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ON PRIME SUBMODULES AND PRIMARY DECOMPOSITIONS IN TWO-GENERATED FREE MODULES

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Abstract. In this paper, we consider the free R-module $R \oplus R$, where R is an arbitrary commutative ring with identity. We give a full characterization for prime submodules of $R \oplus R$ and a useful primeness test for a finitely generated submodule of $R \oplus R$. We study the existence of primary decomposition of a submodule of $R \oplus R$ and characterize the minimal primary decomposition. As applications of our results, we give some examples of primary decompositions in $R \oplus R$.

1. Introduction

Throughout this paper, all rings are commutative with identity and all modules are unitary.

Let R be a ring and M be an R-module. For any submodule N of M we set $(N : M) = \{r \in R : rM \subseteq N\}$. A proper submodule N of M is called a P-prime submodule if $rm \in N$ for $r \in R$ and $m \in M$ implies that either $m \in N$ or $r \in P = (N : M)$. It is well-known that a proper submodule N of M is prime if and only if P is a prime ideal of R and M/N is torsion-free as an R/P-module.

A proper submodule Q of M is called a primary submodule provided that for any $s \in R$ and $m \in M$, $sm \in Q$ implies that $m \in Q$ or $s^n \in (Q : M)$ for some positive integer n. Let Q be a primary submodule of M, then the radical of the ideal $(Q : M)$ is a prime ideal of R. If $P = \sqrt{(Q : M)}$, then Q is called a P-primary submodule of M.

A submodule N of M has a primary decomposition if $N = Q_1 \cap ... \cap Q_k$ with each Q_i a P_i-primary submodule of M for some prime ideal P_i. If no Q_i contains $Q_1 \cap ... \cap Q_{i-1} \cap Q_{i+1} \cap ... \cap Q_k$ and the ideals $P_1, ..., P_k$ are all distinct, then the primary decomposition is said to be minimal and the set $Ass(N) = \{P_1, ..., P_k\}$ is said to be the set of associated prime ideals of N.

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Let I be an ideal of R such that I has a primary decomposition. It is well-known that the minimal members of $Ass(I)$ are precisely the minimal prime ideals of I. These prime ideals are called the minimal primes of I. The remaining associated primes of I, that is, the associated primes of I which are not minimal, are called the embedded primes of I.

Prime submodules and primary decompositions of submodules of a module over a commutative ring have been studied by many authors (see, [6], [7], and [11]). In [10], Tıraş and Harmancı gave some characterizations of prime and primary submodules of $R \oplus R$, where R is a PID (Principal Ideal Domain). Moreover, these submodules of finitely generated free modules over a PID were studied in [3], [4], [5] and [1]. Pusat-Yılmaz in [9] also studied prime submodules of finitely generated free modules over arbitrary commutative domains.

In this paper, we completely determine prime submodules of $R \oplus R$ for an arbitrary commutative ring R, and we generalize some known results in [10] and [9]. We also study the existence of the primary decomposition of a submodule of $R \oplus R$, and characterize the minimal primary decomposition. As applications of our results, we give some examples of primary decompositions in $R \oplus R$, where R is not a PID.

2. Prime Submodules and Primary Decompositions

In the rest of this paper we fix the following notations: Let R be a commutative ring with identity and $F = R \oplus R$. We use N to be a non–zero submodule of F generated by the set $\{(a_i, b_i) \in F : i \in \Lambda\}$ and $\mathcal{L} = \sum_{i,j \in \Lambda} R\Delta_{ij}$ where $\Delta_{ij} = a_i b_j - b_i a_j$ for $i, j \in \Lambda$.

The following Lemma can be found in [2]. But we give its proof for completeness.

Lemma 2.1. *Let F and N be as above. Then $\mathcal{L} \subseteq (N : F) \subseteq \sqrt{\mathcal{L}}$.*

Proof. For all $i, j \in \Lambda$, we have

$$\begin{aligned} \Delta_{ij}(1,0) = & \ (a_i, b_i)b_j - (a_j, b_j)b_i \in N \\ \Delta_{ij}(0,1) = & \ (a_j, b_j)a_i - (a_i, b_i)a_j \in N \end{aligned}$$

and so $\sum_{i,j \in \Lambda} R\Delta_{ij} \subseteq (N : F)$.

Let $x \in (N : F)$. Then there exists a finite subset Υ of Λ such that $x(1,0) = \sum_{i \in \Upsilon} t_i(a_i, b_i)$ and $x(0,1) = \sum_{i \in \Upsilon} k_i(a_i, b_i)$, where $t_i, k_i \in R$ for all $i \in \Upsilon$. Then $x = \sum_{i \in \Upsilon} t_i a_i$, $x = \sum_{i \in \Upsilon} k_i b_i$, $0 = \sum_{i \in \Upsilon} t_i b_i$ and $0 = \sum_{i \in \Upsilon} k_i a_i$. Thus we have

$$\begin{aligned} x^2 &= \sum_{i \in \Upsilon} t_i k_i a_i b_i + \sum_{i \in \Upsilon} (\sum_{j \in \Upsilon,\, i \neq j} t_i a_i k_j b_j) \\ 0 &= \sum_{i \in \Upsilon} t_i k_i a_i b_i + \sum_{i \in \Upsilon} (\sum_{j \in \Upsilon,\, i \neq j} t_i b_i k_j a_j). \end{aligned}$$

Therefore, $x^2 = \sum_{i \in \Upsilon} (\sum_{j \in \Upsilon,\, i \neq j} t_i k_j \Delta_{ij}) \in \sum_{i,j \in \Upsilon} R\Delta_{ij}$ and so $x \in \sqrt{\mathcal{L}}$. ∎

Theorem 2.2. *Let N be a submodule of F with $(N : F) = P$.*

(a) *If P is a prime ideal of R and $a_i, b_i \in P$ for all $i \in \Lambda$, then $N = P \oplus P$ and N is a prime submodule.*

(b) *If P is a maximal ideal of R, $a_i \notin P$ for some $i \in \Lambda$ and $b_j \notin P$ for some $j \in \Lambda$, then $N = (a_i b_j)N + PF$ and N is a prime submodule.*

(c) *If $\{a_i : i \in \Lambda\} \cup \{b_i : i \in \Lambda\} \nsubseteq P$ and N is a prime submodule, then $N = \{(m, n) \in F : mb_i - na_i \in P$ for all $i \in \Lambda\}$. In particular,*

(i) *If $a_i \in P$ for all $i \in \Lambda$ and $b_j \notin P$ for some $j \in \Lambda$ (resp. $b_i \in P$ for all $i \in \Lambda$ and $a_j \notin P$ for some $j \in \Lambda$) then $N = P \oplus R$ (resp. $N = R \oplus P$).*

(ii) *If $a_i \notin P$ and $b_i \in P$ for some $i \in \Lambda$ (resp. $b_i \notin P$ and $a_i \in P$ for some $i \in \Lambda$) then $N = R \oplus P$ (resp. $N = P \oplus R$).*

Proof.

(a) It is clear that PF is a prime submodule of F contained in N. If $a_i, b_i \in P$ for all $i \in \Lambda$ then $PF = P \oplus P$ contains N.

(b) Let P be a maximal ideal of R and let $a_i \notin P$ for some $i \in \Lambda$ and $b_j \notin P$ for some $j \in \Lambda$. Then we get that $(a_i b_j)R + P = R$. Let $(x, y) \in N$. Then $x = ra_i b_j + p_1$ and $y = sa_i b_j + p_2$ for some $r, s \in R$ and $p_1, p_2 \in P$. It follows that $(x, y) = a_i b_j(r, s) + (p_1, p_2)$ and then $a_i b_j(r, s) \in N$ as $(p_1, p_2) \in PF \subseteq N$. Since $(N : F) = P$ is a maximal ideal of R, N is a prime submodule of F and hence $(r, s) \in N$. Thus $(x, y) \in (a_i b_j)N + PF$ and so $N \subseteq (a_i b_j)N + PF$. The other inclusion is clear.

(c) We may assume that $a_1 \notin P$. Consider the submodule

$$T_P = \{(m, n) \in F : mb_i - na_i \in P \text{ for all } i \in \Lambda\}.$$

By Lemma 2.1, it is clear that $N \subseteq T_P$. Let $(m, n) \in T_P$. Then there exists a $p \in P$ such that $na_1 = mb_1 + p$ and so

$a_1(m, n) = (a_1 m, a_1 n) = (a_1 m, b_1 m) + (0, p) = m(a_1, b_1) + (0, p) \in N + PF = N$. Since $a_1 \notin P$, we get that $(m, n) \in N$ and so $T_P = N$.

(i) Let $a_i \in P$ for all $i \in \Lambda$ and $b_j \notin P$ for some $j \in \Lambda$. It is clear that $N \subseteq P \oplus R$. $N = \{(m, n) \in F : mb_i - na_i \in P$ for all $i \in \Lambda\}$ by (c). Let $(x, y) \in P \oplus R$. Then $xb_i - ya_i \in P$ for all $i \in \Lambda$ and so $(x, y) \in N$. Thus $N = P \oplus R$.

By using the same argument as above it can be proved that $N = R \oplus P$ if $b_i \in P$ for all $i \in \Lambda$ and $a_j \notin P$ for some $j \in \Lambda$.

(ii) Let $a_i \notin P$ and $b_i \in P$ for some $i \in \Lambda$. Let $(x, y) \in N$. Then $xb_i - ya_i \in P$ and so $N \subseteq R \oplus P$. Since $xb_j - ya_j \in P$ for all $j \in \Lambda$, we get $xb_j \in P$. If $x \in P$ then $N = P \oplus P$. This is a contradiction as $(a_i, b_i) \in N - (P \oplus P)$. Therefore $b_j \in P$ for all $j \in \Lambda$. Now the result follows from (i).

By using the same argument as above it can be proved that $N = P \oplus R$ if $b_i \notin P$ and $a_i \in P$ for some $i \in \Lambda$. ∎

By using Theorem 2.2, we prove the following corollary which is a generalization of [10, Proposition 2.3] with a different proof.

Corollary 2.3. *Let N be a prime submodule of F. Then*

(a) *If $(1, 0) \in N$ then $N = R \oplus (N : F)$.*

(b) *If $(0, 1) \in N$ then $N = (N : F) \oplus R$.*

Proof. (a) Let $(1, 0) \in N$. It is clear that $N \neq P \oplus P$. Then $b_i \in P$, for all $i \in \Lambda$ by Theorem 2.2 (c). We get that $N = R \oplus (N : F)$ by Theorem 2.2 $(c - i)$. ∎

Let N be a P-prime submodule of a module M. It is said that N has P–height n for some non-negative integer n, if there exists a chain $K_n \subset K_{n-1} \subset ... \subset K_1 \subset K_0 = N$ of P–prime submodules K_i of M, but no longer such chain.

Proposition 2.4. *Let N be a P-prime submodule of F. If $N \neq P \oplus P$ then the P-height of N is* 1.

Proof. Since $N \neq P \oplus P$, we have that $a_i \notin P$ for some $i \in \Lambda$ or $b_j \notin P$ for some $j \in \Lambda$. By Theorem 2.2, $N = \{(m, n) \in F : mb_i - na_i \in P \text{ for all } i \in \Lambda\}$. Let K be a P-prime submodule of F with $K \subseteq N$ and let $\{(c_i, d_i) \in F : i \in \Omega\}$ be a generating set for K. If $c_i, d_i \in P$ for all $i \in \Omega$ then $K = P \oplus P$. Suppose that $c_k \notin P$ for some $k \in \Omega$ or $d_l \notin P$ for some $l \in \Omega$. Then $K = \{(x, y) \in F : xd_i - yc_i \in P$ for all $i \in \Omega\}$ by Theorem 2.2-(c). Since $(c_i, d_i) \in N$ for all $i \in \Omega$ we get that $c_i b_j - d_i a_j \in P$ and so $(a_j, b_j) \in K$ for all $j \in \Lambda$. Hence $K = N$. ∎

Corollary 2.5. *Let N be a P-prime submodule of F. If $(R \oplus P) \cap N \neq PF$ (resp. $(P \oplus R) \cap N \neq PF$), then $N = R \oplus P$ (resp. $P \oplus R$).*

Proof. Let N be a P–prime submodule of F. Then $(R \oplus P) \cap N$ is a P-prime submodule. By Proposition 2.4, we get that $N = (R \oplus P) \cap N$ and so $P \oplus P \subset N \subseteq R \oplus P$. The P-height of $R \oplus P$ is 1 by Proposition 2.4. Thus we have $R \oplus P = N$. ∎

Theorem 2.6. *Let N be a submodule of F which doesn't contain $(1, 0)$ and $(0, 1)$. Let $P = (N : F)$ be a maximal ideal of R and $(a, b) \in N$ with $Ra + Rb \nsubseteq P$. Then $N = \{(x, y) \in F : ay - bx \in P\}$ and N is a prime submodule of F.*

Proof. Since $(N : F) = P$ is a maximal ideal of R, N is prime.

Assume that $a \in P$. Since $(a, 0) \in N$, we have $(0, b) = b(0, 1) \in N$ and so $b \in P$. Thus $Ra + Rb \subseteq P$, a contradiction, and we get that $a, b \notin P$. Therefore, there exist $x_1, y_1 \in R$ and $p_1, p_2 \in P$ such that $ax_1 + p_1 = 1$, $by_1 + p_2 = 1$. Let $K = \{(x, y) \in F : ay - bx \in P\}$. Clearly, K is a P–prime submodule of F.

To show the equality $N = K$, take $(c, d) \in N$. Since $(ad - bc, 0) = d(a, b) - b(c, d) \in N$ and $(1, 0) \notin N$, we get that $ad - bc \in P$ and so $N \subseteq K$. For the reverse inclusion, take $(c, d) \in K$. Then we get that

$$(c, d) = (by_1c + p_2c, ax_1d + p_1d) = (by_1c, ax_1d) + (p_2c, p_1d)$$

Since $(p_2c, p_1d) \in P \oplus P$, it is enough to show that $(by_1c, ax_1d) \in N$.

Since $x_1(a, b) \in N$, it follows that

$$(ax_1, bx_1) + (p_1, 0) = (1, bx_1) = (by_1, bx_1) + (p_2, 0) \in N$$

and so $b(y_1, x_1) \in N$. Then we have $(y_1, x_1) \in N$ as $b \notin P$. On the other hand, there exists $q \in P$ such that $bc = q + ad$. Then we get that

$$(y_1(bc), ax_1d) = (y_1ad + y_1q, ax_1d) = ad(y_1, x_1) + (qy_1, 0).$$

Therefore, $(by_1c, ax_1d) \in N$ and so $K = N$. ■

In [8], Pusat-Yılmaz and Smith defined the submodule $K(N, P) = \{m \in M : cm \in N + PM \text{ for } c \in R\backslash P\}$ for an R-module M and $N \leq M$. Then they showed that $K(N, P) = M$ or $K(N, P)$ is the smallest P-prime submodule containing N. As a consequence of Theorem 2.6, we obtain the following corollary which characterizes $K(N, P)$ and the structure of a prime submodule of F. Corollary 2.7-(2) is a generalization of [10, Theorem 2.7].

Corollary 2.7. (1) *Let N be a submodule of F, $P = (N : F)$ be a prime ideal and $(a, b) \in N$ with $Ra + Rb \nsubseteq P$. If N_P doesn't contain $(\frac{1}{1}, \frac{0}{1})$ and $(\frac{0}{1}, \frac{1}{1})$, then $\{(x, y) \in F : ay - bx \in P\} = K(N, P)$.*

(2) *Let N be a submodule of F which doesn't contain $(1, 0)$ and $(0, 1)$. Suppose that $P = (N : F)$ is a prime ideal of R and $(a, b) \in N$ with $Ra + Rb \nsubseteq P$. Then N is a P-prime submodule of F if and only if $N = \{(x, y) \in F : ay - bx \in P\}$.*

Proof. (1) Since $P_P = (N : F)_P = (N_P : F_P)$ is a maximal ideal of R_P, $N_P = \{(\frac{x}{s}, \frac{y}{t}) \in F_P : say - tbx \in P\}$ by Theorem 2.6. Let $\varphi : F \longrightarrow F_P$, be the natural homomorphism. Then we have $\varphi^{-1}(N_P) = \{(x, y) \in F : (\frac{x}{1}, \frac{y}{1}) \in N_P\} = \{(x, y) \in F : ay - bx \in P\} = \{(x, y) \in F : r(x, y) \in N \text{ for some } r \in R\backslash P\} = K(N, P)$.

(2) Suppose that N is a prime submodule and $(\frac{1}{1}, \frac{0}{1}) \in N_P$. Then $\frac{(1,0)}{1} = \frac{(x,y)}{s}$ for some $(x, y) \in N$ and $s \in R\backslash P$. We have $u(s(1, 0) - (x, y)) = 0$ for some $u \in R\backslash P$. Since $us(1, 0) \in N$ and $(1, 0) \notin N$, we get that $us \in P$, a contradiction. Thus $(\frac{1}{1}, \frac{0}{1}) \notin N_P$. Similarly $(\frac{0}{1}, \frac{1}{1}) \notin N_P$. By (1), $K(N, P) = \{(x, y) \in F : ay - bx \in P\} = N$ as $K(N, P)$ is the smallest P-prime submodule containing N.

Conversely, it can be easily seen that $\{(x, y) \in F : ay - bx \in P\}$ is a P-prime submodule of F. ■

To sum up our results about prime submodules of F, combining Corollary 2.3 and Corollary 2.7, we give the following theorem which characterizes all prime submodules of F.

Theorem 2.8. *Let N be a submodule of F.*

(1) *Assume that N contains $(1,0)$ or $(0,1)$. Then, N is a prime submodule of F if and only if $(N : F) = P$ is a prime ideal of R and $N = R \oplus P$ or $N = P \oplus R$.*

(2) *Assume that N does not contain $(1,0)$ and $(0,1)$. Then N is a prime submodule of F if and only if $(N : F) = P$ is a prime ideal of R and $N = P \oplus P$ or $N = \{(x,y) \in F : ay - bx \in P\}$, where $(a,b) \in N$ with $Ra + Rb \nsubseteq P$.*

In the following theorem, we determine whether N is a prime submodule of F or not, by using primeness of a certain ideal of R. This theorem is a generalization of [9, Proposition 3.4] and a useful primeness test for a finitely generated submodule of F.

Theorem 2.9. *Let N be an n–generated submodule of F with $R = Ra_n + Rb_n$. Then N is a prime submodule of F if and only if $\sum_{i=1}^{n-1} R\Delta_{ni}$ is a prime ideal of R.*

Proof. By the hypothesis, there exist elements $s_1, s_2 \in R$ such that $1 = s_1a_n + s_2b_n$. Let $L = R(a_n, b_n)$ and $L' = \{(x,y) \in F : s_1x + s_2y = 0\}$. Consider the functions $\Psi : R \to F$ defined by $\Psi(r) = r(a_n, b_n)$ and $\Phi : F \to R$ defined by $\Phi((r_1, r_2)) = s_1r_1 + s_2r_2$. Then since Φ is onto and R is projective, we get that $F = Im\Psi \oplus ker\Phi = L \oplus L'$. On the other hand by the modularity law, we have $N = L \oplus (N \cap L')$. Set $c_i = s_1a_i + s_2b_i$ $(1 \le i \le n-1)$. Then $\sum_{i=1}^{n-1} R((a_i, b_i) - c_i(a_n, b_n)) \subseteq N \cap L'$.

To show that $N = \left(\sum_{i=1}^{n-1} R((a_i, b_i) - c_i(a_n, b_n))\right) \oplus L$, take $(x,y) \in N$. Then $(x,y) = \sum_{i=1}^{n} r_i(a_i, b_i)$ for some $r_i \in R$ and so

$$\begin{aligned}(x,y) &= \sum_{i=1}^{n} r_i(a_i,b_i) - \sum_{i=1}^{n-1} r_ic_i(a_n,b_n) \\ &\quad + \sum_{i=1}^{n-1} r_ic_i(a_n,b_n) \in \left(\sum_{i=1}^{n-1} R((a_i,b_i) - c_i(a_n,b_n))\right) \oplus L\end{aligned}$$

and so $N = \left(\sum_{i=1}^{n-1} R((a_i, b_i) - c_i(a_n, b_n))\right) \oplus L$. Therefore we get the equality $N \cap L' = \sum_{i=1}^{n-1} R((a_i, b_i) - c_i(a_n, b_n))$.

Now we show that $F = L + R(-s_2, s_1)$. We have that

$$\begin{aligned}(1,0) &= s_1(a_n,b_n) + (-b_n)(-s_2,s_1) \\ (0,1) &= s_2(a_n,b_n) + a_n(-s_2,s_1)\end{aligned}$$

These imply that $F = L + R(-s_2, s_1)$. Then since $R(-s_2, s_1) \subseteq L'$ and by the modularity law, it follows that $L' = R(-s_2, s_1) + (L \cap L') = R(-s_2, s_1)$. Note that

$$\begin{aligned} -s_2\Delta_{ni} &= -s_2(a_nb_i - b_na_i) = -s_2a_nb_i + s_2b_na_i \\ &= -s_2a_nb_i + (1 - s_1a_n)a_i = a_i - a_n(s_1a_i + s_2b_i) \\ &= a_i - c_ia_n, \end{aligned}$$

and similarly, we get that $s_1\Delta_{ni} = b_i - c_ib_n$.

Then $(a_i, b_i) - c_i(a_n, b_n) = (a_i - c_ia_n, b_i - c_ib_n) = \Delta_{ni}(-s_2, s_1)$ $(1 \le i \le n-1)$.

Let $I = \sum_{i=1}^{n-1} R\Delta_{ni}$. Then $N \cap L' = I(-s_2, s_1)$. Now since $F = L \oplus L'$ and $N = L \oplus (N \cap L')$, it follows that

$$F/N \simeq L'/(N \cap L') = R(-s_2, s_1)/I(-s_2, s_1)$$

On the other hand, if $r \in R$ and $r(-s_2, s_1) = (0, 0)$, then $rs_2 = rs_1 = 0$ and hence $r = r1 = (rs_1)a_n + (rs_2)b_n = 0$. Thus we get that $F/N \simeq R/I = R/\sum_{i=1}^{n-1} R\Delta_{ni}$. Thus N is a prime submodule of F if and only if $\sum_{i=1}^{n-1} R\Delta_{ni}$ is a prime ideal of R. ■

Corollary 2.10. *Let R be a domain and $a, b \in R$ such that $Ra + Rb = R$. Then $N = R(a, b)$ is a prime submodule of F.*

Now we determine a primary decomposition of N when R is a domain.

Lemma 2.11. *Let Q be a P-primary ideal of R containing $\mathcal{L}$ and $T_Q = \{(m_1, m_2) \in F : a_im_2 - b_im_1 \in Q$ for all $i \in \Lambda\}$. Then $T_Q = F$ or T_Q is a P-primary submodule of F containing N.*

Proof. If $a_i, b_i \in Q$ for all $i \in \Lambda$, then $T_Q = F$. Suppose that $a_j \notin Q$ for some $j \in \Lambda$. Now we prove that T_Q is a P-primary submodule of F.

Let $r \in \sqrt{(T_Q : F)}$. Then $r^n(0, 1) \in T_Q$ for some $n \in \mathbb{Z}^+$ and so $r^na_j \in Q$. Since $a_j \notin Q$, we have $r \in P$. Hence $\sqrt{(T_Q : F)} \subseteq P$. Let $r \in P$. $r^n \in Q$ for some $n \in \mathbb{Z}^+$. It follows that $r^n(x, y) \in T_Q$ for all $(x, y) \in F$. Therefore $\sqrt{(T_Q : F)} = P$.

Assume that $rm \in T_Q$ for $r \in R - P$ and $m = (m_1, m_2) \in F$. Then $r(a_im_2 - b_im_1) \in Q$ for all $i \in \Lambda$. Since $r \notin P$, we get that $m \in T_Q$. Thus T_Q is a P-primary submodule of F. Since $\mathcal{L} \subseteq \mathcal{Q}$ we have $N \subseteq T_Q$. ■

Theorem 2.12. *Let R be a domain, N be a proper submodule of F with $\mid \Lambda \mid \geq 2$ and let $\mathcal{L}$ be a non-zero ideal of R such that $\mathcal{L} = R\Delta_{kl}$ for some $k, l \in \Lambda$. Let $\mathcal{L} = \cap_{i=1}^n Q_i$ be a minimal primary decomposition of $\mathcal{L}$ with $Ass(\mathcal{L}) = \{P_i\}_{i=1}^n$. Then,*

- (a) *$\cap_{i=1}^n T_{Q_i}$ is a primary decomposition of N.*
- (b) *If $\{a_j : j \in \Lambda\} \cup \{b_j : j \in \Lambda\} \nsubseteq P_i$ for all $1 \le i \le n$, then $\cap_{i=1}^n T_{Q_i}$ is a minimal primary decomposition of N with $Ass(N) = \{P_i\}_{i=1}^n$*
- (c) *If $\mathcal{L}$ has no embedded prime ideal, then $\cap_{i=1}^n T_{Q_i}$ is a minimal primary decomposition of N with $Ass(N) = \{P_i\}_{i=1}^n$.*

Proof.

(a) Since $N \subseteq T_{Q_i}$ for all $i \in \{1,..,n\}$, we have $N \subseteq \cap_{i=1}^{n} T_{Q_i}$. Take an element $(x,y) \in \cap_{i=1}^{n} T_{Q_i}$. Then $a_j y - b_j x \in Q_i$ for all $j \in \Lambda$ and $i \in \{1,...,n\}$ and so $a_j y - b_j x \in \cap_{i=1}^{n} Q_i = \mathcal{L} = R\Delta_{kl}$. In particular, there exist $t_1, t_2 \in R$ such that $a_k y - b_k x = t_1 \Delta_{kl}$ and $a_l y - b_l x = t_2 \Delta_{kl}$. It is easily seen that $(x,y) = (a_k, b_k)(-t_2) + (a_l, b_l)t_1 \in N$. Hence $N = \cap_{i=1}^{n} T_{Q_i}$. Let $S = \{s \in \{1,...,n\} : a_i \notin P_s$ for some $i \in \Lambda$ or $b_j \notin P_s$ for some $j \in \Lambda\}$ and $i \in \{1,...,n\} - S$. Then $N \subseteq Q_i \oplus Q_i$ and $T_{Q_i} = F$. Therefore, $N = \cap_{s\in S} T_{Q_s} \subseteq \cap_{i\notin S}(Q_i \oplus Q_i)$. Then we get that $\cap_{s\in S} Q_s \subseteq (\cap_{s\in S} T_{Q_s} : F) \subseteq \cap_{i\notin S}(Q_i \oplus Q_i : F) = \cap_{i\notin S} Q_i$. It follows that $\cap_{i=1, i\neq j}^{n} Q_i \subseteq Q_j$ for every $j \notin S$, a contradiction. Thus $S = \{1,...,n\}$ and so $N = \cap_{i=1}^{n} T_{Q_i}$ is a primary decomposition of N.

(b) Suppose that $\cap_{i=1,\, i\neq j}^{n} T_{Q_i} \subseteq T_{Q_j}$ for some $1 \leq j \leq n$. Take an element $r \in \cap_{i=1,\, i\neq j}^{n} Q_i - Q_j$. Then $(0,r) \in \cap_{i=1,\, i\neq j}^{n} T_{Q_i}$. We can assume that $a_t \notin P_j$ for some $t \in \Lambda$. Since $(0,r) \in T_{Q_j}$ we have $(-ra_t) \in Q_j$ and so $r \in Q_j$. But this is a contradiction. Thus $\cap_{i=1,\, i\neq j}^{n} T_{Q_i} \nsubseteq T_{Q_j}$ for all $1 \leq j \leq n$. So $\cap_{i=1}^{n} T_{Q_i}$ is a minimal primary decomposition of N with $Ass(N) = \{P_i\}_{i=1}^{n}$.

(c) Suppose that $\cap_{i=1, i\neq j}^{n} T_{Q_i} \subseteq T_{Q_j}$ for some $1 \leq j \leq n$. Then $\sqrt{(\cap_{i=1, i\neq j}^{n} T_{Q_i} : F)} \subseteq \sqrt{(T_{Q_j} : F)}$ and so $\cap_{i=1, i\neq j}^{n} P_i \subseteq P_j$. It follows that $P_i \subseteq P_j$ for some $1 \leq i \leq n$, $i \neq j$. Since $\mathcal{L}$ has no embedded prime, we get that $P_i = P_j$, a contradiction. ■

Note that the first condition on $\mathcal{L}$ in Theorem 2.12 is satisfied if N is two-generated or N is finitely generated and R is a valuation domain.

Corollary 2.13. *Let R be a domain, N be a proper submodule of F with $\mid \Lambda \mid \geq 2$ and let $\mathcal{L}$ be a non-zero ideal of R such that $\mathcal{L} = R\Delta_{kl}$ for some $k,l \in \Lambda$. If $\mathcal{L}$ has the unique prime ideal factorization $P_1^{t_1}...P_n^{t_n}$ with distinct maximal ideals P_i, $(1 \leq i \leq n)$, then $\cap_{i=1}^{n} T_{P_i^{t_i}}$ is a minimal primary decomposition of N with $Ass(N) = \{P_i\}_{i=1}^{n}$.*

Proof. $\mathcal{L} = P_1^{t_1} \cap ... \cap P_n^{t_n}$ is a minimal primary decomposition of $\mathcal{L}$ with $Ass(\mathcal{L}) = \{P_i\}_{i=1}^{n}$. Suppose that there exists an $i \in \{1,...,n\}$ such that $a_j, b_j \in P_i^{t_i}$ for all $j \in \Lambda$. Then we get that $\mathcal{L} = P_1^{t_1}...P_i^{2t_i}...P_n^{t_n}$. But this contradicts with the unique prime ideal factorization of $\mathcal{L}$. So $\cap_{i=1}^{n} T_{P_i^{t_i}}$ is a minimal primary decomposition of $\mathcal{L}$ by Theorem 2.12-(b). ■

Finally we give two examples as applications of our results for free modules with two generators over domains which are not principal ideal domains.

Example 2.14. Let R be the polynomial ring $\mathbb{Z}[X]$ and N the submodule $R(X-2, X-2) + R(1, X)$. Then $\mathcal{L} = R\Delta_{12} = R(X-2)(X-1)$ and $R(X-2) \cap R(X-$

1) is a minimal primary decomposition of $\mathcal{L}$. By applying Theorem 2.12-(b), we get that $N = T_{R(X-2)} \cap T_{R(X-1)}$ is a minimal primary decomposition of N, where $T_{R(X-2)} = \{(f,g) \in F : Xf - g \in R(X-2)\}$ and $T_{R(X-1)} = \{(f,g) \in F : f-g, Xf - g \in R(X-1)\}$.

Example 2.15. Let $R = \mathbb{Z}[\sqrt{-5}]$ and $N = R(1+\sqrt{-5}, 3) + R(1, 1-\sqrt{-5})$. Then $\mathcal{L} = R3$. It is well-known that R is a Dedekind domain and the unique prime ideal factorization of $\mathcal{L}$ is P_1P_2, where $P_1 = R3 + R(1+\sqrt{-5})$, $P_2 = R3 + R(1-\sqrt{-5})$. By applying Corollary 2.13, we get that $N = T_{P_1} \cap T_{P_2}$ is a minimal primary decomposition of N, where $T_{P_1} = \{(x,y) \in F : y - x(1-\sqrt{-5}) \in P_1\}$ and $T_{P_2} = R \oplus P_2$.

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