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OPENNESS OF MULTIPLICATION IN SOME FUNCTION SPACES

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Abstract. We show that, for several function Banach spaces, multiplication considered as a bilinear continuous surjection is an open mapping. In particular, we prove that multiplication from $L_p \times L_q$ to L_1 (for $p, q \in [1, \infty]$, $1/p+1/q = 1$) is open.

1. Introduction

Let X and Y be topological spaces. A mapping $f\colon X \to Y$ is called *open* if the image $f[U]$ is open for each open set $U \subseteq X$. We say that f is *open at a point* $x_0 \in X$ (cf. [1]) whenever $f(x_0) \in \operatorname{int} f[U]$ for every open neighbourhood U of x_0. It easily follows that f is open if and only if f is open at every point of X.

The Banach open mapping principle, a classical result in functional analysis, states that every continuous linear surjection between two Banach spaces is an open mapping. This theorem has been generalized in several papers (see [9]). One can ask about an extension of the Banach principle to the bilinear case. Such an extension is not valid in general. See [11, Chapter 2, Exercise 11] where a simple counterexample is given, compare also with [4, 6] and [5]. Thus it would be interesting to establish which bilinear continuous surjections $T\colon X \times Y \to Z$ (for Banach spaces X, Y, Z) are open mappings. In some function spaces, multiplication is a natural bilinear continuous surjection, however it need not be an open mapping. Namely, if $X = C[0, 1]$ denotes the Banach space of all real-valued continuous functions on $[0, 1]$, with the supremum norm, then multiplication from X^2 into X is not open at (f, f) where $f(x) = x-(1/2)$, $x \in [0, 1]$ (see [2]). For some further discussion on that topic, see [7, 13, 8, 3, 1].

The aim of this paper is to show several examples of function spaces in which multiplication being a bilinear continuous surjection is an open mapping. In fact, we also consider a strong version of openness.

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If X and Y are metric spaces, the openness of $f\colon X \to Y$ at $x_0 \in X$ means that

$$\forall \varepsilon > 0\ \exists \delta > 0\ B(f(x_0), \delta) \subseteq f[B(x_0, \varepsilon)]$$

where $B(z, \eta)$ denotes the ball with centre z and radius η in the respective space. We say that f is *uniformly open* whenever

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in X\ B(f(x), \delta) \subseteq f[B(x, \varepsilon)].$$

Note that $\arctan$ is a function from $\mathbb{R}$ into $\mathbb{R}$ which is open but not uniformly open. Indeed, for every $\delta > 0$ we can find $x \in \mathbb{R}$ such that $(\arctan x - \delta, \arctan x + \delta)$ is not included in $J_x = (\arctan(x-1), \arctan(x+1))$ since the length of J_x tends to 0 if x tends to ∞.

It follows from [2, Prop. 1] that, for every normed space X, addition is a uniformly open mapping from X^2 into X. Also by [2, Prop. 2], minimum and maximum are uniformly open mappings from $C[0,1] \times C[0,1]$ into $C[0,1]$ (the same holds when they are considered as functions from $\mathbb{R}^2$ into $\mathbb{R}$). Note that, in the Banach open mapping principle, we can state the uniform openness in its assertion since the global openness of a linear operator is equivalent to the openness at zero.

2. Results

First, we will show that multiplication as a function from $\mathbb{R}^2$ into $\mathbb{R}$ is a uniformly open mapping. The idea of this proof will be then repeated in a modified way. For $U, V \subseteq \mathbb{R}$, write $U \cdot V = \{xy\colon x \in U,\ y \in V\}$. The same notation will be used for the respective Banach spaces.

Proposition 1. *Multiplication as a function from $\mathbb{R}^2$ into $\mathbb{R}$ is a uniformly open mapping.*

Proof. Fix $\varepsilon > 0$, $(x_0, y_0) \in \mathbb{R}^2$ and put $U = (x_0 - \varepsilon, x_0 + \varepsilon)$, $V = (y_0 - \varepsilon, y_0 + \varepsilon)$. Define $\delta = \varepsilon^2/4$ and let $z \in (x_0 y_0 - \delta, x_0 y_0 + \delta)$. Consider three cases:

1^0 $|x_0| > \varepsilon/4$. Put $x = x_0$ and $y = z/x_0$. Then $z = xy$ and $x \in U$. Also $y \in V$ since

$$|y - y_0| = \frac{|z - x_0 y_0|}{|x_0|} < \frac{\delta}{\varepsilon/4} = \varepsilon.$$

2^0 $|y_0| > \varepsilon/4$ – analogous to 1^0.

3^0 $|x_0| \le \varepsilon/4$ and $|y_0| \le \varepsilon/4$. Put $x = \sqrt{|z|}$, $y = \sqrt{|z|}\operatorname{sgn} z$. Then $z = xy$ and

$$|x - x_0| \le |x| + |x_0| \le \sqrt{|z|} + \frac{\varepsilon}{4} \le \sqrt{|z - x_0 y_0|} + \sqrt{|x_0 y_0|} + \frac{\varepsilon}{4} < \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon.$$

Hence $x \in U$ and similarly, $y \in V$.

So, we have $(x_0y_0 - \delta, x_0y_0 + \delta) \subseteq U \cdot V$ which ends the proof. ■

Now, we will show that multiplication is an open mapping in several Banach spaces of real-valued bounded functions equipped with the norm $||f|| = \sup_{x \in X} |f(x)|$.

Theorem 2. *Multiplication is a uniformly open mapping as a function from Y^2 into Y where Y denotes the Banach space of all real-valued bounded functions measurable with respect to a given σ-algebra $\mathcal{S}$ of subsets of a nonempty set X. In particular, Y can be considered as:*

- *the space of all real-valued bounded functions on a nonempty set X;*
- *the space of all bounded Borel measurable functions on a metrizable space X.*

Proof. Fix $\varepsilon > 0$ and $f_0, g_0 \in Y$. Put $U = B(f_0, \varepsilon)$, $V = B(g_0, \varepsilon)$ and $h_0 = f_0g_0$. Define $\delta = \varepsilon^2/5$. We will show that $B(h_0, \delta) \subseteq U \cdot V$. So, let $h \in B(h_0, \delta)$. Define $F = \{x \in X : |f_0(x)| > \varepsilon/4\}$, $G = \{x \in X \setminus F : |g_0(x)| > \varepsilon/4\}$, $H = X \setminus (F \cup G)$. These sets are in $\mathcal{S}$ and they form a partition of X. Then define functions f and g on X as follows:

- for each $x \in F$ put $f(x) = f_0(x)$ and $g(x) = h(x)/f_0(x)$;
- for each $x \in G$ put $f(x) = h(x)/g_0(x)$ and $g(x) = g_0(x)$;
- for each $x \in H$ put $f(x) = \sqrt{|h(x)|}$ and $g(x) = \sqrt{|h(x)|} \cdot \operatorname{sgn}(h(x))$.

We have $h = fg$. We infer that $||f - f_0|| < \varepsilon$ and $||g - g_0|| < \varepsilon$ which shows that $h \in U \cdot V$. Indeed, if $x \in F$ then $|f(x) - f_0(x)| = 0$ and $|g(x) - g_0(x)| = |h(x) - h_0(x)|/|f_0(x)| < (\varepsilon^2/5)/(\varepsilon/4) = 4\varepsilon/5$. If $x \in G$, we proceed similarly. Now, let $x \in H$. We have

$$\begin{aligned} |f(x) - f_0(x)| &\leq |f(x)| + |f_0(x)| \leq \sqrt{|h(x)|} + \varepsilon/4 \\ &\leq \sqrt{||h - h_0||} + \sqrt{|h_0(x)|} + \varepsilon/4 < \varepsilon/\sqrt{5} + \varepsilon/4 + \varepsilon/4. \end{aligned}$$

Similarly, for $|g(x) - g_0(x)|$. ■

Of course, multiplication considered in Theorem 2 is a continuous surjection. Now, let X be a fixed metrizable space. By Σ^0_α, $\alpha < \omega_1$, we denote the respective countably additive classes of Borel subsets of X. So, $\Sigma^0_1 =$ open sets, $\Sigma^0_2 = F_\sigma$, $\Sigma^0_3 = G_{\delta\sigma}$, etc. (see [12]). We say that a function $f\colon X \to \mathbb{R}$ is *Borel measurable of class α* whenever the preimage $f^{-1}[U]$ is in $\Sigma^0_{1+\alpha}$ for every open set $U \subseteq \mathbb{R}$ (cf. [12]). For an ordinal α, $1 \leq \alpha < \omega_1$, consider the Banach space bBor_α of all bounded functions on X that are Borel measurable of class α. It is known that $fg \in \mathrm{bBor}_\alpha$ for all $f, g \in \mathrm{bBor}_\alpha$, and multiplication is a continuous surjection from $\mathrm{bBor}_\alpha \times \mathrm{bBor}_\alpha$ into bBor_α.

In the proof of the following theorem, we mimic some trick of Komisarski [7, p. 150]. In fact, from the proof of his result it follows that multiplication from $C(K) \times C(K)$ into $C(K)$ is a uniformly open mapping, provided that K is a zero-dimensional compact space.

Theorem 3. *For an arbitrary* α, $1 \leq \alpha < \omega_1$, *let* $Y = \mathrm{bBor}_\alpha$. *Then multiplication as a function from* Y^2 *into* Y *is a uniformly open mapping.*

Proof. We start with the same notation that was used in the previous proof. Let $\varepsilon > 0$. Again put $\delta = \varepsilon^2/5$. We will show that $B(h_0, \delta) \subseteq U \cdot V$. So, let $h \in B(h_0, \delta)$. Define

$$F_0 = \{x \in X\colon |f_0(x)| > \varepsilon/4\},\ G_0 = \{x \in X\colon |g_0(x)| > \varepsilon/4\},$$

$$H_0 = \{x \in X\colon |f_0(x)| < \varepsilon/3 \text{ and } |g_0(x)| < \varepsilon/3\}.$$

The sets F_0, G_0, H_0 are in $\Sigma^0_{1+\alpha}$ and $F_0 \cup G_0 \cup H_0 = X$. By the reduction theorem (see [12, Thm 3.6.10]) pick pairwise disjoint sets F, G, H in $\Sigma^0_{1+\alpha}$ such that $F \subseteq F_0$, $G \subseteq G_0$, $H \subseteq H_0$ and $F \cup G \cup H = X$. Define functions f and g on the sets F, G and H as in the previous proof. We have $h = fg$. The argument for $||f - f_0|| < \varepsilon$ and $||g - g_0|| < \varepsilon$ is similar to that in the previous proof but if $x \in H$, the calculation is a bit subtler:

$$\begin{aligned} |f(x) - f_0(x)| &\leq |f(x)| + |f_0(x)| \leq \sqrt{|h(x)|} + \varepsilon/3 \\ &\leq \sqrt{||h - h_0|| + |h_0(x)|} + \varepsilon/3 < \varepsilon\sqrt{14/45} + \varepsilon/3. \end{aligned}$$

It remains to show that f and g are in bBor_α. It suffices to prove that their restrictions to the sets F, G, H are Borel measurable of class α. In fact, we should check what happens with $f|_H$ and $g|_H$. Recall that the composition $\psi \circ \varphi$ of a function φ being Borel measurable of class α with a continuous function ψ is Borel measurable of class α. Thus we need only to check $g|_H$. For $c \in \mathbb{R}$ define $A_c = (g|_H)^{-1}[(-\infty, c)]$ and $A^c = (g|_H)^{-1}[(c, \infty)]$. Then A_c equals to $\{x \in H\colon h(x) < c^2\}$ if $c > 0$, and it equals to $\{x \in H\colon h(x) < 0 \text{ and } -\sqrt{|h(x)|} < c\}$ if $c \leq 0$. Hence A_c is in $\Sigma^0_{1+\alpha}$. The argument for A^c is similar. ■

From now on, fix a measure space $(X, \mathcal{S}, \mu)$ where μ is a measure on the σ-algebra $\mathcal{S}$ of subsets of X. Let $p, q \in (1, \infty)$, $1/p + 1/q = 1$. By Hölder's inequality

$$\int_X |fg| \leq \left(\int_X |f|^p\right)^{1/p} \left(\int_X |g|^q\right)^{1/q}$$

for $f \in L_p$, $g \in L_q$, it follows that multiplication $\Phi\colon L_p \times L_q \to L_1$, $\Phi(f, g) = fg$, is a bilinear continuous mapping. Also Φ is a surjection since for every $h \in L_1$ we pick $f = |h|^{1/p}$, $g = |h|^{1/q}\,\mathrm{sgn}(h)$, and then $f \in L_p$, $g \in L_q$, $fg = h$. Similarly, one can show that multiplication $\Phi\colon L_1 \times L_\infty \to L_1$ is a continuous bilinear surjection.

If $Z \in \mathcal{S}$, we will denote by $L_p(Z)$ the respective Banach space of functions defined on Z.

Theorem 4. *For any $p, q \in [1, \infty]$ with $1/p + 1/q = 1$, multiplication $\Phi\colon L_p \times L_q \to L_1$, $\Phi(f, g) = fg$, is an open mapping.*

Proof. For $p \in [1, \infty]$, denote by $B_p(f, r)$ a ball in L_p. Let $p, q \in [1, \infty]$ with $1/p + 1/q = 1$. Fix $(f_0, g_0) \in L_p \times L_q$ and $\varepsilon > 0$. We will find $\delta > 0$ such that $B_1(f_0 g_0, \delta) \subseteq B_p(f_0, \varepsilon) \cdot B_q(g_0, \varepsilon)$ which shows that Φ is open at (f_0, g_0).

Case 1. Assume that $0 < \mu(X) < \infty$. For simplicity, let $\mu(X) = 1$, $\varepsilon \in (0, 1)$.

First assume that $p, q \in (1, \infty)$. We will find $\delta > 0$ such that for each $h \in L_1$ with $\int_\Omega |h - f_0 g_0| < \delta$ we have $h = fg$ for some $f \in L_p$, $g \in L_q$ with $(\int_X |f - f_0|^p)^{1/p} < \varepsilon$, $(\int_X |g - g_0|^q)^{1/q} < \varepsilon$. By the absolute continuity of integrals, pick $\delta_0 \in (0, 1)$ such that for each $H \in \mathcal{S}$ with $\mu(H) < \delta_0$ we have

$$\begin{aligned}(1)\qquad &\left(\int_H |f_0|^p\right)^{1/p} < \frac{\varepsilon}{13}, \quad \left(\int_H |g_0|^q\right)^{1/q} < \frac{\varepsilon}{13},\\ &\int_H |f_0 g_0| < \min\left\{\left(\frac{\varepsilon}{13}\right)^p, \left(\frac{\varepsilon}{13}\right)^q\right\}.\end{aligned}$$

Define

$$\delta = \delta_0 \min\left\{\left(\frac{\varepsilon}{13}\right)^{p^2}, \left(\frac{\varepsilon}{13}\right)^{q^2}\right\}.$$

Let $h \in L_1$, $\int_X |h - f_0 g_0| < \delta$. Consider the following sets in $\mathcal{S}$ which form a partition of X:

$$\begin{aligned}
A &= \left\{x \in X\colon |f_0(x)| \le \left(\frac{\varepsilon}{13}\right)^p \text{ and } |g_0(x)| \le \left(\frac{\varepsilon}{13}\right)^q\right\},\\
B &= \left\{x \in X\colon |f_0(x)| > \left(\frac{\varepsilon}{13}\right)^p \text{ and } |h(x) - (f_0 g_0)(x)|^{q-1} > |f_0(x)|^q\right\},\\
C &= \left\{x \in X\colon |f_0(x)| > \left(\frac{\varepsilon}{13}\right)^p \text{ and } |h(x) - (f_0 g_0)(x)|^{q-1} \le |f_0(x)|^q\right\},\\
D &= \left\{x \in X \setminus (B \cup C)\colon |g_0(x)| > \left(\frac{\varepsilon}{13}\right)^q \text{ and } |h(x) - (f_0 g_0)(x)|^{p-1} > |g_0(x)|^p\right\},\\
E &= \left\{x \in X \setminus (B \cup C)\colon |g_0(x)| > \left(\frac{\varepsilon}{13}\right)^q \text{ and } |h(x) - (f_0 g_0)(x)|^{p-1} \le |g_0(x)|^p\right\}.
\end{aligned}$$

For each $x \in A \cup B \cup D$ define $f(x) = |h(x)|^{1/p}$ and $g(x) = |h(x)|^{1/q} \operatorname{sgn}(h(x))$. Then $h(x) = f(x)g(x)$. Since $\varepsilon \in (0, 1)$, for each $x \in A$ we have

$$|(f_0 g_0)(x)| \le \min\left\{\left(\frac{\varepsilon}{13}\right)^p, \left(\frac{\varepsilon}{13}\right)^q\right\}.$$

Hence

$$\begin{aligned}(2)\qquad &\left(\int_A |f - f_0|^p\right)^{1/p} \le \left(\int_A |f|^p\right)^{1/p} + \left(\int_A |f_0|^p\right)^{1/p} \le \left(\int_A |h|\right)^{1/p} + \frac{\varepsilon}{13}\\ &\le \left(\int_A |h - f_0 g_0|\right)^{1/p} + \left(\int_A |f_0 g_0|\right)^{1/p} + \frac{\varepsilon}{13} < \delta^{1/p} + \frac{2\varepsilon}{13} \le \frac{3\varepsilon}{13}.\end{aligned}$$

Similarly,

$$\left(\int_A |g - g_0|^q\right)^{1/q} < \frac{3\varepsilon}{13}. \tag{3}$$

Note that $q/(q-1) = p$. Hence for each $x \in B$ we have

$$|h(x) - (f_0 g_0)(x)| > |f_0(x)|^{q/(q-1)} > \left(\frac{\varepsilon}{13}\right)^{p^2} \text{ and } \left(\frac{\varepsilon}{13}\right)^{p^2} \mu(B) \leq \int_B |h - f_0 g_0| < \delta.$$

Thus $\mu(B) < (13/\varepsilon)^{p^2}\delta \leq \delta_0$. Then by (1) we have

$$\begin{aligned} &\left(\int_B |f - f_0|^p\right)^{1/p} \leq \left(\int_B |f|^p\right)^{1/p} + \left(\int_B |f_0|^p\right)^{1/p} < \left(\int_B |h|\right)^{1/p} + \frac{\varepsilon}{13} \\ &\leq \left(\int_B |h - f_0 g_0|\right)^{1/p} + \left(\int_B |f_0 g_0|\right)^{1/p} + \frac{\varepsilon}{13} < \delta^{1/p} + \frac{2\varepsilon}{13} \leq \frac{3\varepsilon}{13}. \end{aligned} \tag{4}$$

Analogously,

$$\left(\int_B |g - g_0|^q\right)^{1/q} < \frac{3\varepsilon}{13}. \tag{5}$$

On the set D we proceed similarly and we obtain

$$\left(\int_D |f - f_0|^p\right)^{1/p} < \frac{3\varepsilon}{13}. \tag{6}$$

$$\left(\int_D |g - g_0|^q\right)^{1/q} < \frac{3\varepsilon}{13}. \tag{7}$$

For $x \in C$ define $f(x) = f_0(x)$ and $g(x) = h(x)/f_0(x)$. Then $h(x) = f(x)g(x)$ and obviously,

$$\left(\int_C |f - f_0|^p\right)^{1/p} = 0. \tag{8}$$

Also

$$\left(\int_C |g - g_0|^q\right)^{1/q} = \left(\int_C \frac{|h - f_0 g_0|^q}{|f_0|^q}\right)^{1/q} \leq \left(\int_C |h - f_0 g_0|\right)^{1/q} < \delta^{1/q} \leq \frac{\varepsilon}{13}. \tag{9}$$

For $x \in E$ define $g(x) = g_0(x)$ and $f(x) = h(x)/g_0(x)$. Then $h(x) = f(x)g(x)$ and analogously as above,

$$\left(\int_E |f - f_0|^p\right)^{1/p} < \frac{\varepsilon}{13}. \tag{10}$$

$$\left(\int_E |g - g_0|^q\right)^{1/q} = 0. \tag{11}$$

Clearly, $h = fg$ on X. Finally, from (2), (4), (6), (8), (10) it follows that $(\int_X |f - f_0|^p)^{1/p} < \varepsilon$ and by (3), (5), (7), (9), (11) we have $(\int_X |g - g_0|^q)^{1/q} < \varepsilon$.

Now, let $p = 1$, $q = \infty$. We will find $\delta > 0$ such that for each $h \in L_1$ with $\int_X |h - f_0 g_0| < \delta$ we have $h = fg$ for some $f \in L_1$, $g \in L_\infty$ with $\int_X |f - f_0| < \varepsilon$ and $\operatorname{ess\,sup}_{x\in X} |g(x) - g_0(x)| < \varepsilon$. By the absolute continuity of integrals, pick $\delta_0 > 0$ such that for each $H \in \mathcal{S}$ with $\mu(H) < \delta_0$ we have

$$\int_H |f_0| < \frac{\varepsilon}{8}, \quad \int_H |f_0 g_0| < \frac{\varepsilon^2}{16}. \tag{12}$$

Define $\delta = \min\{\varepsilon^2/64, (\delta_0\varepsilon^2)/32\}$. Let $h \in L_1$, $\int_X |h - f_0 g_0| < \delta$. Consider the following sets in $\mathcal{S}$ which form a partition of X:

$$\begin{aligned}
A &= \left\{x \in X \colon |g_0(x)| > \frac{\varepsilon}{8}\right\},\\
B &= \left\{x \in X \colon |g_0(x)| \le \frac{\varepsilon}{8} \text{ and } |f_0(x)| \le \frac{\varepsilon}{8}\right\},\\
C &= \left\{x \in X \colon |g_0(x)| \le \frac{\varepsilon}{8} \text{ and } |f_0(x)| > \frac{\varepsilon}{8} \text{ and } |h(x) - (f_0g_0)(x)| \le \frac{\varepsilon}{4}|f_0(x)|\right\},\\
D &= \left\{x \in X \colon |g_0(x)| \le \frac{\varepsilon}{8} \text{ and } |f_0(x)| > \frac{\varepsilon}{8} \text{ and } |h(x) - (f_0g_0)(x)| > \frac{\varepsilon}{4}|f_0(x)|\right\}.
\end{aligned}$$

For $x \in A$ define $g(x) = g_0(x)$, $f(x) = h(x)/g_0(x)$. Then $h(x) = f(x)g(x)$ and $\operatorname{ess\,sup}_{x\in A} |g(x) - g_0(x)| = 0$. Also we have

$$\int_A |f - f_0| = \int_A \frac{|h - f_0 g_0|}{|g_0|} < \frac{8\delta}{\varepsilon} \le \frac{\varepsilon}{8}. \tag{13}$$

For $x \in B$ define $g(x) = \varepsilon/8$, $f(x) = h(x)/(\varepsilon/8)$. Then $h(x) = f(x)g(x)$ and

$$\operatorname*{ess\,sup}_{x\in B} |g(x) - g_0(x)| \le \frac{\varepsilon}{8} + \frac{\varepsilon}{8} = \frac{\varepsilon}{4}, \tag{14}$$

$$\begin{aligned}
\int_B |f - f_0| &= \int_B \frac{|h - (\varepsilon/8)f_0|}{\varepsilon/8} \le \int_B \frac{|h - f_0g_0|}{\varepsilon/8} + \int_B \frac{|f_0g_0 - (\varepsilon/8)f_0|}{\varepsilon/8}\\
&< \frac{8\delta}{\varepsilon} + \frac{8}{\varepsilon} \operatorname*{ess\,sup}_{x\in B} \left|g_0(x) - \frac{\varepsilon}{8}\right| \int_B |f_0| \le \frac{\varepsilon}{8} + \frac{8}{\varepsilon}\cdot\frac{\varepsilon}{4}\cdot\frac{\varepsilon}{8} = \frac{3}{8}\varepsilon.
\end{aligned} \tag{15}$$

For $x \in C$ define $f(x) = f_0(x)$, $g(x) = h(x)/f_0(x)$. Then $h(x) = f(x)g(x)$ and $\int_C |f - f_0| = 0$. Also

$$\operatorname*{ess\,sup}_{x\in C} |g(x) - g_0(x)| = \operatorname*{ess\,sup}_{x\in C} \frac{|h(x) - (f_0g_0)(x)|}{|f_0(x)|} \le \frac{\varepsilon}{4}. \tag{16}$$

For $x \in D$ define $g(x) = \varepsilon/4$, $f(x) = h(x)/(\varepsilon/4)$. Then $h(x) = f(x)g(x)$ and

$$\operatorname*{ess\,sup}_{x\in D} |g(x) - g_0(x)| \le \frac{\varepsilon}{4} + \frac{\varepsilon}{8} = \frac{3}{8}\varepsilon. \tag{17}$$

We have

$$\int_D |h - f_0 g_0| \ge \frac{\varepsilon}{4} \int_D |f_0| \ge \frac{\varepsilon^2}{32}\mu(D) \text{ and thus } \mu(D) < \frac{32\delta}{\varepsilon^2} \le \delta_0.$$

Consequently, by (12) we obtain

$$\begin{aligned} \int_D |f - f_0| &\le \int_D |f| + \int_D |f_0| \le \frac{4}{\varepsilon}\int_D |h| + \frac{\varepsilon}{8} \le \frac{4}{\varepsilon}\int_D |h - f_0 g_0| \\ &+ \frac{4}{\varepsilon}\int_D |f_0 g_0| + \frac{\varepsilon}{8} \le \frac{4\delta}{\varepsilon} + \frac{4}{\varepsilon}\cdot\frac{\varepsilon^2}{16} + \frac{\varepsilon}{8} \le \frac{\varepsilon}{16} + \frac{3\varepsilon}{8} < \frac{\varepsilon}{2}. \end{aligned} \tag{18}$$

By (13)-(18) the proof is finished.

Case 2. Assume that $\mu(X) = \infty$ and that measure μ is σ-finite. Fix a partition $\{X_n : n \ge 1\}$ of X into pairwise disjoint sets in $\mathcal{S}$ of finite measure. For an integer $k \ge 1$, denote $X_k^- = \bigcup_{n\le k} X_n$ and $X_k^+ = \bigcup_{n>k} X_n$.

Let $p, q \in (1, \infty)$. Since $f_0 \in L_p$, $g_0 \in L_q$, $f_0 g_0 \in L_1$, the series $\sum_n \int_{X_n} |f_0|^p$, $\sum_n \int_{X_n} |g_0|^q$, $\sum_n \int_{X_n} |f_0 g_0|$ are convergent. So, by the σ-additivity of integral, we can pick an index k such that

$$\left(\int_{X_k^+} |f_0|^p\right)^{1/p} < \frac{\varepsilon}{6}, \quad \left(\int_{X_k^+} |g_0|^q\right)^{1/q} < \frac{\varepsilon}{6} \tag{19}$$

$$\left(\int_{X_k^+} |f_0 g_0|\right)^{1/p} < \frac{\varepsilon}{6}, \quad \left(\int_{X_k^+} |f_0 g_0|\right)^{1/q} < \frac{\varepsilon}{6}. \tag{20}$$

By Case 1 we can find $\delta_0 > 0$ such that for every $\varphi \in L_1(X_k^-)$ with $\int_{X_k^-} |\varphi - f_0 g_0| < \delta_0$ we have $\varphi = f_* g_*$ for some $f_* \in L_p(X_k^-)$, $g_* \in L_q(X_k^-)$ with

$$\left(\int_{X_k^-} |f_* - f_0|^p\right)^{1/p} < \frac{\varepsilon}{2}, \quad \left(\int_{X_k^-} |g_* - g_0|^q\right)^{1/q} < \frac{\varepsilon}{2}. \tag{21}$$

Put $\delta = \min\{\delta_0, (\varepsilon/6)^p, (\varepsilon/6)^q\}$. Let $h \in L_1$ and $\int_X |h - f_0 g_0| < \delta$. For $\varphi = h|_{X_k^-}$ find the respective functions f_* and g_* defined on X_k^- and fulfilling $\varphi = f_* g_*$ and (21). For $x \in X_k^+$ put $f^*(x) = |h(x)|^{1/p}$, $g^*(x) = |h(x)|^{1/q}\operatorname{sgn}(h(x))$. Define f and g on X as follows

$$f(x) = \begin{cases} f_*(x) & \text{if } x \in X_k^- \\ f^*(x) & \text{if } x \in X_k^+ \end{cases} \tag{22}$$

(23) $$g(x) = \begin{cases} g_*(x) & \text{if } x \in X_k^- \\ g^*(x) & \text{if } x \in X_k^+. \end{cases}$$

Then $h = fg$ on X. Also, by (19), using the choice of δ and the evaluations anologous to (2) and (3) on X_k^+, we obtain

(24) $$\left(\int_{X_k^+} |f - f_0|^p\right)^{1/p} < \frac{\varepsilon}{2}, \quad \left(\int_{X_k^+} |g - g_0|^q\right)^{1/q} < \frac{\varepsilon}{2}.$$

This together with (21) yields the assertion.

Now, let $p = 1$, $q = \infty$. We proceed similarly as before. So, we pick an index k such that

(25) $$\int_{X_k^+} |f_0| < \frac{\varepsilon}{8}$$

where X_k^+, X_k^- are defined as before. By Case 1, find $\delta_0 > 0$ such that for every $\varphi \in L_1(X_k^-)$ with $\int_{X_k^-} |\varphi - f_0 g_0| < \delta_0$ we have $\varphi = f_* g_*$ for some $f_* \in L_1(X_k^-)$, $g_* \in L_\infty(X_k^-)$ with

(26) $$\int_{X_k^-} |f_* - f_0| < \frac{\varepsilon}{2}, \quad \operatorname*{ess\,sup}_{x \in X_k^-} |g_*(x) - g_0(x)| < \varepsilon.$$

Put $\delta = \min\{\delta_0, \varepsilon^2/8\}$. Let $h \in L_1$ and $\int_X |h - f_0 g_0| < \delta$. For $\varphi = h|_{X_k^-}$ find the respective functions f_* and g_* defined on X_k^- and fulfilling $\varphi = f_* g_*$ and (26). Let $M = \operatorname*{ess\,sup}_{x \in X_k^+} |f(x)|$ and put $w = \min\{n > k \colon \varepsilon n/2 > M\}$. Define g^* on X_k^+ by the formula

(27) $$g^*(x) = \begin{cases} \dfrac{\varepsilon(m+1)}{2} & \text{if } \frac{\varepsilon m}{2} \le g_0(x) < \frac{\varepsilon(m+1)}{2} \text{ and } m = 0, \dots, w-1 \\ -\dfrac{\varepsilon(m+1)}{2} & \text{if } -\frac{\varepsilon(m+1)}{2} \le g_0(x) < -\frac{\varepsilon m}{2} \text{ and } m = 0, \dots, w-1 \\ 1 & \text{otherwise (this holds on a set of measure zero).} \end{cases}$$

Also put $f^*(x) = h(x)/g^*(x)$ for $x \in X_k^+$. Then define f and g on X by (22) and (23). By (26) and (27), it is clear that $\operatorname*{ess\,sup}_{x \in X} |g(x) - g_0(x)| < \varepsilon$. By (25), (27) and the choice of δ we have

$$\begin{aligned} \textstyle\int_{X_k^+} |f - f_0| &= \int_{X_k^+} \frac{|h - f_0 g^*|}{|g^*|} \le \frac{2}{\varepsilon} \int_{X_k^*} |h - f_0 g^*| \\ &\le \frac{2}{\varepsilon} \int_{X_k^+} |h - f_0 g_0| + \frac{2}{\varepsilon} \int_{X_k^*} |f_0| \cdot |g_0 - g^*| < \frac{2}{\varepsilon}\delta + \frac{2\varepsilon}{\varepsilon} \int_{X_k^+} |f_0| \\ &< \frac{2}{\varepsilon} \cdot \frac{\varepsilon^2}{8} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}. \end{aligned}$$

This together with (26) yields the assertion.

Case 3. Assume that μ is not σ-finite. Let $p, q \in (1, \infty)$. Given $f_0 \in L_p$, $g_0 \in L_q$ and $\varepsilon > 0$, denote $K = \{x \in X \colon f_0(x) \neq 0 \text{ or } g_0(x) \neq 0\}$ and observe that μ restricted to K is σ-finite. So, by Case 2, pick $\delta_0 > 0$ such that each $\widetilde{h} \in L_1(K)$ with $||\widetilde{h} - f_0 g_0||_{L_1(K)} < \delta_0$ can be written as $\widetilde{h} = \widetilde{f}\widetilde{g}$ with $\widetilde{f} \in L_p(K)$ and $\widetilde{g} \in L_q(K)$ such that $||\widetilde{f} - f_0||_{L_p(K)} < \varepsilon/2$ and $||\widetilde{g} - g_0||_{L_q(K)} < \varepsilon/2$. Let $\delta = \min\{\delta_0, (\varepsilon/2)^p, (\varepsilon/2)^q\}$ and assume that $h \in L_1(X)$, $||h - fg||_{L_1(X)} < \delta$. Let $\widetilde{h} = h|_K$ and pick $\widetilde{f}$, $\widetilde{g}$ as above. Next extend $\widetilde{f}$ to f and $\widetilde{g}$ to g, where f, g are defined in X, by letting $f(x) = |h(x)|^{1/p}$ and $g(x) = |h(x)|^{1/q} \operatorname{sgn} h(x)$ for $x \in X \setminus K$. Then $h = fg$, $f \in L_p(X)$, $g \in L_q(X)$ and

$$||f - f_0||_{L_p(X)} \leq ||\widetilde{f} - f_0||_{L_p(K)} + ||h||^{1/p}_{L_p(X\setminus K)} < \frac{\varepsilon}{2} + \delta^{1/p} \leq \varepsilon.$$

Similarly $||g - g_0||_{L_q(X)} < \varepsilon$.

If $p = 1$, $q = \infty$, an analogous argument works with $K = \{x \in X \colon f_0(x) = 0\}$, δ_0 chosen as before and $\delta = \min\{\delta_0, \varepsilon^2/4\}$. Taking h, $\widetilde{h}$, $\widetilde{f}$, $\widetilde{g}$ as before, we produce the respective extensions f and g of $\widetilde{f}$ and $\widetilde{g}$ by letting $f(x) = (2/\varepsilon)h(x)$ and $g(x) = \varepsilon/2$ for $x \in X \setminus K$. ∎

Note that the Banach space ℓ_p, for $p \in [1, \infty]$, can be treated as a special case of the space L_p associated with the σ-finite counting measure on the power set of positive integers. So, from Theorem 4 we deduce the following corollary.

Corollary 5. *For any $p, q \in [1, \infty]$ with $1/p + 1/q = 1$, multiplication $\Phi \colon \ell_p \times \ell_q \to \ell_1$ is an open mapping.*

In the case if $p = 1$, we have $\ell_1 \cdot \ell_\infty = \ell_1$ and the above result shows that multiplication is an open mapping. It turns out that we also have $\ell_1 \cdot c_0 = \ell_1$, in other words, the multiplication $\Phi \colon \ell_1 \times c_0 \to \ell_1$ is a surjection. Indeed, let $z = (z_n) \in \ell_1$. We may suppose that there are infinitely many nonzero terms z_n. Put $r_n = \sum_{i \geq n} |z_i|$ for $n \geq 1$. Then $(\sqrt{r_n}) \in c_0$ and $(z_n/\sqrt{r_n}) \in \ell_1$ since $|z_n|/\sqrt{r_n} \leq 2(\sqrt{r_n} - \sqrt{r_{n+1}})$ for all n and $\sum_n 2(\sqrt{r_n} - \sqrt{r_{n+1}}) = 2\sqrt{r_1} < \infty$ (cf. [10, Exercise 12, Chapter 2]). Clearly, Φ is continuous. In this case, we have the following result which cannot be deduced directly from Corollary 5.

Theorem 6. *Multiplication Φ from $\ell_1 \times c_0$ into ℓ_1 is an open mapping.*

Proof. Fix $a^0 = (a^0_n) \in \ell_1$, $b^0 = (b^0_n) \in c_0$ and $\varepsilon > 0$. Pick an index k such that

$$\sum_{n>k} |a^0_n b^0_n| < \frac{\varepsilon^2}{64}, \quad \sup_{n>k} |b^0_n| < \frac{\varepsilon}{2}, \quad \sum_{n>k} |a^0_n| < \frac{\varepsilon}{4}.$$

By Proposition 1, for $\varepsilon/(4k)$ pick $\delta_0 > 0$ witnessing that multiplication from $\mathbb{R}^2$ to $\mathbb{R}$ is a uniformly open mapping. Define $\delta = \min\{\delta_0, \varepsilon^2/64\}$. Let $z = (z_n) \in \ell_1$ and

$\sum_n |z_n - a_n^0 b_n^0| < \delta$. For $n \in \{1, \ldots, k\}$, from $|z_n - a_n^0 b_n^0| < \delta \leq \delta_0$ it follows that we can find $a_n, b_n \in \mathbb{R}$ such that $z_n = a_n b_n$ and $|a_n - a_n^0| < \varepsilon/(4k)$, $|b_n - b_n^0| < \varepsilon/(4k)$.

Now, let $n > k$. Define $r_n = \sum_{i \geq n} |z_i|$. If $r_n = 0$, put $b_n = b_n^0$ and $a_n = 0$ (this case is easy and we will ignore it in further calculations). Otherwise, put $b_n = \sqrt{r_n}$, $a_n = z_n/\sqrt{r_n}$. Then $z_n = a_n b_n$ and we have

$$|b_n - b_n^0| \leq |b_n| + |b_n^0| \leq \sqrt{\sum_{i>k} |z_i - a_i^0 b_i^0|} + \sqrt{\sum_{i>k} |a_i^0 b_i^0|} + |b_n^0| < \frac{\varepsilon}{8} + \frac{\varepsilon}{8} + \frac{\varepsilon}{2} = \frac{3}{4}\varepsilon.$$

Hence $\sup_{n \geq 1} |b_n - b_n^0| < \varepsilon$. Also we have

$$\begin{aligned}
\sum_n |a_n - a_n^0| &= \sum_{n \leq k} |a_n - a_n^0| + \sum_{n>k} |a_n| + \sum_{n>k} |a_n^0| < \frac{\varepsilon}{2} + \sum_{n>k} \frac{|z_n|}{\sqrt{r_n}} \\
&\leq \frac{\varepsilon}{2} + 2 \sum_{n>k} (\sqrt{r_n} - \sqrt{r_{n+1}}) \\
&= \frac{\varepsilon}{2} + 2 \cdot \sqrt{\sum_{n>k} |z_n|} \leq \frac{\varepsilon}{2} + 2 \left(\sqrt{\sum_{n>k} |z_n - a_n^0 b_n^0|} + \sqrt{\sum_{n>k} |a_n^0 b_n^0|} \right) \\
&< \frac{\varepsilon}{2} + 2 \left(\frac{\varepsilon}{8} + \frac{\varepsilon}{8} \right) = \varepsilon.
\end{aligned}$$

■

In the forthcoming paper we will study the (nontrivial) problem whether Φ in Theorems 4 and 6 is uniformly open.

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