}_Y$ and $\mathfrak{X}$ (see [5]) Does $\mathfrak{X}_Y$ exert even greater control upon the structure of $\mathfrak{X}$? The answer appears to be a resounding no. Looking at (2.3) and (2.4) and taking W of type A_5 with $Y = W_{1234}$ we have that the X-posets for $\langle[12324321]\rangle$ and $\langle[21321432]\rangle$ in Y are isomorphic (both are $\mathcal{C}_2$) but the X-posets in W are not isomorphic. Note that both $\langle[12324321]\rangle$ and $\langle[21321432]\rangle$ also have the same standard parabolic closure in Y. There are other examples like this for W of type A_5 as well as for W of type B_3.

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