TAIWANESE JOURNAL OF MATHEMATICS
Vol. 17, No. 2, pp. 501-515, April 2013
DOI: 10.11650/tjm.17.2013.2081
This paper is available online at http://journal.taiwanmathsoc.org.tw

EXISTENCE OF WEAK SOLUTIONS FOR p-LAPLACIAN PROBLEM WITH IMPULSIVE EFFECTS

Jiafa Xu, Zhongli Wei and Youzheng Ding

Abstract. In this paper, we shall adopt topological degree theory and critical point theory to study the existence of weak solutions for the p-Laplacian Dirichlet boundary value problem

$$\begin{cases} -(|u'|^{p-2}u')' = f(t,u), \text{ in } \Omega, \\ u(0) = u(1) = 0, \end{cases}$$

with impulsive conditions $u(t_j^+)-u(t_j^-) = 0$, $\Delta|u'(t_j)|^{p-2}u'(t_j) = I_j(u(t_j)),\ j = 1, 2, \ldots, n,$ where $p \in (1, +\infty)$, $\Omega = (0, 1)\backslash\{t_1, \ldots, t_n\},\ f \in C([0, 1] \times \mathbb{R}, \mathbb{R})$ and $I_j \in C(\mathbb{R}, \mathbb{R})$ $(j = 1, 2, \ldots, n)$.

1. Introduction

In this work, we will investigate the existence of weak solutions for the following Dirichlet boundary value problem with p-Laplacian operator

$$\begin{cases} -(|u'|^{p-2}u')' = f(t,u), \text{ in } \Omega, \\ u(0) = u(1) = 0, \end{cases} \tag{1.1}$$

subject to impulsive conditions $u(t_j^+)-u(t_j^-) = 0$, $\Delta|u'(t_j)|^{p-2}u'(t_j) = I_j(u(t_j)),\ j = 1, 2, \ldots, n,$ where $p \in (1, +\infty)$, $\Omega = (0, 1)\backslash\{t_1, \ldots, t_n\},\ f \in C([0, 1] \times \mathbb{R}, \mathbb{R})$, $I_j \in C(\mathbb{R}, \mathbb{R})$ $(j = 1, 2, \ldots, n)$ and $\Delta|u'(t_j)|^{p-2}u'(t_j) = |u'(t_j^+)|^{p-2}u'(t_j^+) - |u'(t_j^-)|^{p-2} u'(t_j^-)$, here $u'(t_j^+)$ and $u'(t_j^-)$ denote the right and left limits of $u'(t)$ at t, respectively.

The theory of impulsive differential equations is emerging as an important area of investigation since it is a lot richer than the corresponding theory of non-impulsive

Received May 29, 2012, accepted August 17, 2012.
Communicated by Eiji Yanagida.
2010 *Mathematics Subject Classification*: 35A15, 35A16, 47H11.
Key words and phrases: Impulsive differential equation, p-Laplacian, Topological degree theory, Critical point theory, Weak solution.
Research supported by the NNSF-China (10971046), Shandong Provincial Natural Science Foundation (ZR2012AQ007), Graduate Independent Innovation Foundation of Shandong University (yzc12063) and IIFSDU (2012TS020).

differential equations. Many evolutionary processes in nature are characterized by the fact that at certain moments in time an abrupt change of state is experienced. That is the reason for the rapid development of the theory of impulsive differential equations, for instance, see the two books [1, 2].

Recently, many researchers pay their attention to impulsive differential equations by variational method and critical point theory, to the best our knowledge, we refer the reader to [3, 4, 5, 6, 7, 8] and references cited therein. Meanwhile, some people begin to study p-Laplacian differential equations with impulsive effects, for example, see [9, 10, 11, 12, 13, 14].

In [9, 10], Bai and Dai utilize B. Ricceri's three critical point theorem and mountain pass theorem to investigate the existence of solutions for an impulsive boundary value problem involving the p-Laplacian operator. Chen and Tang [11] adopt the least action principle and the saddle point theorem to obtain some existence theorems for second-order p-Laplacian systems with or without impulsive effects under weak sublinear growth conditions. In [12], They also consider that a class of second-order impulsive differential equations with Dirichlet problems has one or infinitely many solutions under more relaxed assumptions on their nonlinearity f, which satisfies a kind of new superquadratic and subquadratic condition. Wang et al. [13] study the critical point theory and the method of lower and upper solutions to obtain the existence of solutions to a p-Laplacian impulsive problem. As applications, they also get unbounded sequences of solutions and sequences of arbitrarily small positive solutions of the p-Laplacian impulsive problem. Ivan Bogun [14] discusses the existence of weak solutions for a p-Laplacian problem with superlinear impulses by virtue of mountain pass theorem and symmetric mountain pass theorem. Moreover, when the case $p = 2$, he also offers the existence of at least one non-positive and one non-negative solution.

Motivated by the works cited above, in particular [14], in this paper, we shall discuss the problem (1.1). Firstly, we adopt topological degree theory to prove that the problem (1.1) has at least one weak solution. Secondly, we shall utilize Fountain theorem under Cerami condition (C), which is introduced in [15], to investigate the problem (1.1) has infinitely many weak solutions. The results obtained here improve some existing results in the literature.

2. Variational Structure and the Existence of at Least One Weak Solution

In this paper, we use the Hilbert space $X := W_0^{1,p}(0,1)$, the norm in $W_0^{1,p}(0,1)$ is $\|x\| = (\int_0^1 |x'(t)|^p)^{\frac{1}{p}}$. We denote by $\|x\|_p$ the norm in $L^p(0,1)$ and $\|x\|_\infty$ in $C[0,1]$. For $u \in W^{1,p}(0,1)$, we have that u and u' are both absolutely continuous. Hence $u'(t^+) - u'(t^-) = 0$ for any $t \in [0,1]$.

If $u \in X$, then u is absolutely continuous. In this case, the one-sided derivatives $u'(t^+), u'(t^-)$ may not exist. It leads to the impulsive effects. As a result, we need to introduce a different concept of solution. Suppose that $u \in C[0,1]$ satisfies the Dirichlet condition $u(0) = u(1) = 0$. Assume that, for every $j = 1, 2, \ldots, n$, $u_j = u|_{(t_j, t_{j+1})}$ and $u_j \in W^{1,p}(0,1)$. In what follows, we first translate (1.1) into its equivalent integral equation. For any $v \in X$, multiply (1.1) by v and integrate between 0 and 1 to obtain

$$\int_0^1 -(|u'|^{p-2}u')'v\mathrm{d}t = \int_0^1 f(t,u)v\mathrm{d}t. \tag{2.1}$$

Note that the impulsive effects, for the left integral of (2.1), define $t_0 = 0, t_{n+1} = 1$, we have

$$\begin{aligned}
&\int_0^1 -(|u'|^{p-2}u')'v\mathrm{d}t \\
&= \sum_{j=0}^{n} \int_{t_j^+}^{t_{j+1}^-} -(|u'|^{p-2}u')'v\mathrm{d}t = \sum_{j=0}^{n} \int_{t_j^+}^{t_{j+1}^-} -v\mathrm{d}\left(|u'|^{p-2}u'\right) \\
&= \sum_{j=0}^{n} -|u'|^{p-2}u'v\Big|_{t_j^+}^{t_{j+1}^-} + \sum_{j=0}^{n} \int_{t_j^+}^{t_{j+1}^-} |u'|^{p-2}u'v'\mathrm{d}t \\
&= \sum_{j=0}^{n} \Big[|u'(t_j^+)|^{p-2}u'(t_j^+)v(t_j^+) - |u'(t_{j+1}^-)|^{p-2}u'(t_{j+1}^-)v(t_{j+1}^-)\Big] \qquad (2.2)\\
&\quad + \int_0^1 |u'|^{p-2}u'v'\mathrm{d}t \\
&= \sum_{j=1}^{n} \Big[|u'(t_j^+)|^{p-2}u'(t_j^+) - |u'(t_j^-)|^{p-2}u'(t_j^-)\Big]\, v(t_j) + \int_0^1 |u'|^{p-2}u'v'\mathrm{d}t \\
&= \sum_{j=1}^{n} I_j(u(t_j))v(t_j) + \int_0^1 |u'|^{p-2}u'v'\mathrm{d}t.
\end{aligned}$$

By (2.1) and (2.2), we find that if for all $v \in X$, there exists $u \in X$ such that

$$\int_0^1 |u'|^{p-2}u'v'\mathrm{d}t + \sum_{j=1}^{n} I_j(u(t_j))v(t_j) - \int_0^1 f(t,u)v\mathrm{d}t = 0, \tag{2.3}$$

then u is called a weak solution of (1.1). Meanwhile, we can obtain the weak solutions for (1.1) coincide with critical points of the energy functional

$$\varphi(u) = \frac{1}{p}\int_0^1 |u'|^p\mathrm{d}t + \sum_{j=1}^{n} \int_0^{u(t_j)} I_j(t)\mathrm{d}t - \int_0^1 F(t,u)\mathrm{d}t, \tag{2.4}$$

where $F(t,u) = \int_0^u f(t,x)\mathrm{d}x$. Clearly, φ is class of C^1 and it's derivative is

$$(\varphi'(u), v) = \int_0^1 |u'|^{p-2}u'v'\mathrm{d}t + \sum_{j=1}^{n} I_j(u(t_j))v(t_j) - \int_0^1 f(t,u)v\mathrm{d}t, \quad \forall v \in X. \tag{2.5}$$

The parameter $\lambda \in \mathbb{R}$ for which there is a nontrivial weak solution $\varphi(t),\ t \in [0,1]$ of the problem

$$
\begin{cases} -(|u'|^{p-2}u')' - \lambda|u|^{p-2}u = 0,\ t \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{2.6}
$$

is called an eigenvalue of the eigenvalue problem (2.6) and the function φ an eigenfunction associated with the eigenvalue λ. It is known that the problem (2.6) has a countable set of simple eigenvalues $0 < \lambda_1 < \lambda_2 < \cdots$, $\lim_{n\to\infty} \lambda_n = \infty$ and the values of $\lambda_n, n = 1, 2, \ldots$, can be explicitly calculated in terms of p and π, i.e., $\lambda_n = (p-1)\left(\frac{2n\pi}{p\sin\frac{\pi}{p}}\right)^p$ for $n = 1, 2, \ldots$, see [12, the below of (2.1)]. The eigenfunction φ_n associated with λ_n is continuously differentiable and has exactly $n-1$ zero points in $(0,1)$. In particular, we can choose $\varphi_1(t) > 0, t \in (0,1)$.

In the following, we introduce the fundamental definitions and theorems for topological degree theory needed in the ensuing results.

Definition 2.1. (see [16, Page 122], [17, Page 2340]). Let X be a reflexive real Banach space and X^* its dual. The operator $T : X \to X^*$ is said to satisfy the (S_+) condition if the assumptions $u_n \rightharpoonup u_0$ weakly in X and $\limsup_{n\to\infty}(T(u_n), u_n - u_0) \le 0$ imply $u_n \to u_0$ strongly in X.

Definition 2.2. (see [18, the top of page 912]). The operator $T : X \to X^*$ is said to be demicontinuous if T maps strongly convergent sequences in X to weakly convergent sequences in X^*.

Lemma 2.1. (see [19]). *Let $T : X \to X^*$ satisfy the (S_+) condition and let $K : X \to X^*$ be a compact operator. Then the sum $T + K : X \to X^*$ satisfies the (S_+) condition.*

Lemma 2.2. (see [19]). *Let $T : X \to X^*$ be a bounded and demicontinuous operator satisfying the (S_+) condition. Let $\mathcal{D} \subset X$ be an open, bounded and nonempty set with the boundary $\partial\mathcal{D}$ such that $T(u) \neq 0$ for $u \in \partial\mathcal{D}$. Then there exists an integer $\deg(T, \mathcal{D}, 0)$ such that*

(1) $\deg(T, \mathcal{D}, 0) \neq 0$ *implies that there exists an element $u_0 \in \mathcal{D}$ such that $T(u_0) = 0$.*
(2) *If $\mathcal{D}$ is symmetric with respect to the origin and T satisfies $T(u) = -T(-u)$ for any $u \in \partial\mathcal{D}$, then $\deg(T, \mathcal{D}, 0)$ is an odd number.*
(3) *Let T_λ be a family of bounded and demicontinuous mappings which satisfy the (S_+) condition and which depend continuously on a real parameter $\lambda \in [0,1]$, and let $T_\lambda(u) \neq 0$ for any $u \in \partial\mathcal{D}$ and $\lambda \in [0,1]$. Then $\deg(T_\lambda, \mathcal{D}, 0)$ is constant with respect to $\lambda \in [0,1]$.*

Now, we list our assumptions on f and I_j $(j = 1, 2, \ldots, n)$ in this section.
(H1) $\lim_{|u|\to\infty} \frac{f(t,u)}{|u|^{p-2}u} = \lambda$, where $\lambda_n < \lambda < \lambda_{n+1}$ for any $n = 1, 2, \ldots$.
(H2) There exist $a_j, b_j > 0$ and $\gamma_j \in [1, p)$ such that $|I_j(u)| \le a_j + b_j|u|^{\gamma_j - 1}$, $\forall u \in \mathbb{R}$ and $j = 1, 2, \ldots, n$.

Theorem 2.1. *Let* (*H1*) *and* (*H2*) *hold. Then* (1.1) *has at least one weak solution.*

Proof. From (2.3), let us define the three operators $J, G, Q : X \to X^*$ by

$$\begin{aligned}(J(u), v) &= \int_0^1 |u'|^{p-2}u'v'\mathrm{d}t,\ (G(u), v)\\ &= \int_0^1 f(t,u)v\mathrm{d}t,\ (Q(u), v) = \sum_{j=1}^n I_j(u(t_j))v(t_j),\ \forall u, v \in X.\end{aligned} \tag{2.7}$$

We first show the two operators G, Q are compact. If $\{u_n\}_{n=1}^\infty$ is a bounded sequence in X. By $X \hookrightarrow\hookrightarrow C[0, 1]$, passing to a subsequence, we may assume that the sequence $\{u_n\}_{n=1}^\infty$ is converged. Then $\{u_n\}_{n=1}^\infty$ is a Cauchy sequence in $C[0, 1]$. Therefore,

$$\begin{aligned}&\|G(u_n) - G(u_m)\|\\ &= \sup_{\|v\|\le 1} |(G(u_n) - G(u_m), v)|\\ &= \sup_{\|v\|\le 1} \left|\int_0^1 (f(t,u_n) - f(t,u_m))v\mathrm{d}t\right|\\ &\le \sup_{\|v\|\le 1}\int_0^1 |(f(t,u_n) - f(t,u_m))v|\mathrm{d}t \to 0,\ \text{as } m, n \to \infty,\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}&\|Q(u_n) - Q(u_m)\|\\ &= \sup_{\|v\|\le 1} |(Q(u_n) - Q(u_m), v)|\\ &= \sup_{\|v\|\le 1} \left|\sum_{j=1}^n [I_j(u_n(t_j)) - I_j(u_m(t_j))]\, v(t_j)\right|\\ &\le \sup_{\|v\|\le 1} \sum_{j=1}^n |\,[I_j(u_n(t_j)) - I_j(u_m(t_j))]\, v(t_j)| \to 0,\ \text{as } m, n \to \infty.\end{aligned} \tag{2.9}$$

In what follows, we shall sketch the properties of the operator J. For J, it is clear that $(J(u), u) = \|u\|^p$. Furthermore, J is an odd mapping, bounded and continuous.

Indeed, by Hölder inequality,

$$
\begin{aligned}
|(J(u), v)| &= \left|\int_0^1 |u'|^{p-2}u'v'\mathrm{d}t\right| \\
&\leq \left(\int_0^1 |u'|^p\mathrm{d}t\right)^{\frac{p-1}{p}} \left(\int_0^1 |v'|^p\mathrm{d}t\right)^{\frac{1}{p}} \leq \|u\|^{p-1}\|v\| < \infty,\ \forall u, v \in X.
\end{aligned}
\tag{2.10}
$$

Hence, J is bounded. We assume that a sequence $\{u_n\}_{n=1}^{\infty} \subset X$ weakly converges to a element u_0 in X, and then $X \hookrightarrow\hookrightarrow L^p(0,1)$, $\int_0^1 |u_n'|^{p-1}\mathrm{d}t \leq \|u_n\|^{p-1} < \infty$, for $u_n \in X$ and Hölder inequality enable us to find

$$
\begin{aligned}
\|J(u_n) - J(u_0)\| &= \sup_{\|v\|\leq 1} |(J(u_n) - J(u_0), v)| \\
&= \sup_{\|v\|\leq 1} \left|\int_0^1 [|u_n'|^{p-2}u_n' - |u_0'|^{p-2}u_0']v'\mathrm{d}t\right| \\
&\leq \left(\int_0^1 \left||u_n'|^{p-2}u_n' - |u_0'|^{p-2}u_0'\right|^{\frac{p}{p-1}}\mathrm{d}t\right)^{\frac{p-1}{p}} \left(\int_0^1 |v'|^p\mathrm{d}t\right)^{\frac{1}{p}} \\
&\leq \left(\int_0^1 \left||u_n'|^{p-2}u_n' - |u_0'|^{p-2}u_0'\right|^{\frac{p}{p-1}}\mathrm{d}t\right)^{\frac{p-1}{p}} \to 0, \quad \text{as } n \to \infty,
\end{aligned}
\tag{2.11}
$$

where the last limit follows from Lebesgue's dominated convergence theorem. Consequently, J is continuous and so demicontinuous.

Secondly, we prove J satisfies the (S_+) condition. Let $u_n \rightharpoonup u$ weakly in X and $\limsup_{n\to\infty}(J(u_n), u_n - u_0) \leq 0$. Then $\lim_{n\to\infty}(J(u_0), u_n - u_0) = 0$ and so

$$
\begin{aligned}
0 &\geq \limsup_{n\to\infty}(J(u_n) - J(u_0), u_n - u_0) \\
&= \limsup_{n\to\infty} \int_0^1 \left(|u_n'|^{p-2}u_n' - |u_0'|^{p-2}u_0'\right)(u_n' - u_0')\mathrm{d}t \\
&\geq \limsup_{n\to\infty} \left\{\int_0^1 |u_n'|^p\mathrm{d}t - \left(\int_0^1 |u_n'|^p\mathrm{d}t\right)^{\frac{1}{p'}} \left(\int_0^1 |u_0'|^p\mathrm{d}t\right)^{\frac{1}{p}} \right. \\
&\qquad \left. - \left(\int_0^1 |u_n'|^p\mathrm{d}t\right)^{\frac{1}{p}} \left(\int_0^1 |u_0'|^p\mathrm{d}t\right)^{\frac{1}{p'}} + \int_0^1 |u_0'|^p\mathrm{d}t\right\} \\
&= \limsup_{n\to\infty} \left(\|u_n\|^{p-1} - \|u_0\|^{p-1}\right)(\|u_n\| - \|u_0\|) \geq 0.
\end{aligned}
\tag{2.12}
$$

Hence, $\|u_n\| \to \|u_0\|$, and note that the uniform convexity of X, then $u_n \to u_0$ in X.

Finally, we prove that the inverse operator J^{-1} of J exists and J^{-1} is bounded and continuous. The strict monotonicity of $s \mapsto |s|^{p-1}$ implies that $(J(u) - J(v), u - v) > 0$

for $u \neq v$. Hence, J is injective. By (2.12), we know $(J(u) - J(v), u - v) \geq \left(\|u\|^{p-1} - \|v\|^{p-1}\right)(\|u\| - \|v\|)$ and thus J^{-1} is bounded. Next, we claim that J^{-1} is continuous. If the claim is false. Indeed, there exists $\{f_n\}_{n=1}^{\infty}$, $f_n \to f$ in X^* and $\|J^{-1}(f_n) - J^{-1}(f)\| \geq \delta$, for a $\delta > 0$. Let $x_n := J^{-1}(f_n)$, $x := J^{-1}(f)$. Then we have $\|f_n\|\|x_n\| \geq (f_n, x_n) = (J(x_n), x_n) = \|x_n\|^p \Longrightarrow \|x_n\|^{p-1} \leq \|f_n\|$. We may assume $x_n \rightharpoonup \widetilde{x}$ in X due to the reflexivity of X. Hence

$$(2.13)\quad (J(x_n) - J(\widetilde{x}), x_n - \widetilde{x}) = (J(x_n) - J(x), x_n - \widetilde{x}) + (J(x) - J(\widetilde{x}), x_n - \widetilde{x}) \to 0.$$

Since $J(x_n) \to J(x)$ and $\|x_n\| \to \|\widetilde{x}\|$. Hence $x_n \to \widetilde{x}$. Since J is continuous and injective, $\widetilde{x} = x$, a contradiction. Up to now, we have discussed the properties of the three operators J, G, Q.

Let us denote an operator $S : X \to X^*$ by

$$(2.14)\qquad (S(u), v) = \int_0^1 |u|^{p-2} uv \mathrm{d}t, \quad \forall u, v \in X.$$

Next, we show S is compact. Indeed, Hölder inequality and $X \hookrightarrow\hookrightarrow L^p(0,1)$ imply that $\int_0^1 |u|^{p-1} \mathrm{d}t \leq \|u\|_p^{p-1} < \infty$, for $u \in X$. Thus we know that the mapping $u \mapsto |u|^{p-2}u$ (i.e., $X \to L^{p'}$) is continuous. We take the operator S as a functional from $L^{p'}$ to X^*, so the operator S is compact from the fact that the compactness of $L^{p'} \to X^*$.

We define a homotopy

$$(2.15)\quad T_\tau(u) = J(u) - (1-\tau)G(u) - \tau\lambda S(u) + (1-\tau)Q(u), \quad \text{for } \tau \in [0,1], u \in X.$$

From Lemma 2.1, we know that T_τ satisfies the (S_+) condition. We shall prove that there exists a large enough $R > 0$ such that this homotopy is admissible with respect to the ball $B(0, R) \subset X$. If the claim is false, for any $k \in \mathbb{N}$, there exists $\tau_k \in [0,1]$ and $u_k \in X$, $\|u_k\| \geq k$ such that $T_{\tau_k}(u_k) = 0$, i.e.,

$$(2.16)\qquad J(u_k) - (1 - \tau_k)G(u_k) - \tau_k \lambda S(u_k) + (1 - \tau_k)Q(u_k) = 0.$$

It is equivalent to the integral equation

$$(2.17)\qquad \begin{aligned} &\int_0^1 |u_k'|^{p-2} u_k' v' \mathrm{d}t - (1 - \tau_k) \int_0^1 f(t, u_k) v \mathrm{d}t \\ &-\tau_k \lambda \int_0^1 |u_k|^{p-2} u_k v \mathrm{d}t + (1 - \tau_k) \sum_{j=1}^n I_j(u_k(t_j)) v(t_j) = 0. \end{aligned}$$

Set $\omega_k = \frac{u_k}{\|u_k\|^{p-1}}$ and divided (2.17) by $\|u_k\|^{p-1}$ to get

$$(2.18)\qquad \begin{aligned} &\int_0^1 |\omega_k'|^{p-2} \omega_k' v' \mathrm{d}t - (1 - \tau_k) \int_0^1 \frac{f(t, u_k)}{\|u_k\|^{p-1}} v \mathrm{d}t \\ &-\tau_k \lambda \int_0^1 |\omega_k|^{p-2} \omega_k v \mathrm{d}t + (1 - \tau_k) \sum_{j=1}^n \frac{I_j(u_k(t_j))}{\|u_k\|^{p-1}} v(t_j) = 0. \end{aligned}$$

Due to the reflexivity of X and the compactness of the interval $[0,1]$, we may assume that $\omega_k \rightharpoonup \omega$ weakly in X and $\tau_k \to \tau \in [0,1]$. The continuity of G and S, combining with (H1), enables us to obtain

$$\begin{aligned}&(1-\tau_k)\int_0^1 \frac{f(t,u_k)}{\|u_k\|^{p-1}} v\mathrm{d}t \to (1-\tau)\lambda\int_0^1 |\omega|^{p-2}\omega v\mathrm{d}t,\\ &\tau_k\lambda\int_0^1 |\omega_k|^{p-2}\omega_k v\mathrm{d}t \to \tau\lambda\int_0^1 |\omega|^{p-2}\omega v\mathrm{d}t.\end{aligned}\tag{2.19}$$

On the other hand, (H2) and $X \hookrightarrow\hookrightarrow C[0,1]$ yield that there exists $d>0$ such that

$$\begin{aligned}\left|(1-\tau_k)\sum_{j=1}^n \frac{I_j(u_k(t_j))}{\|u_k\|^{p-1}}v(t_j)\right| &\le |(1-\tau_k)|\sum_{j=1}^n \frac{|I_j(u_k(t_j))|}{\|u_k\|^{p-1}}|v(t_j)|\\ &\le |(1-\tau_k)|\sum_{j=1}^n \frac{a_j+b_j|u_k|^{\gamma_j-1}}{\|u_k\|^{p-1}}|v(t_j)|\\ &\le |(1-\tau_k)|\sum_{j=1}^n \frac{a_j+b_jd\ \|u_k\|^{\gamma_j-1}}{\|u_k\|^{p-1}}|v(t_j)| \to 0.\end{aligned}\tag{2.20}$$

Passing to the limit in (2.18), together with (2.19) and (2.20), we find

$$(J(\omega_k),v) \to (1-\tau)\lambda(S(\omega),v)+\tau\lambda(S(\omega),v), \quad \forall v\in X, \text{ as } k\to\infty,\tag{2.21}$$

i.e., $\omega_k \to J^{-1}(\lambda S(\omega))$. Since $\omega_k \rightharpoonup \omega$ in X, we have $J(\omega_k)\to J(\omega)$, i.e., $J(\omega)-\lambda S(\omega)=0$.

Since $\|w_k\|=1$ for $k=1,2,\ldots$, we have $\|\omega\|=1$ and it contradicts the fact that λ is not an eigenvalue of (2.6). This prove that the homotopy T_τ is admissible with respect to the ball $B(0,R)$ if R is large enough. Hence, Lemma 2.2 (3) yields that

$$\deg(J-G-Q,B(0,R),0)=\deg(J-\lambda S,B(0,R),0),\tag{2.22}$$

Note that the value of the degree on the right-hand side of (2.22) is an odd number by Lemma 2.2 (2). Hence $\deg(J-G-Q,B(0,R),0)\neq 0$, and Lemma 2.2 (1) indicate that the existence of at least one weak solution $u\in X$ of (1.1) which satisfies $\|u\|\le R$.

3. Infinitely Many Weak Solutions for (1.1)

In this section, we shall adopt Fountain theorem under Cerami condition (C) (see [15]) to study the existence of infinitely many weak solutions for (1.1). We first give the definition of Cerami condition (C) (see [15, Definition 1.1]).

Definition 3.1. Assume that X is a Banach space with norm $\|\cdot\|$, we say that $\varphi\in C^1(X,\mathbb{R})$ satisfies Cerami condition (C), if for all $d\in\mathbb{R}$:

(i) any bounded sequence $\{u_n\} \subset X$ satisfying $\varphi(u_n) \to d$, $\varphi'(u_n) \to 0$ possesses a convergent subsequence;
(ii) there exist $\delta, \xi, \rho > 0$ such that for any $u \in \varphi^{-1}([d-\delta, d+\delta])$ with $\|u\| \geq \xi$, $\|\varphi'(u)\| \cdot \|u\| \geq \rho$.

As X is a Hilbert space, there exist (see [20]) $\{e_n\}_{n=1}^{\infty} \subset X$ and $\{f_n\}_{n=1}^{\infty} \subset X^*$ such that $f_n(e_m) = \delta_{n,m}$, $X = \overline{\text{span}}\{e_n : n = 1, 2, \ldots\}$ and $X^* = \overline{\text{span}}^{W^*}\{f_n : n = 1, 2, \ldots\}$. For $j, k \in \mathbb{N}$, denote $X_j := \text{span}\{e_j\}$, $Y_k := \bigoplus_{j=1}^{k} X_j$ and $Z_k := \overline{\bigoplus_{j=k}^{\infty} X_j}$. Clearly, $X = \overline{\bigoplus_{j\in\mathbb{N}} X_j}$ with $\dim X_j < \infty$ for all $j \in \mathbb{N}$. Denote $S_\rho := \{u \in X : \|u\| = \rho\}$. We will introduce the following Fountain theorem under condition (C).

Lemma 3.1. (see [15, Proposition 1.2]). *Let X, Y_k, Z_k define above. Assume that $\varphi \in C^1(X, \mathbb{R})$ satisfies condition (C), and $\varphi(-u) = \varphi(u)$. For each $k \in \mathbb{N}$, there exist $\rho_k > r_k > 0$ such that*

(i) $b_k := \inf_{u\in Z_k\cap S_{r_k}} \varphi(u) \to +\infty, k \to \infty$;

(ii) $a_k := \max_{u\in Y_k\cap S_{\rho_k}} \varphi(u) \leq 0$.

Then φ has a sequence of critical points u_n, such that $\varphi(u_n) \to +\infty$ as $n \to \infty$.

Nowadays, we list our assumptions on f and $I_j (j = 1, 2, \ldots, n)$ in this section.

$(H3)$ *There exist $c_1, c_2 > 0$ such that $f(t, u) \leq c_1 + c_2|u|^{p-1}$, $\forall u \in \mathbb{R},\ t \in [0, 1]$.*

$(H4)$ *There is a positive constant $a > 0$ such that $\lim_{|u|\to\infty} \frac{-pF(t,u)+f(t,u)u}{|u|} \geq a$, uniformly on $t \in [0, 1]$.*

$(H5)$ $p\int_0^u I_j(s)\mathrm{d}s - I_j(u)u \geq 0$, $\int_0^u I_j(s)\mathrm{d}s \geq 0$, $\forall u \in \mathbb{R}$, $j = 1, 2, \ldots, n$.

$(H6)$ $\lim_{|u|\to\infty} \frac{F(t,u)}{|u|^p} = +\infty$, *uniformly on $t \in [0, 1]$.*

$(H7)$ *$f(t, u)$ and $I_j(u)$ $(j = 1, 2, \ldots, n)$ are odd functions about u, for all $t \in [0, 1]$.*

Lemma 3.2. *Let $(H3)$-$(H5)$ hold. Then φ satisfies Cerami condition (C).*

Proof. For all $d \in \mathbb{R}$, we assume that $\{u_n\}_{n=1}^{\infty} \subset X$ is bounded and

$$\varphi(u_n) \to d, \quad \varphi'(u_n) \to 0, \quad n \to \infty. \tag{3.1}$$

Going, if necessary, to a subsequence, we can assume that $u_n \rightharpoonup u$ weakly in X, then

$$\begin{aligned}(\varphi'(u_n) - \varphi'(u))(u_n - u) &= \int_0^1 \left(|u_n'|^{p-2}u_n' - |u'|^{p-2}u'\right)(u_n' - u')\mathrm{d}t \\ &+ \sum_{j=1}^{n}(I_j(u_n(t_j)) - I_j(u(t_j)))(u_n(t_j) - u(t_j)) \\ &- \int_0^1 (f(t, u_n) - f(t, u))(u_n - u)\mathrm{d}t.\end{aligned} \tag{3.2}$$

$X \hookrightarrow\hookrightarrow C[0,1]$ enables us to obtain that

$$\begin{aligned}
&\sum_{j=1}^{n}(I_j(u_n(t_j)) - I_j(u(t_j)))(u_n(t_j) - u(t_j)) \to 0, \\
&\int_0^1 (f(t,u_n) - f(t,u))(u_n - u)\mathrm{d}t \to 0 \text{ as } n \to \infty.
\end{aligned} \tag{3.3}$$

It follows from $u_n \rightharpoonup u$ weakly in X and $(\varphi'(u_n) - \varphi'(u))(u_n - u) \to 0$ that

$$\int_0^1 \left(|u_n'|^{p-2}u_n' - |u'|^{p-2}u'\right)(u_n' - u')\mathrm{d}t \to 0 \text{ as } n \to \infty. \tag{3.4}$$

Note that (2.12), we have

$$\left(\|u_n\|^{p-1} - \|u\|^{p-1}\right)(\|u_n\| - \|u\|) \le \int_0^1 \left(|u_n'|^{p-2}u_n' - |u'|^{p-2}u'\right)(u_n' - u')\mathrm{d}t, \tag{3.5}$$

and thus $\|u_n - u\| \to 0$ as $n \to \infty$. Hence, condition (i) of Definition 3.1 holds. Next, we prove condition (ii) of Definition 3.1, if not, there exists a sequence $\{u_n\} \subset X$ such that

$$\varphi(u_n) \to d, \quad \|\varphi'(u_n)\| \cdot \|u_n\| \to 0, \ n \to \infty, \tag{3.6}$$

and

$$\|u_n\| \to \infty, \quad n \to \infty. \tag{3.7}$$

By (3.6), there exists a constant $\varepsilon_1 > 0$ such that

$$\varphi(u_n) - \frac{1}{p}\varphi'(u_n)u_n \le \varepsilon_1. \tag{3.8}$$

On the other hand, (H4) implies that there is a $M > 0$ such that $-pF(t,u) + f(t,u)u \ge a|u|$, $\forall |u| > M$ and $t \in [0,1]$. Furthermore, $-pF(t,u) + f(t,u)u$ is bounded for $|u| \le M$ and $t \in [0,1]$. Therefore, there exists $c > 0$ such that $-F(t,u) + \frac{1}{p}f(t,u)u \ge \frac{a}{p}|u| - c$, $\forall u \in \mathbb{R}$, $t \in [0,1]$. This, together with (H5), yields

$$\begin{aligned}
&\varphi(u_n) - \frac{1}{p}\varphi'(u_n)u_n \\
&= \sum_{j=1}^{n}\int_0^{u_n(t_j)} I_j(t)\mathrm{d}t - \frac{1}{p}I_j(u_n(t_j))u_n(t_j) \\
&\quad + \int_0^1 \left(-F(t,u_n) + \frac{1}{p}f(t,u_n)u_n\right)\mathrm{d}t \\
&\ge \int_0^1 \left(-F(t,u_n) + \frac{1}{p}f(t,u_n)u_n\right)\mathrm{d}t \ge \int_0^1 \left(\frac{a}{p}|u_n| - c\right)\mathrm{d}t,
\end{aligned} \tag{3.9}$$

which implies $\int_0^1 |u_n|\mathrm{d}t \le pa^{-1}(c+\varepsilon_1)$. Therefore, there is a $\varepsilon_2 > 0$ such that $\|u_n\|_\infty \le \varepsilon_2$.

It follows from (H3) that there are $c_3, c_4 > 0$ such that

$$F(t,u) \le c_3|u| + c_4|u|^p, \ \forall u \in \mathbb{R} \text{ and } t \in [0,1]. \tag{3.10}$$

By this and (H5), we can find

$$\begin{aligned}
&\varphi(u_n)\\
&= \frac{1}{p}\int_0^1 |u_n'|^p \mathrm{d}t + \sum_{j=1}^n \int_0^{u_n(t_j)} I_j(t)\mathrm{d}t - \int_0^1 F(t,u_n)\mathrm{d}t\\
&\ge \frac{1}{p}\|u_n\|^p - \int_0^1 (c_3|u_n| + c_4|u_n|^p)\mathrm{d}t \ge \frac{1}{p}\|u_n\|^p - c_3\|u_n\|_\infty - c_4\|u_n\|_\infty^p\\
&\ge \frac{1}{p}\|u_n\|^p - c_3\varepsilon_2 - c_4\varepsilon_2^p
\end{aligned} \tag{3.11}$$

and thus $\varphi(u_n) \to \infty$ if (3.7) holds, which contradicts $\varphi(u_n) \to d$ in (3.6). This proves that φ satisfies condition (C).

Theorem 3.1. *Suppose* (*H2*)-(*H7*) *hold, then* (1.1) *has infinitely many weak solutions.*

Proof. By (H7), we easily have φ is even. Denote $l_k = \sup_{u \in Z_k \cap S_1} \|u\|_2$, $\beta_k = \sup_{u \in Z_k \cap S_1} \|u\|_p$, by the compactness of the embedding $X \hookrightarrow\hookrightarrow L^p(0,1)$ $(p > 1)$, we know that $l_k, \beta_k \to 0$ as $k \to \infty$ (see [21, Lemma 3.8]). Note that (3.10), we have by (H5) and Schwarz inequality, for any $u \in Z_k$ and $\|u\| = r_k := (l_k + \beta_k)^{-1}$, we get

$$\begin{aligned}
\varphi(u) &\ge \frac{1}{p}\|u\|^p - \int_0^1 (c_3|u| + c_4|u|^p)\mathrm{d}t \ge \frac{1}{p}\|u\|^p - c_3\|u\|_2 - c_4\|u\|_p^p\\
&\ge \frac{1}{p}\|u\|^p - c_3 l_k\|u\| - c_4\beta_k^p\|u\|^p \ge \frac{(l_k+\beta_k)^{-p}}{p} - c_3 - c_4.
\end{aligned} \tag{3.12}$$

We easily have $r_k \to \infty$ as $k \to \infty$, then we have

$$\varphi(u) \ge \frac{(l_k+\beta_k)^{-p}}{p} - c_3 - c_4 \to \infty, \text{ as } k \to \infty. \tag{3.13}$$

Hence, $b_k := \inf_{u \in Z_k, \|u\| = r_k} \varphi(u) \to \infty$ as $k \to \infty$.

On the other hand, by (H6), we find there are $b, c > 0$ such that $F(t,u) \ge b|u|^p - c$, $\forall u \in \mathbb{R}$, $t \in [0,1]$. Since all the norms of a finite dimensional normed space are equivalent, note that $\|\cdot\|_p$ is a norm of Y_k, so there exists a $\xi > 0$ such that

$$\|u\|_p^p \ge \xi\|u\|^p, \ \forall u \in Y_k. \tag{3.14}$$

Moreover, $X \hookrightarrow\hookrightarrow C[0,1]$ implies that there is a $d > 0$ such that $\|u\|_\infty \le d\|u\|, \forall u \in X$. Those imply

$$\begin{aligned}
\varphi(u) &= \frac{1}{p}\int_0^1 |u'|^p \mathrm{d}t + \sum_{j=1}^n \int_0^{u(t_j)} I_j(t)\mathrm{d}t - \int_0^1 F(t,u)\mathrm{d}t \\
&\le \frac{1}{p}\|u\|^p - \int_0^1 (b|u|^p - c)\mathrm{d}t + \sum_{j=1}^n \int_0^{u(t_j)} (a_j + b_j|s|^{\gamma_j - 1})\mathrm{d}t \\
&\le \frac{1}{p}\|u\|^p - b\|u\|_p^p + c + \sum_{j=1}^n \left[a_j|u(t_j)| + \frac{b_j}{\gamma_j}|u(t_j)|^{\gamma_j}\right] \qquad (3.15)\\
&\le \frac{1}{p}\|u\|^p - b\xi\|u\|^p + \sum_{j=1}^n \left[a_j\|u\|_\infty + \frac{b_j}{\gamma_j}\|u\|_\infty^{\gamma_j}\right] + c \\
&\le \left(\frac{1}{p} - b\xi\right)\|u\|^p + \sum_{j=1}^n \left[a_j d\|u\| + \frac{b_j}{\gamma_j} d^{\gamma_j}\|u\|^{\gamma_j}\right] + c.
\end{aligned}$$

Note that we can choose a large enough b such that $\frac{1}{p} - b\xi < 0$ by (H6) and $p > \gamma_j$ by (H2), then there exists positive constants d_k such that

$$\varphi(u) \le 0, \text{ for each } u \in Y_k \text{ and } \|u\| \ge d_k. \qquad (3.16)$$

Combining this and (3.16), we can take $\rho_k := \max\{d_k, r_k + 1\}$, and thus $a_k := \max_{u \in Y_k, \|u\| = \rho_k} \varphi(u) \le 0$. Up until now, we have proved the functional φ satisfies all the conditions of Lemma 3.1, then φ has infinitely many solutions.

Remark 3.1. (H6). can be weaken that $\lim_{|u|\to\infty} \frac{F(t,u)}{|u|^p} > (p\xi)^{-1}$, uniformly on $t \in [0,1]$, where ξ is determined by (3.14). Indeed, by this, we can obtain that there is a $b > (p\xi)^{-1}$ and $M > 0$ such that $F(t,u) \ge b|u|^p$, $\forall |u| \ge M$. As a result of this, we can also have (3.16).

In what follows, we shall give two interesting examples to illustrate our results.

Example 3.1. Let $t_1 = \cdots = t_n = 1$. Then we investigate the problem

$$\begin{cases}
-(|u'|^{p-2}u')' = (p-1)\left(\dfrac{2(n+\frac{1}{2})\pi}{p\sin\frac{\pi}{p}}\right)^p |u|^{p-2}u, \text{ in } \Omega, \\
u(0) = u(1) = 0, \\
u(t_j^+) - u(t_j^-) = 0, \\
\Delta|u'(t_j)|^{p-2}u'(t_j) = I_j(u(t_j)),\ j = 1, 2, \ldots, n,
\end{cases} \qquad (3.17)$$

where $I_j(u(t_j)) = \sqrt[5]{u(t_1)}$. Clearly, (H2) holds. By direct computation, we have $\lim_{|u|\to\infty} \frac{f(t,u)}{|u|^{p-2}u} = (p-1)\left(\frac{2(n+\frac{1}{2})\pi}{p\sin\frac{\pi}{p}}\right)^p \in (\lambda_n, \lambda_{n+1})$ by the fact that $n^p < (n+\frac{1}{2})^p <$

$(n+1)^p$ for $p > 1$ and $n \in \mathbb{N}$. Thus, (H1) is satisfied. By Theorem 2.1, (3.17) has at least one weak solution.

Example 3.2. Let $p = 4$ and $t_1 = \cdots = t_n = 1$. Consider the following problem

$$\begin{cases} -(|u'|^{p-2}u')' = f(t,u), \text{ in } \Omega, \\ u(0) = u(1) = 0, \\ u(t_j^+) - u(t_j^-) = 0, \\ \Delta |u'(t_j)|^{p-2}u'(t_j) = I_j(u(t_j)), \ j = 1, 2, \ldots, n, \end{cases} \tag{3.18}$$

where $f(t,u) = -\eta + (\xi^{-1} + 4)u^3$ and $I_j(u(t_j)) = \sqrt[3]{u(t_1)}$, where $\eta > 0$ and ξ is defined by (3.14).

For $I_j(u)$, we can easily have (H2) and (H7) hold. By computation, $\int_0^u I_j(s)\mathrm{d}s = \frac{3}{4}\sqrt[3]{u^4} \geq 0$, $p\int_0^u I_j(s)\mathrm{d}s - I_j(u)u = 4 \times \frac{3}{4}\sqrt[3]{u^4} - \sqrt[3]{u^4} = 2\sqrt[3]{u^4} \geq 0$, $\forall u \in \mathbb{R}$, hence we obtain (H5) is satisfied. For $f(t,u)$ and $F(t,u)$, we see $f(t,u) = -\eta + (\xi^{-1}+4)u^3 \leq \eta + (\xi^{-1}+4)|u|^3$, $\forall u \in \mathbb{R}$, $t \in [0,1]$. Moreover, $f(t,u)$ is an odd function about u, for all $t \in [0,1]$. Therefore, (H3) and (H7) hold.

It is obvious that $F(t,u) = \int_0^u f(t,x)\mathrm{d}x = \int_0^u (-\eta + (\xi^{-1}+4)x^3)\mathrm{d}x = -\eta|u| + \frac{\xi^{-1}+4}{4}|u|^4$, furthermore,

$$\lim_{|u|\to\infty} \frac{F(t,u)}{|u|^p} = \lim_{|u|\to\infty} \frac{-\eta|u| + \frac{\xi^{-1}+4}{4}|u|^4}{|u|^4} = \frac{\xi^{-1}+4}{4} > (p\xi)^{-1}.$$

As a result of this, (H6) is true by Remark 3.1. It follows from $-pF(t,u) + f(t,u)u = -4[-\eta|u| + \frac{\xi^{-1}+4}{4}|u|^4] + (-\eta + (\xi^{-1}+4)u^3)u = 4\eta|u| - \eta u \geq 3\eta|u|$ that $\lim_{|u|\to\infty} \frac{-pF(t,u)+f(t,u)u}{|u|} \geq \lim_{|u|\to\infty} \frac{3\eta|u|}{|u|} = 3\eta$, uniformly on $t \in [0,1]$. Consequently, (H4) holds. Nowadays, we have proved that (H2)-(H7) hold, then (3.18) has infinitely many solutions by Theorem 3.1.

Acknowledgments

Thank you for the referee's useful comments and suggestions for our manuscript. We are grateful to Prof. Zhilin Yang for his careful reading and for helpful remarks.

References

1. D. Bainov and P. Simeonov, *Systems with Impulse Effect*, Ellis Horwood Series: Mathematics and Its Applications, Ellis Horwood, Chichester, 1989.

2. V. Lakshmikantham, D. Bainov and P. Simeonov, *Theory of Impulsive Differential Equations*, Series in Modern Applied Mathematics, Vol. 6, World Scientific, New Jersy, 1989.

3. P. Chen and X. Tang, New existence and multiplicity of solutions for some Dirichlet problems with impulsive effects, *Math. Comput. Modelling*, **55** (2012), 723-739.

4. J. Nieto and D. O'Regan, Variational approach to impulsive differential equations, *Nonlinear Anal. Real World Appl.*, **10** (2009), 680-690.

5. J. Sun, H. Chen, J. Nieto and M. Otero-Novoa, The multiplicity of solutions for perturbed second order Hamiltonian systems with impulsive effects, *Nonlinear Anal.*, **72** (2010), 4575-4586.

6. Y. Tian, W. Ge and D. Yang, Existence results for second order system with impulsive effects via variational methods, *J. Appl. Math. Comput.*, **31** (2009), 255-265.

7. J. Xiao, J. Nieto and Z. Luo, Multiplicity of solutions for nonlinear second order impulsive differential equations with linear derivative dependence via variational methods, *Commun. Nonlinear Sci. Numer. Simul.*, **17** (2012), 426-432.

8. J. Zhou and Y. Li, Existence of solutions for a class of second order Hamiltonian systems with impulsive effects, *Nonlinear Anal.*, **72** (2010), 1594-1603.

9. L. Bai and B. Dai, Three solutions for a p-Laplacian boundary value problem with impulsive effects, *Appl. Math. Comput.*, **217** (2011), 9895-9904.

10. L. Bai and B. Dai, Existence and multiplicity of solutions for an impulsive boundary value problem with a parameter via critical point theory, *Math. Comput. Modelling*, **53** (2011), 1844-1855.

11. P. Chen and X. Tang, Existence of solutions for a class of p-Laplacian systems with impulsive effects, *Taiwanese J. Math.*, **16** (2012), 803-828.

12. P. Chen and X. Tang, Existence and multiplicity of solutions for second-order impulsive differential equations with Dirichlet problems, *Appl. Math. Comput.*, **218** (2012), 11775-11789.

13. L. Wang, W. Ge and M. Pei, Infinitely many solutions of a second-order p-Laplacian problem with impulsive condition, *Appl. Math.*, **55** (2010), 405-418.

14. I. Bogun, Existence of weak solutions for impulsive p-Laplacian problem with superlinear impulses, *Nonlinear Anal. Real World Appl.*, **13** (2012), 2701-2707.

15. A. Qian and C. Li, *Infinitely many solutions for a robin boundary value problem*, Int. J. Differ. Equ. Vol. 2010, 9 pages.

16. A. Kartsatos and I. Skrypnik, A new topological degree theory for densely defined quasibounded $(\widetilde{S}_+)$-perturbations of multivalued maximal monotone operators in reflexive Banach spaces, *Abstr. Appl. Anal.*, **2** (2005), 121-158.

17. A. Kartsatos and J. Quarcoo, A new topological degree theory for densely defined $(S_+)_L$-perturbations of multivalued maximal monotone operators in reflexive separable Banach spaces, *Nonlinear Anal.*, **69** (2008), 2339-2354.

18. K. Lan, A variational inequality theory for demicontinuous S-contractive maps with applications to semilinear elliptic inequalities, *J. Differential Equations*, **246** (2009), 909-928.

19. I. Skrypnik, *Nonlinear Elliptic Boundary Value Problems*, Teubner, Leipzig, 1986.

20. X. Fan and X. Han, Existence and multiplicity of solutions for $p(x)$-Laplacian equations in $\mathbb{R}^N$, *Nonlinear Anal.*, **59** (2004), 173-188.

21. M. Willem, *Minimax Theorems*, Birkhäuser, Boston, 1996.

Jiafa Xu[a], Zhongli Wei[a,b] and Youzheng Ding[a,b]
[a] School of Mathematics
Shandong University
Jinan, Shandong 250100
P. R. China
E-mail: xujiafa292@sina.com
[b] Department of Mathematics
Shandong Jianzhu University
Jinan, Shandong 250101
P. R. China
E-mail: jnwzl@yahoo.com.cn
dingyouzheng@163.com