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GENERAL DECAY RATE ESTIMATE FOR THE ENERGY OF A WEAK VISCOELASTIC EQUATION WITH AN INTERNAL TIME-VARYING DELAY TERM

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Abstract. In this paper we consider the weak viscoelastic equation with an internal time-varying delay term

$$u_{tt}(x,t)-\Delta u(x,t)+\alpha(t)\int_0^t g(t-s)\Delta u(x,s)\,\mathrm{d}s+a_0u_t(x,t)+a_1u_t(x,t-\tau(t))=0$$

in a bounded domain. By introducing suitable energy and Lyapunov functionals, under suitable assumptions, we establish a general decay rate estimate for the energy, which depends on the behavior of both α and g.

1. Introduction

In this work, we investigate the following weak viscoelastic equation with a linear damping and a time-varying delay term in the internal feedback

$$(1.1)\quad \begin{cases} u_{tt}(x,t)-\Delta u(x,t)+\alpha(t)\displaystyle\int_0^t g(t-s)\Delta u(x,s)\,\mathrm{d}s \\ \qquad +a_0u_t(x,t)+a_1u_t(x,t-\tau(t))=0, & (x,t)\in\Omega\times(0,\infty), \\ u(x,t)=0, & (x,t)\in\partial\Omega\times[0,\infty), \\ u(x,0)=u_0(x),\quad u_t(x,0)=u_1(x), & x\in\Omega, \\ u_t(x,t)=f_0(x,t) & (x,t)\in\Omega\times[-\tau(0),0), \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^n (n\ge 2)$ with a boundary $\partial\Omega$ of class C^2, α and g are positive non-increasing functions defined on $\mathbb{R}^+$, a_0 and a_1 are real numbers with

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$a_0 > 0$, $\tau(t) > 0$ represents the time-varying delay, and the initial datum u_0, u_1, f_0 are given functions belonging to suitable spaces.

Time delays so often arise in many physical, chemical, biological, thermal and economical phenomena. In recent years, the control of PDEs with time delay effects has become an active area of research, see for instance [1, 15, 17, 28, 34] and the references therein. The presence of delay may be a source of instability. For example, it was proved in [9, 10, 22, 23, 30] that an arbitrarily small delay may destabilize a system which is uniformly asymptotically stable in the absence of delay unless additional conditions or control terms have been used. In [22], Nicaise and Pignotti examined (1.1) with $g \equiv 0, a_0 > 0, a_1 > 0$ and $\tau(t) \equiv \tau$ be a constant delay in the case of mixed homogeneous Dirichlet-Neumann boundary conditions, under a geometric condition on the Neumann part of the boundary. Assuming that $0 \leq a_1 < a_0$, a stabilization result is given, by using a suitable observability estimate and inequalities obtained from Carleman estimates for the wave equation due to Lasiecka et al. in [14]. However, for the opposite case $a_1 \geq a_0$, they were able to construct a sequence of delays for which the corresponding solution is unstable. The same results were obtained for the case when both the damping and the delay act on the boundary, see also [2] for the treatment of this problem in more general abstract form. Kirane and Said-Houari [13] considered (1.1) with $\alpha(t) \equiv 1, a_0 > 0, a_1 > 0$ and $\tau(t) \equiv \tau$ be a constant delay. They established general energy decay results under the condition that $0 \leq a_1 \leq a_0$. The stability of PDEs with time-varying delays was studied in [5, 11, 24, 25]. In [25], Nicaise et al. analyzed the exponential stability of the heat and wave equations with time-varying boundary delay in one space dimension, under the condition $0 \leq a_1 < \sqrt{1-d}\, a_0$, where d is a constant such that $\tau'(t) \leq d < 1, \forall\, t > 0$. In [24], Nicaise and Pignotti studied the stabilization problem by interior damping of the wave equation with internal time-varying delay feedback and obtained exponential stability estimates by introducing suitable Lyapunov functionals, under the condition $|a_1| < \sqrt{1-d}\, a_0$ in which the positivity of the coefficient a_1 is not necessary. More recently, the present author [17] considered (1.1) with $\alpha(t) \equiv 1$ and established a general energy decay result from which the exponential and polynomial types of decay are only special cases.

We also recall some results regarding the viscoelastic equation without delay (i.e., $a_1 = 0$). Cavalcanti et al. [7] studied

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)\mathrm{d}\tau + a(x)u_t + |u|^\gamma u = 0, \quad (x,t) \in \Omega \times (0,\infty),$$

for $a : \Omega \to \mathbb{R}^+$, a function, which may be null on a part of the domain Ω. Under the conditions that $a(x) \geq a_0 > 0$ on $\omega \subset \Omega$, with ω satisfying some geometry restrictions and $-\xi_1 g(t) \leq g'(t) \leq -\xi_2 g(t),\ t \geq 0$, the authors established an exponential rate of decay. Berrimi and Messaoudi [3] improved Cavalcanti's result by introducing a different functional which allowed to weak the conditions on both a and g. In [8], Cavalcanti et al. considered

$$u_{tt} - k_0\Delta u + \int_0^t \mathrm{div}[a(x)g(t-\tau)\nabla u(\tau)]\mathrm{d}\tau + b(x)h(u_t) + f(u) = 0,$$

under similar conditions on the relaxation function g and $a(x) + b(x) \geq \rho > 0$, for all $x \in \Omega$. They improved the result of [7] by establishing exponential stability for g decaying exponentially and h linear and polynomial stability for g decaying polynomially and h nonlinear. Berrimi and Messaoudi [4] considered

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)\mathrm{d}\tau = |u|^{p-2}u, \quad p > 2$$

in a bounded domain. They showed, under weaker conditions than those in [8], that the solution is global and decay in a polynomial or exponential fashion when the initial data is small enough. Then Messaoudi [19] improved this result by establishing a general decay of energy which is similar to the relaxation function. Recently, Messaoudi [21] considered problem (1.1) without the linear damping and the time-varying delay term, and proved a general decay result which depends both on the behavior of α and g. For other related works, we refer the readers to [6, 16, 18, 26, 27, 20, 29, 31, 32, 33] and the references therein.

Motivatied by these results, we investigate in this paper system (1.1) under suitable assumptions and prove a general decay rate estimate for the energy, which depends on the behavior of both α and g. This work extends the previous results in [13, 17, 21, 22] to the weak viscoelastic equation and to time-varying delay with not necessarily positive coefficient a_1 of the delay term. For our purpose, we use the idea of Nicaise and Pignotti in [24] (see also [17]) to take into account the dependence of the delay with respect to time, and some techniques of Messaoudi in [21] to deal with the weak viscoelastic term.

The paper is organized as follows. In Section 2 we present some assumptions and state the main result. The general decay result is proved in Sections 3.

2. Preliminaries and Main Results

In this section, we present some assumptions and state the main result. We use the standard Lebesgue space $L^2(\Omega)$ and the Sobolev space $H_0^1(\Omega)$ with their usual scalar products and norms. Throughout this paper, C_i is used to denote a generic positive constant.

For the relaxation function g and the potential α, we assume that (see [21]):

(G1) $g, \alpha : \mathbb{R}^+ \to \mathbb{R}^+$ are non-increasing differentiable functions satisfying

$$g(0) > 0, \quad \int_0^{+\infty} g(s)\mathrm{d}s < +\infty, \quad \alpha(t) > 0, \quad 1-\alpha(t)\int_0^t g(s)\,\mathrm{d}s \geq l > 0.$$

(G2) There exists a non-increasing differentiable function $\xi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$\xi(t) > 0, \quad g'(t) \leq -\xi(t)g(t), \quad \forall\, t \geq 0, \quad \lim_{t\to+\infty} \frac{-\alpha'(t)}{\xi(t)\alpha(t)} = 0.$$

For the time-varying delay, we assume as in [24] that there exist positive constants τ_0, $\overline{\tau}$ such that

$$0 < \tau_0 \leq \tau(t) \leq \overline{\tau}, \quad \forall\, t > 0. \tag{2.1}$$

Moreover, we assume that the speed of the delay satisfies

$$\tau'(t) \leq d < 1, \quad \forall\, t > 0, \tag{2.2}$$

that

$$\tau \in W^{2,\infty}([0,T]), \quad \forall\, T > 0 \tag{2.3}$$

and that a_0, a_1 satisfy

$$|a_1| < \sqrt{1-d}\, a_0. \tag{2.4}$$

As in [24], let us introduce the function

$$z(x,\rho,t) = u_t(x, t-\tau(t)\rho), \quad x \in \Omega,\ \rho \in (0,1),\ t > 0. \tag{2.5}$$

Then, problem (1.1) is equivalent to

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) + \alpha(t)\displaystyle\int_0^t g(t-s)\Delta u(x,s)\,\mathrm{d}s & \\ \qquad\qquad + a_0 u_t(x,t) + a_1 z(x,1,t) = 0, & (x,t) \in \Omega\times(0,\infty), \\ \tau(t) z_t(x,\rho,t) + (1-\tau'(t)\rho) z_\rho(x,\rho,t) = 0, & (x,\rho,t) \in \Omega\times(0,1)\times(0,\infty), \\ u(x,t) = 0, & (x,t)\in\partial\Omega\times[0,\infty), \\ z(x,0,t) = u_t(x,t), & (x,t)\in\Omega\times(0,\infty), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \Omega, \\ z(x,\rho,0) = f_0(x, -\rho\tau(0)), & (x,\rho)\in\Omega\times(0,1). \end{cases} \tag{2.6}$$

We now state, without a proof, a well-posedness result, which can be established by combining the arguments of [12, 13].

Lemma 2.1. *Let* (2.1)-(2.4) *be satisfied and* g, α *satisfy* (G1)*. Then given* $u_0 \in H_0^1(\Omega)$*,* $u_1 \in L^2(\Omega)$*,* $f_0 \in L^2(\Omega \times (0,1))$ *and* $T > 0$*, there exists a unique weak solution* (u, z) *of the problem* (2.6) *on* $(0, T)$ *such that*

$$u \in C(0,T; H_0^1(\Omega)) \cap C^1(0,T; L^2(\Omega)), \quad u_t \in L^2(0,T; H_0^1(\Omega)) \cap L^2((0,T) \times \Omega).$$

Inspired by [17, 21, 24], we define the energy functional as

$$\begin{aligned} E(t) : &= \frac{1}{2}\int_\Omega \left[u_t^2 + \left(1 - \alpha(t)\int_0^t g(s)\mathrm{d}s\right)|\nabla u|^2\right]\mathrm{d}x + \frac{1}{2}\alpha(t)(g \circ \nabla u)(t) \\ &+\frac{\xi}{2}\int_{t-\tau(t)}^t \int_\Omega e^{\lambda(s-t)} u_t^2(s)\mathrm{d}x\mathrm{d}s, \end{aligned} \tag{2.7}$$

where ξ, λ are suitable positive constants, and

$$(g \circ v)(t) = \int_\Omega \int_0^t g(t-s)|v(t) - v(s)|^2 \mathrm{d}s\mathrm{d}x, \quad \forall\, v \in L^2(\Omega).$$

We will fix ξ such that

$$2a_0 - \frac{|a_1|}{\sqrt{1-d}} - \xi > 0 \quad \text{and} \quad \xi - \frac{|a_1|}{\sqrt{1-d}} > 0, \tag{2.8}$$

and

$$\lambda < \frac{1}{\overline{\tau}}\left|\log \frac{|a_1|}{\xi\sqrt{1-d}}\right|. \tag{2.9}$$

In fact, the existence of such a constant ξ is guaranteed by the assumption (2.4).

Our main result reads as follows.

Theorem 2.2. *Let* (2.1)–(2.4) *be satisfied and* g, α *satisfy* (G1)-(G2)*. Then there exist two positive constants* K, k *such that, for any solution of problem* (1.1)*, the energy satisfies*

$$E(t) \le K e^{-k\int_0^t \alpha(s)\xi(s)\mathrm{d}s}, \quad \forall\, t \ge 0. \tag{2.10}$$

3. Proof of the General Decay Result

In this section, we give the proof of the general decay result. We have the following lemmas.

Lemma 3.1. *Let* (2.1)-(2.4) *be satisfied and* g, α *satisfy* (G1). *Then for all regular solution of problem* (1.1)*, the energy functional defined by* (2.7) *satisfies*

$$
\begin{aligned}
E'(t) \le\ & \frac{1}{2}\alpha(t)(g' \circ \nabla u)(t) - \frac{1}{2}\alpha(t)g(t)\int_\Omega |\nabla u|^2 \mathrm{d}x \\
& -C_1 \int_\Omega \left[u_t^2(t) + u_t^2(t-\tau(t))\right] \mathrm{d}x - \frac{\lambda\xi}{2}\int_{t-\tau(t)}^t \int_\Omega e^{\lambda(s-t)} u_t^2(s)\mathrm{d}x\mathrm{d}s \\
& + \frac{1}{2}\alpha'(t)(g \circ \nabla u)(t) - \frac{1}{2}\alpha'(t)\left(\int_0^t g(s)\mathrm{d}s\right)\int_\Omega |\nabla u|^2 \mathrm{d}x \\
\le\ & \frac{1}{2}\alpha(t)(g' \circ \nabla u)(t) - \frac{1}{2}\alpha'(t)\left(\int_0^t g(s)\mathrm{d}s\right)\int_\Omega |\nabla u|^2 \mathrm{d}x,
\end{aligned}
\tag{3.1}
$$

for some positive constant C_1.

Proof. Differentiating (2.7) and by (1.1), we obtain

$$
\begin{aligned}
E'(t) =& \int_\Omega \left[u_t u_{tt} + \left(1 - \alpha(t)\int_0^t g(s)\mathrm{d}s\right)\nabla u \cdot \nabla u_t - \frac{1}{2}\alpha(t)g(t)|\nabla u|^2\right]\mathrm{d}x \\
& + \frac{1}{2}\alpha'(t)(g \circ \nabla u)(t) - \frac{1}{2}\alpha'(t)\left(\int_0^t g(s)\mathrm{d}s\right)\int_\Omega |\nabla u|^2 \mathrm{d}x \\
& + \alpha(t)\int_0^t g(t-s)\int_\Omega \nabla u_t(t)\cdot[\nabla u(t) - \nabla u(s)]\mathrm{d}x\mathrm{d}s \\
& + \frac{1}{2}\alpha(t)\int_0^t g'(t-s)\int_\Omega |\nabla u(t) - \nabla u(s)|^2 \mathrm{d}x\mathrm{d}s + \frac{\xi}{2}\int_\Omega u_t^2(t)\mathrm{d}x \\
& - \frac{\xi}{2}\int_\Omega e^{-\lambda\tau(t)} u_t^2(t-\tau(t))(1-\tau'(t))\mathrm{d}x - \lambda\frac{\xi}{2}\int_{t-\tau(t)}^t \int_\Omega e^{-\lambda(t-s)} u_t^2(s)\mathrm{d}x\mathrm{d}s \\
=& \int_\Omega \left[u_t u_{tt} + \nabla u \cdot \nabla u_t - \alpha(t)\int_0^t g(t-s)\nabla u(s)\cdot \nabla u_t(t)\mathrm{d}s\right]\mathrm{d}x \\
& + \frac{1}{2}\alpha'(t)(g \circ \nabla u)(t) - \frac{1}{2}\alpha'(t)\left(\int_0^t g(s)\mathrm{d}s\right)\int_\Omega |\nabla u|^2 \mathrm{d}x \\
& - \frac{1}{2}\alpha(t)g(t)\int_\Omega |\nabla u|^2 \mathrm{d}x + \frac{1}{2}\alpha(t)(g' \circ \nabla u)(t) + \frac{\xi}{2}\int_\Omega u_t^2(t)\mathrm{d}x \\
& - \frac{\xi}{2}\int_\Omega e^{-\lambda\tau(t)} u_t^2(t-\tau(t))(1-\tau'(t))\mathrm{d}x - \lambda\frac{\xi}{2}\int_{t-\tau(t)}^t \int_\Omega e^{-\lambda(t-s)} u_t^2(s)\mathrm{d}x\mathrm{d}s,
\end{aligned}
$$

and then, using integration by parts, the assumptions (2.1)-(2.2) and some manipulations as in [24],

$$
\begin{aligned}
&E'(t)\\
= &-a_0\int_\Omega u_t^2(t)\mathrm{d}x-a_1\int_\Omega u_t(t)\int_\Omega u_t(t-\tau(t))\mathrm{d}x-\frac{1}{2}\alpha(t)g(t)\int_\Omega|\nabla u|^2\mathrm{d}x\\
&+\frac{1}{2}\alpha(t)(g'\circ\nabla u)(t)+\frac{\xi}{2}\int_\Omega u_t^2(t)\mathrm{d}x-\frac{\xi}{2}\int_\Omega e^{-\lambda\tau(t)}u_t^2(t-\tau(t))(1-\tau'(t))\mathrm{d}x\\
&-\lambda\frac{\xi}{2}\int_{t-\tau(t)}^t\int_\Omega e^{-\lambda(t-s)}u_t^2(s)\mathrm{d}x\mathrm{d}s\\
&+\frac{1}{2}\alpha'(t)(g\circ\nabla u)(t)-\frac{1}{2}\alpha'(t)\left(\int_0^t g(s)\mathrm{d}s\right)\int_\Omega|\nabla u|^2\mathrm{d}x\\
\le &-a_0\int_\Omega u_t^2(t)\mathrm{d}x-a_1\int_\Omega u_t(t)\int_\Omega u_t(t-\tau(t))\mathrm{d}x-\frac{1}{2}\alpha(t)g(t)\int_\Omega|\nabla u|^2\mathrm{d}x\\
&+\frac{1}{2}\alpha(t)(g'\circ\nabla u)(t)+\frac{\xi}{2}\int_\Omega u_t^2(t)\mathrm{d}x-\frac{\xi}{2}(1-d)e^{-\lambda\overline{\tau}}\int_\Omega u_t^2(t-\tau(t))\mathrm{d}x\\
&-\lambda\frac{\xi}{2}\int_{t-\tau(t)}^t\int_\Omega e^{-\lambda(t-s)}u_t^2(s)\mathrm{d}x\mathrm{d}s\\
&+\frac{1}{2}\alpha'(t)(g\circ\nabla u)(t)-\frac{1}{2}\alpha'(t)\left(\int_0^t g(s)\mathrm{d}s\right)\int_\Omega|\nabla u|^2\mathrm{d}x\\
\le &\ \frac{1}{2}\alpha(t)(g'\circ\nabla u)(t)-\frac{1}{2}\alpha(t)g(t)\int_\Omega|\nabla u|^2\mathrm{d}x-\left(a_0-\frac{|a_1|}{2\sqrt{1-d}}-\frac{\xi}{2}\right)\int_\Omega u_t^2(t)\mathrm{d}x\\
&-\left(e^{-\lambda\overline{\tau}}\frac{\xi}{2}(1-d)-\frac{|a_1|}{2}\sqrt{1-d}\right)\int_\Omega u_t^2(t-\tau(t))\mathrm{d}x\\
&-\lambda\frac{\xi}{2}\int_{t-\tau(t)}^t\int_\Omega e^{-\lambda(t-s)}u_t^2(s)\mathrm{d}x\mathrm{d}s\\
&+\frac{1}{2}\alpha'(t)(g\circ\nabla u)(t)-\frac{1}{2}\alpha'(t)\left(\int_0^t g(s)\mathrm{d}s\right)\int_\Omega|\nabla u|^2\mathrm{d}x.
\end{aligned}\tag{3.2}
$$

Combining (2.8)–(2.9), (3.2) and hypothese (G1), (3.1) is established. ∎

Remark 1. Since $-\alpha'(t)\left(\int_0^t g(s)\mathrm{d}s\right)\int_\Omega|\nabla u|^2\mathrm{d}x\ge 0$, $E(t)$ may not be non-increasing.

Now we are going to construct a Lyapunov functional L equivalent to E. For this purpose, we use the following functionals:

$$I(t):=\int_\Omega uu_t\mathrm{d}x,\tag{3.3}$$

$$K(t):=-\int_\Omega u_t\int_0^t g(t-s)(u(t)-u(s))\mathrm{d}s\mathrm{d}x,\tag{3.4}$$

Set

$$L(t)=NE(t)+\varepsilon\alpha(t)I(t)+\alpha(t)K(t)\tag{3.5}$$

where N and ε are suitable positive constants to be determined later. Similar as in [19, 21], we can prove that, for ε small enough while N large enough, there exist two positive constants β_1, β_2 such that

$$\beta_1 E(t) \leq L(t) \leq \beta_2 E(t), \quad \forall\, t \geq 0. \tag{3.6}$$

The following estimates hold true.

Lemma 3.2. ([21, Lemma 3.2]). *For $u \in H_0^1(\Omega)$, we have*

$$\int_\Omega \left(\int_0^t g(t-s)(u(t)-u(s))\mathrm{d}s \right)^2 \mathrm{d}x \leq C_p^2 \left(\int_0^t g(s)\mathrm{d}s \right) (g \circ \nabla u)(t),$$

where C_p is the Poincaré constant.

Lemma 3.3. *Under the assumption* (G1)*, the functional I satisfies, along the solution, the estimate*

$$I'(t) \leq -\frac{l}{2}\int_\Omega |\nabla u|^2 \mathrm{d}x + C_2 \int_\Omega [u_t^2(t) + u_t^2(t-\tau(t))]\mathrm{d}x + C_3 \alpha(t)(g \circ \nabla u)(t). \tag{3.7}$$

Proof. Differentiating and integrating by parts

$$\begin{aligned} I'(t) &= \int_\Omega u_t^2 \mathrm{d}x \\ &\quad + \int_\Omega u \left(\Delta u - \alpha(t)\int_0^t g(t-s)\Delta u(s)\mathrm{d}s - a_0 u_t(t) - a_1 u_t(t-\tau(t)) \right) \mathrm{d}x \\ &\leq \int_\Omega u_t^2 \mathrm{d}x - l\int_\Omega |\nabla u|^2 \mathrm{d}x + \alpha(t)\int_\Omega \nabla u \cdot \int_0^t g(t-s)(\nabla u(s) \\ &\quad - \nabla u(t))\mathrm{d}s\mathrm{d}x - a_0 \int_\Omega u(t)u_t(t)\mathrm{d}x - a_1\int_\Omega u(t)u_t(t-\tau(t))\mathrm{d}x. \end{aligned} \tag{3.8}$$

Now, using Young's inequality and (G1), we obtain (see [19])

$$\begin{aligned} &\alpha(t)\int_\Omega \nabla u \cdot \int_0^t g(t-s)(\nabla u(s) - \nabla u(t))\mathrm{d}s\mathrm{d}x \\ &\leq \delta \int_\Omega |\nabla u|^2 \mathrm{d}x + \frac{\alpha^2(t)}{4\delta}\int_\Omega \left(\int_0^t g(t-s)|\nabla u(s) - \nabla u(t)|\mathrm{d}s \right)^2 \mathrm{d}x \\ &\leq \delta \int_\Omega |\nabla u|^2 \mathrm{d}x + \frac{(1-l)\alpha(t)}{4\delta}(g \circ \nabla u)(t), \quad \forall\, \delta > 0. \end{aligned} \tag{3.9}$$

Also, using Young's and Poincaré's inequalities gives

$$-a_0\int_\Omega u(t)u_t(t)\mathrm{d}x \leq \delta\int_\Omega |\nabla u|^2 \mathrm{d}x + C(\delta)\int_\Omega u_t^2 \mathrm{d}x, \tag{3.10}$$

$$-a_1\int_\Omega u(t)u_t(t-\tau(t))\mathrm{d}x \leq \delta\int_\Omega |\nabla u|^2 \mathrm{d}x + C(\delta)\int_\Omega u_t^2(t-\tau(t))\mathrm{d}x. \tag{3.11}$$

Combining (3.8)–(3.11) and choosing δ small enough, we obtain (3.7). ∎

Lemma 3.4. *Under the assumption* (G1)*, the functional* K *satisfies, along the solution, the estimate*

$$
\begin{aligned}
K'(t) \leq & -\left(\int_0^t g(s)\mathrm{d}s - 2\delta\right)\int_\Omega u_t^2\mathrm{d}x + \delta\left[1+2(1-l)^2\right]\int_\Omega |\nabla u|^2\mathrm{d}x \\
& + \left(\frac{2+C_4}{4\delta} + 2\delta\alpha^2(t)\right)\left(\int_0^t g(s)\mathrm{d}s\right)(g\circ\nabla u)(t) \\
& - \frac{g(0)}{4\delta}C_p^2(g'\circ\nabla u)(t) + \delta\int_\Omega u_t^2(t-\tau(t))\mathrm{d}x.
\end{aligned} \tag{3.12}
$$

Proof. Using (1.1) and (3.4), we have

$$
\begin{aligned}
& K'(t) \\
&= -\int_\Omega u_{tt}\int_0^t g(t-s)(u(t)-u(s))\mathrm{d}s\mathrm{d}x \\
&\quad - \int_\Omega u_t\int_0^t g'(t-s)(u(t)-u(s))\mathrm{d}s\mathrm{d}x - \left(\int_0^t g(s)\mathrm{d}s\right)\|u_t\|_2^2 \\
&= \int_\Omega \nabla u(t)\cdot\left(\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\mathrm{d}s\right)\mathrm{d}x \\
&\quad -\alpha(t)\int_\Omega\left(\int_0^t g(t-s)\nabla u(s)\mathrm{d}s\right)\cdot\left(\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\mathrm{d}s\right)\mathrm{d}x \\
&\quad - \int_\Omega u_t\int_0^t g'(t-s)(u(t)-u(s))\mathrm{d}s\mathrm{d}x - \left(\int_0^t g(s)\mathrm{d}s\right)\|u_t\|_2^2 \\
&\quad + \int_\Omega\left(\int_0^t g(t-s)(u(t)-u(s))\mathrm{d}s\right)[a_0u_t(t)+a_1u_t(t-\tau(t))]\mathrm{d}x \\
&:= I_1+I_2+I_3+I_4+I_5.
\end{aligned} \tag{3.13}
$$

The first to the third terms on the right-hand side of (3.13) can be estimated as in [21] as follows, for any $\delta > 0$:

$$
\begin{aligned}
I_1 &\leq \delta\|\nabla u\|_2^2 + \frac{1}{4\delta}\left(\int_0^t g(s)\mathrm{d}s\right)(g\circ\nabla u)(t), \\
I_2 &\leq \delta\alpha^2(t)\int_\Omega\left(\int_0^t g(t-s)(|\nabla u(t)-\nabla u(s)|+|\nabla u(t)|)\mathrm{d}s\right)^2\mathrm{d}x \\
&\quad + \frac{1}{4\delta}\int_\Omega\left(\int_0^t g(t-s)|\nabla u(t)-\nabla u(s)|\mathrm{d}s\right)^2\mathrm{d}x \\
&\leq \left(2\delta\alpha^2(t)+\frac{1}{4\delta}\right)\left(\int_0^t g(s)\mathrm{d}s\right)(g\circ\nabla u)(t) + 2\delta(1-l)^2\int_\Omega|\nabla u|^2\mathrm{d}x, \\
I_3 &\leq \delta\int_\Omega u_t^2\mathrm{d}x - \frac{g(0)}{4\delta}C_p^2(g'\circ\nabla u)(t).
\end{aligned} \tag{3.14}
$$

As for the fifth term, we have

$$I_5 \le \frac{C_4}{4\delta}(g\circ\nabla u)(t) + \delta\int_\Omega u_t^2 \mathrm{d}x + \delta\int_\Omega u_t^2(t-\tau(t))\mathrm{d}x.$$

Summarizing these estimates with (3.13), we get (3.12). ■

Now, we are ready to prove the general decay result.

Proof of Theorem 2.2. Since g is positive, we have, for any $t_0 > 0$,

$$\int_0^t g(s)\mathrm{d}s \ge \int_0^{t_0} g(s)\mathrm{d}s := g_0 > 0, \quad t \ge t_0.$$

By using (3.1), (3.5), (3.7), (3.12) and (G1), a series of computations yields, for $t \ge t_0$,

$$\begin{aligned}
&L'(t)\\
\le\ & \frac{N}{2}\alpha(t)(g'\circ\nabla u)(t) - \frac{N}{2}\alpha'(t)\left(\int_0^t g(s)\mathrm{d}s\right)\int_\Omega |\nabla u|^2\mathrm{d}x\\
&-NC_1\int_\Omega [u_t^2(t)+u_t^2(t-\tau(t))]\mathrm{d}x - \frac{\lambda\xi N}{2}\int_{t-\tau(t)}^t\int_\Omega e^{\lambda(s-t)}u_t^2(s)\mathrm{d}x\mathrm{d}s\\
&+\varepsilon C_2\alpha(t)\int_\Omega [u_t^2(t)+u_t^2(t-\tau(t))]\mathrm{d}x - \frac{\varepsilon l}{2}\alpha(t)\int_\Omega |\nabla u|^2\mathrm{d}x\\
&+\varepsilon C_3\alpha^2(t)(g\circ\nabla u)(t) - \left(\int_0^t g(s)\mathrm{d}s - 2\delta\right)\alpha(t)\int_\Omega u_t^2\mathrm{d}x\\
&+\delta[1+2(1-l)^2]\alpha(t)\int_\Omega |\nabla u|^2\mathrm{d}x\\
&+\left(\frac{2+C_4}{4\delta}+2\delta\alpha^2(t)\right)\left(\int_0^t g(s)\mathrm{d}s\right)\alpha(t)(g\circ\nabla u)(t)\\
&-\frac{g(0)C_p^2}{4\delta}\alpha(t)(g'\circ\nabla u)(t)+\delta\alpha(t)\int_\Omega u_t^2(t-\tau(t))\mathrm{d}x+\varepsilon\alpha'(t)I(t)+\alpha'(t)K(t)\\
\le\ & -\alpha(t)(g_0-2\delta-\varepsilon C_2)\int_\Omega u_t^2\mathrm{d}x + \alpha(t)\left(\frac{N}{2}-\frac{g(0)C_p^2}{4\delta}\right)(g'\circ\nabla u)(t)\\
&-\frac{N}{2}\alpha'(t)\left(\int_0^t g(s)\mathrm{d}s\right)\int_\Omega |\nabla u|^2\mathrm{d}x - \alpha(t)\left(\frac{\varepsilon l}{2}-\delta[1+2(1-l)^2]\right)\int_\Omega |\nabla u|^2\mathrm{d}x\\
&-\frac{\lambda\xi N}{2}\int_{t-\tau(t)}^t\int_\Omega e^{\lambda(s-t)}u_t^2(s)\mathrm{d}x\mathrm{d}s\\
&-(NC_1-\delta\alpha(0)-\varepsilon C_2\alpha(0))\int_\Omega u_t^2(t-\tau(t))\mathrm{d}x\\
&+\alpha(t)\left(\varepsilon C_3\alpha(t)+\left(\frac{2+C_4}{4\delta}+2\delta\alpha^2(t)\right)\left(\int_0^t g(s)\mathrm{d}s\right)\right)(g\circ\nabla u)(t)\\
&+\varepsilon\alpha'(t)I(t)+\alpha'(t)K(t).
\end{aligned} \tag{3.15}$$

By using (3.3), (3.4), Young's and Poincaré's inequalities, we have

$$\begin{aligned}
&\varepsilon\alpha'(t)I(t)+\alpha'(t)K(t)\\
&\leq -\frac{\varepsilon\alpha'(t)C_p^2}{2}\int_\Omega|\nabla u|^2\mathrm{d}x-\frac{(1+\varepsilon)\alpha'(t)}{2}\int_\Omega u_t^2\mathrm{d}x\\
&\quad-\frac{\alpha'(t)}{2}\left(\int_0^t g(s)\mathrm{d}s\right)C_p^2(g\circ\nabla u)(t).
\end{aligned}$$

Hence, (3.15) takes the form

$$\begin{aligned}
L'(t)\leq{}& -\alpha(t)\left[(g_0-2\delta-\varepsilon C_2)+\frac{(1+\varepsilon)\alpha'(t)}{2\alpha(t)}\right]\int_\Omega u_t^2\mathrm{d}x\\
&+\alpha(t)\left(\frac{N}{2}-\frac{g(0)C_p^2}{4\delta}\right)(g'\circ\nabla u)(t)\\
&-\alpha(t)\left[\frac{\varepsilon l}{2}-\delta[1+2(1-l)^2]\right.\\
&\left.+\frac{N\alpha'(t)}{2\alpha(t)}\left(\int_0^t g(s)\mathrm{d}s\right)+\frac{\varepsilon\alpha'(t)C_p^2}{2\alpha(t)}\right]\int_\Omega|\nabla u|^2\mathrm{d}x\\
&-\frac{\lambda\xi N}{2}\int_{t-\tau(t)}^t\int_\Omega e^{\lambda(s-t)}u_t^2(s)\mathrm{d}x\mathrm{d}s-(NC_1\\
&-\delta\alpha(0)-\varepsilon C_2\alpha(0))\int_\Omega u_t^2(t-\tau(t))\mathrm{d}x\\
&+\alpha(t)\left[\varepsilon C_3\alpha(0)+\left(\frac{2+C_4}{4\delta}+2\delta\alpha^2(0)\right.\right.\\
&\left.\left.-\frac{C_p^2\alpha'(t)}{2\alpha(t)}\right)\left(\int_0^t g(s)\mathrm{d}s\right)\right](g\circ\nabla u)(t).
\end{aligned}\tag{3.16}$$

At this point, we choose ε small enough such that $\varepsilon<\frac{g_0}{4C_2}$ and (3.6) still hold, and δ sufficiently small such that

$$\alpha_1=\frac{\varepsilon l}{2}-\delta[1+2(1-l)^2]>0 \text{ and } \frac{g_0}{4}-2\delta>0.$$

As long as ε and δ are fixed, we choose N large enough such that

$$NC_1-\delta\alpha(0)-\varepsilon C_2\alpha(0)>0 \quad\text{and}\quad \frac{N}{2}-\frac{g(0)C_p^2}{4\delta}>0.$$

Thus, it follows from (3.16) that, for all $t\geq t_0$,

$$
\begin{aligned}
L'(t) \le & -\alpha(t)\left[\frac{g_0}{2}+\frac{(1+\varepsilon)\alpha'(t)}{2\alpha(t)}\right]\int_\Omega u_t^2 \mathrm{d}x \\
& -\frac{\lambda\xi N}{2\alpha(0)}\alpha(t)\int_{t-\tau(t)}^t\int_\Omega e^{\lambda(s-t)}u_t^2(s)\mathrm{d}x\mathrm{d}s \\
& -\alpha(t)\left[\alpha_1+\frac{N\alpha'(t)}{2\alpha(t)}\left(\int_0^t g(s)\mathrm{d}s\right)+\frac{\varepsilon\alpha'(t)C_p^2}{2\alpha(t)}\right]\int_\Omega |\nabla u|^2\mathrm{d}x \\
& +\alpha(t)\left[\varepsilon C_3\alpha(0)+\left(\frac{2+C_4}{4\delta}+2\delta\alpha^2(0)\right.\right. \\
& \left.\left. -\frac{C_p^2\alpha'(t)}{2\alpha(t)}\right)\left(\int_0^t g(s)\mathrm{d}s\right)\right](g\circ\nabla u)(t).
\end{aligned}
\tag{3.17}
$$

We then use $\lim\limits_{t\to\infty}\frac{\alpha'(t)}{\alpha(t)}=0$ (which can be deduced from (G2)) to choose $t_1\ge t_0$ so that (3.17) takes the form

$$
L'(t)\le -C_5\alpha(t)E(t)+C_6\alpha(t)(g\circ\nabla u)(t),\quad \forall\, t\ge t_1, \tag{3.18}
$$

where C_5 and C_6 are positive constants.

As in [21], multiplying (3.18) by $\xi(t)$ and using (G2) and (3.1), we obtain

$$
\begin{aligned}
\xi(t)L'(t) \le & -C_5\alpha(t)\xi(t)E(t)+C_6\alpha(t)\xi(t)(g\circ\nabla u)(t) \\
\le & -C_5\alpha(t)\xi(t)E(t)-C_6\alpha(t)(g'\circ\nabla u)(t) \\
\le & -C_5\alpha(t)\xi(t)E(t)-C_6\alpha'(t)\left(\int_0^t g(s)\mathrm{d}s\right)\int_\Omega|\nabla u|^2\mathrm{d}x \\
& -2C_6E'(t),\quad \forall\, t\ge t_1,
\end{aligned}
\tag{3.19}
$$

Since $\xi'(t)\le 0$, by (2.7), we have

$$
\begin{aligned}
& (\xi(t)L(t)+2C_6E(t))' \\
\le & -C_5\alpha(t)\xi(t)E(t)-C_6\alpha'(t)\left(\int_0^t g(s)\mathrm{d}s\right)\int_\Omega|\nabla u|^2\mathrm{d}x \\
\le & -\alpha(t)\xi(t)\left[C_5+\frac{2\alpha'(t)}{l\alpha(t)\xi(t)}\left(\int_0^t g(s)\mathrm{d}s\right)\right]E(t),\quad \forall\, t\ge t_1.
\end{aligned}
\tag{3.20}
$$

By (G2), we can choose $t_2\ge t_1$ and then (3.20) gives

$$
(\xi(t)L(t)+2C_6E(t))'\le -\frac{C_5}{2}\alpha(t)\xi(t)E(t),\quad \forall\, t\ge t_2.
$$

Set $\mathcal{L}(t)=\xi(t)L(t)+2C_6E(t)$. Since $\xi'(t)\le 0$, we can easily get $\mathcal{L}(t)\sim E(t)$ and

$$
\mathcal{L}'(t)\le -k\alpha(t)\xi(t)\mathcal{L}(t),\quad \forall\, t\ge t_2, \tag{3.21}
$$

for some positive constant k. Integrating (3.22) over $[t_2, t]$, we have

$$\mathcal{L}(t) \le \mathcal{L}(t_2)e^{-k\int_{t_2}^{t}\alpha(s)\xi(s)\mathrm{d}s} \le Ke^{-k\int_0^t\alpha(s)\xi(s)\mathrm{d}s}, \quad \forall\, t \ge t_2, \tag{3.22}$$

for some positive constant K. Using the equivalence of $\mathcal{L}(t)$ and $E(t)$ again, we have

$$E(t) \le Ke^{-k\int_0^t\alpha(s)\xi(s)\mathrm{d}s}, \quad \forall\, t \ge t_2.$$

By the virtue of the continuity and boundedness of $E(t)$ in the interval $[0, t_2]$, we complete the proof. ■

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