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STRONG CONVERGENCE THEOREMS BY MONOTONE HYBRID METHOD FOR A FAMILY OF GENERALIZED NONEXPANSIVE MAPPINGS IN BANACH SPACES

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Abstract. In this paper, we study monotone hybrid method for finding a common fixed point of a family of generalized nonexpansive mappings and then prove a strong convergence theorem for a family of generalized nonexpansive mappings in Banach spaces. Using this theorem, we obtain some new results for a generalized nonexpansive mapping and two generalized nonexpansive mappings in Banach spaces. Moreover, we apply our main result to obtain a strong convergence theorem for a family of nonexpansive mappings in a Hilbert space.

1. Introduction

Let E be a real Banach space with $\|\cdot\|$ and let C be a nonempty subset of E. Then a mapping T of C into E is called *nonexpansive* if $\|Tx-Ty\| \leq \|x-y\|$ for all $x, y \in C$. We use $F(T)$ to denote the set of fixed points of T; that is $F(T) = \{x \in C : x = Tx\}$. A mapping T of C into E is called *quasi-nonexpansive* if $F(T)$ is nonempty and $\|Tx - y\| \leq \|x - y\|$ for all $x \in C$ and $y \in F(T)$. It is easy to see that if T is nonexpansive with $F(T) \neq \emptyset$, then it is quasi-nonexpansive.

The theory of nonexpansive mappings is an important subject which can be applied widely in applied areas, in particular, in image recovery and signal processing; see, for instance, [1, 25]. However, the Picard's sequence $\{T^n x\}_{n=1}^{\infty}$ of iterates of mapping T at a point $x \in C$ may not converge even in the weak topology. In 1953, Mann [15] introduced an iterative scheme which is now known as Mann's iteration process. This iteration is defined as follows:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \quad n \geq 0, \tag{1.1}$$

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where the initial guess $x_0 \in C$ is chosen arbitrarily and the sequence $\{\alpha_n\}$ is in the interval $[0,1]$. However, we note that Mann's iteration has only weak convergence even in a Hilbert space.

In 2003, Nakajo and Takahashi [20] proposed the following modification of Mann's iteration process (1.1), by using hybrid method in mathematical programming, for a single nonexpansive mapping T in a Hilbert space H:

$$(1.2)\qquad \begin{cases} x_1 = x \in C, \\ u_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_n = \{z \in C : \|z-u_n\| \le \|z-x_n\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n\rangle \ge 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x \end{cases}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [0,1]$ and Π is the metric projection of H onto $C_n \cap Q_n$. They proved that the sequence $\{x_n\}$ generated by (1.2) converges strongly to a fixed point of T under an appropriate control condition on the sequence $\{\alpha_n\}$.

In 2008, Takahashi, Takeuchi and Kubota [30] proposed the following modification of the iteration method (1.2) for a family of nonexpansive mappings $\{T_n\}$ in a Hilbert space H:

$$(1.3)\qquad \begin{cases} x_1 = x \in C, \\ u_n = \alpha_n x_n + (1-\alpha_n)T_n x_n, \\ C_{n+1} = \{z \in C_n : \|z-u_n\| \le \|z-x_n\|\}, \\ x_{n+1} = \Pi_{C_{n+1}} x \end{cases}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [0,1]$. They proved strong convergence of the sequence $\{x_n\}$ generated by (1.3) under an appropriate control condition on the sequence $\{\alpha_n\}$ and under the condition that the family $\{T_n\}_{n=1}^{\infty}$ satisfies NST-condition.

In 2008, Qin and Su [21] modified the iteration (1.2) by the following method called the *monotone hybrid method*, for a nonexpansive mapping T in a Hilbert space, as follows:

$$(1.4)\qquad \begin{cases} x_1 = x \in C,\ C_0 = Q_0 = C, \\ u_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_n = \{z \in C_{n-1} \cap Q_{n-1} : \|z-u_n\| \le \|z-x_n\|\}, \\ Q_n = \{z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, x - x_n\rangle \ge 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x \end{cases}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [0,1]$. By using this method, they proved a strong convergence theorem under a control condition on the sequence $\{\alpha_n\}$, but the technic they used in this paper is different from Nakajo and Takahashi [20]. More precisely, they showed that the sequence $\{x_n\}$ generated by (1.4) is a Cauchy sequence, without the use of demiclosedness principle, Opial's condition and the Kadec-Klee property.

Recently, by using generalized projections, Su, Wang and Shang [27] proposed the following monotone hybrid method for a hemi-relatively noexpansive mapping T in a Banach space:

$$
(1.5) \qquad \begin{cases} x_1 = x \in C, \ C_0 = Q_0 = C, \\ u_n = J^{-1}\big(\alpha_n Jx_n + (1-\alpha_n)JTx_n\big), \\ C_n = \{z \in C_{n-1} \cap Q_{n-1} : \phi(z, u_n) \le \phi(z, x_n)\}, \\ Q_n = \{z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, Jx - Jx_n \rangle \ge 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x \end{cases}
$$

where J is the duality mapping on E and $\{\alpha_n\} \subset [0,1]$. They proved that if $\limsup_{n\to\infty} \alpha_n < 1$, then the sequence $\{x_n\}$ generated by (1.5) converges strongly to $\Pi_{F(T)}x_0$, where $\Pi_{F(T)}$ is the generalized projection from C onto $F(T)$.

Employing the ideas of Qin and Su [21], Takahashi, Takeuchi and Kubota [30] and Su, Wang and Shang [27], we modify iterations (1.3), (1.4) and (1.5) for finding a common fixed point a countable family of generalized nonexpansive mappings by using monotone hybrid method and then prove a strong convergence theorem in a Banach space. Using this theorem, we obtain some new results for a generalized nonexpansive mapping and two generalized nonexpansive mappings in Banach spaces. Moreover, we apply our main result to obtain a strong convergence theorem for a family of nonexpansive mappings in a Hilbert space.

2. Preliminaries

Throughout this paper, all linear spaces are real. Let $\mathbb{N}$ and $\mathbb{R}$ be the sets of all positive integers and real numbers, respectively. Let E be a Banach space and let E^* be the dual space of E. For a sequence $\{x_n\}$ of E and a point $x \in E$, the *weak* convergence of $\{x_n\}$ to x and the *strong* convergence of $\{x_n\}$ to x are denoted by $x_n \rightharpoonup x$ and $x_n \to x$, respectively. The duality mapping J from E into 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}, \ \forall x \in E.$$

Let $S(E)$ be the unit sphere centered at the origin of E. Then the space E is said to be *smooth* if the limit

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t}$$

exists for all $x, y \in S(E)$. It is also said to be *uniformly smooth* if the limit exists uniformly in $x, y \in S(E)$. A Banach space E is said to be *strictly convex* if $\|\frac{x+y}{2}\| < 1$ whenever $x, y \in S(E)$ and $x \neq y$. It is said to be *uniformly convex* if for each $\epsilon \in (0,2]$, there exists $\delta > 0$ such that $\|\frac{x+y}{2}\| < 1-\delta$ whenever $x, y \in S(E)$ and $\|x-y\| \geq \epsilon$. From [28] we know the following:

(i) If E in smooth, then J is single-valued.

(ii) If E is reflexive, then J is onto.

(iii) If E is strictly convex, then J is one-to-one.

(iv) If E is strictly convex, then J is strictly monotone.

(v) If E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E.

Let E be a smooth Banach space. Throughout this paper, define the function $\phi : E \times E \to \mathbb{R}$ by

$$\phi(y, x) = \|y\|^2 - 2\langle y, Jx\rangle + \|x\|^2, \ \forall y, x \in E. \tag{2.1}$$

Observe that, in a Hilbert space H, (2.1) reduces to $\phi(x, y) = \|x-y\|^2$ for all $x, y \in H$. It is obvious from the definition of the function ϕ that for all $x, y \in E$,

(P1) $(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2$,

(P2) $\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy\rangle$,

(P3) $\phi(x, y) = \langle x, Jx - Jy\rangle + \langle y - x, Jy\rangle \leq \|x\|\|Jx - Jy\| + \|y - x\|\|y\|$.

Let C be a closed subset of a Banach space E, and let T be a mapping from C into E. We use $F(T)$ to denote the set of fixed points of T, that is, $F(T) = \{x \in C : x = Tx\}$. Recall that a mapping $T : C \to E$ is *generalized nonexpansive* if $F(T) \neq \emptyset$ and $\phi(Tx, u) \leq \phi(x, u)$ for all $x \in C$ and $u \in F(T)$. Let R be a mapping from E onto C. Then R is said to be a *retraction* if $R^2 = R$. The mapping R from E onto C is said to be *sunny* if $R(Rx + t(x - Rx)) = Rx$ for all $x \in E$ and $t \geq 0$.

A nonempty closed subset C of a smooth Banach space E is said to be a sunny generalized nonexpansive retract of E if there exists a sunny generalized nonexpansive retraction R from E onto C. We know the following lemmas for sunny generalized nonexpansive retractions.

Lemma 2.1. (Ibaraki and Takahashi [3]). *Let C be a nonempty closed subset of a smooth and strictly convex Banach space E and let R be a retraction from E onto C. Then the following are equivalent:*

(*i*) *R is sunny generalized nonexpansive;*

(*ii*) $\langle x - Rx, Jy - JRx\rangle \leq 0, \ \forall x \in E, \ y \in C$.

Lemma 2.2. (Ibaraki and Takahashi [3]). *Let C be a nonempty closed sunny generalized nonexpansive retract of a smooth and strictly convex Banach space E. Then the sunny generalized nonexpansive retraction from E onto C is uniquely determined.*

Lemma 2.3. (Ibaraki and Takahashi [3]). *Let C be a nonempty closed subset of a smooth and strictly convex Banach space E such that there exists a sunny generalized nonexpansive retraction R from E onto C, let $x \in E$ and $z \in C$. Then the following hold:*

(i) $z = Rx$ *if and only if* $\langle x - z, Jy - Jz \rangle \leq 0$ *for all* $y \in C$;

(ii) $\phi(x, Rx) + \phi(Rx, z) \leq \phi(x, z)$.

Lemma 2.4. (Kohsaka and Takahashi [13]). *Let C be a nonempty closed subset of a smooth, strictly convex and reflexive Banach space E. Then the following are equivalent:*

(i) *C is a sunny generalized nonexpansive retract of E;*

(ii) *JC is closed and convex.*

Lemma 2.5. (Kohsaka and Takahashi [13]). *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed sunny generalized nonexpansive retract of E . Let R be the sunny generalized nonexpansive retraction from E onto C, let $x \in E$ and $z \in C$. Then the following are equivalent:*

(i) $z = Rx$;

(ii) $\phi(x, z) = \min_{y \in C} \phi(x, y)$.

Lemma 2.6. (Kamimura and Takahashi [9]). *Let E be a uniformly convex and smooth Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences in E such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $\lim_{n\to\infty} \phi(x_n, y_n) = 0$, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 2.7. (Kamimura and Takahashi [9]). *Let E be a uniformly convex and smooth Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, \infty) \to [0, \infty)$ such that $g(0) = 0$ and*

$$g(\|x - y\|) \leq \phi(x, y)$$

for all $x, y \in B_r(0)$, where $B_r(0) = \{z \in E : \|z\| \leq 1\}$.

Lemma 2.8. (Zalinescu [31]). *Let E be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, \infty) \to [0, \infty)$ such that $g(0) = 0$ and*

$$\|tx + (1 - t)y\|^2 \leq t\|x\|^2 + (1 - t)\|y\|^2 - t(1 - t)g(\|x - y\|)$$

for all $x, y \in B_r(0)$ and $t \in [0, 1]$, where $B_r(0) = \{z \in E : \|z\| \leq r\}$.

Lemma 2.9. (Ibaraki and Takahashi [4]). *Let E be a smooth and strictly convex Banach space, let $z \in E$ and let $\{t_i\}_{i=1}^m \subset (0,1)$ with $\sum_{i=1}^m t_i = 1$. If $\{x_i\}_{i=1}^m$ is a finite sequence in E such that*

$$\phi\big(\sum_{i=1}^m t_i x_i, z)\big) = \sum_{i=1}^m t_i \phi(x_i, z),$$

then $x_1 = x_2 = ... = x_m$.

3. NST-condition

Let E be a real Banach space and C be a closed subset of E. Motivated by Nakajo, Shimoji and Takahashi [19], we give the following definitions: Let $\{T_n\}$ and $\mathcal{T}$ be two families of generalized noexpansive mappings of C into E such that $\bigcap_{n=1}^\infty F(T_n) = F(\mathcal{T}) \neq \emptyset$, where $F(T_n)$ is the set of all fixed points of T_n and $F(\mathcal{T})$ is the set of all common fixed points of $\mathcal{T}$. Then, $\{T_n\}$ is said to satisfy the *NST-condition* with $\mathcal{T}$ if for each bounded sequence $\{x_n\} \subset C$,

$$\lim_{n\to\infty} \|x_n - T_n x_n\| = 0 \Rightarrow \lim_{n\to\infty} \|x_n - T x_n\| = 0, \;\; for\ all\ T \in \mathcal{T}.$$

In particular, if $\mathcal{T} = \{T\}$, i.e., $\mathcal{T}$ consists of one mapping T, then $\{T_n\}$ is said to satisfy the NST-condition with T. It is obvious that $\{T_n\}$ with $T_n = T$ for all $n \in \mathbb{N}$ satisfies NST-condition with $\mathcal{T} = \{T\}$.

Lemma 3.1. *Let C be a subset of a uniformly smooth and uniformly convex Banach space E and let T be a generalized nonexpansive mapping from C into E with $F(T) \neq \emptyset$. Let $\{\beta_n\} \subset (0,1)$ satisfy $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$. For $n \in \mathbb{N}$, define the mapping T_n from C into E by*

$$T_n x = \beta_n x + (1-\beta_n)Tx,$$

for all $x \in C$. Then, $\{T_n\}$ is a countable family of generalized nonexpansive mappings satisfying the NST-condition with T.

Proof. First, we can easily show that $F(T_n) = F(T)$ for all $n \in \mathbb{N}$. Then $\bigcap_{n=1}^\infty F(T_n) = F(T)$. For $x \in C$ and $u \in F(T_n)$, we have

$$\begin{aligned}
\phi(T_n x, u) &= \phi(\beta_n x + (1-\beta_n)Tx, u) \\
&= \|\beta_n x + (1-\beta_n)Tx\|^2 - 2\langle \beta_n x + (1-\beta_n)Tx, Ju\rangle + \|u\|^2 \\
&\leq \beta_n\|x\|^2 + (1-\beta_n)\|Tx\|^2 - 2\beta_n\langle x, Ju\rangle - 2(1-\beta_n)\langle Tx, Ju\rangle + \|u\|^2 \\
&= \beta_n\phi(x,u) + (1-\beta_n)\phi(Tx,u) \\
&\leq \beta_n\phi(x,u) + (1-\beta_n)\phi(x,u) = \phi(x,u).
\end{aligned}$$

Hence T_n is generalized nonexpansive. Next, we show that $\{T_n\}$ satisfies the NST-condition with T. To show this, suppose that $\{x_n\}$ is a bounded sequence in C such that $\lim_{n\to\infty}\|x_n - T_n x_n\| = 0$. Since $\{x_n\}$ is bounded, we obtain that $\{Tx_n\}$ is also bounded. Put $r = \max\{\sup_n \|x_n\|, \sup_n \|Tx_n\|\}$. Then there exists $r > 0$ such that $\{x_n\}, \{Tx_n\} \subset B_r(0)$, where $B_r(0) = \{z \in E : \|z\| \leq r\}$. We have from Lemma 2.8 that for $u \in \bigcap_{n=1}^{\infty} F(T_n)$,

$$\begin{aligned}
\phi(T_n x_n, u) &= \phi(\beta_n x_n + (1-\beta_n)Tx_n, u)\\
&= \|\beta_n x_n + (1-\beta_n)Tx_n\|^2 - 2\langle \beta_n x_n + (1-\beta_n)Tx_n, Ju\rangle + \|u\|^2\\
&\leq \beta_n\|x_n\|^2 + (1-\beta_n)\|Tx_n\|^2 - 2\beta_n\langle x_n, Ju\rangle - 2(1-\beta_n)\langle Tx_n, Ju\rangle\\
&\quad + \|u\|^2 - \beta_n(1-\beta_n)g(\|x_n - Tx_n\|)\\
&= \beta_n\phi(x_n, u) + (1-\beta_n)\phi(Tx_n, u) - \beta_n(1-\beta_n)g(\|x_n - Tx_n\|)\\
&\leq \beta_n\phi(x_n, u) + (1-\beta_n)\phi(x_n, u) - \beta_n(1-\beta_n)g(\|x_n - Tx_n\|)\\
&= \phi(u, x_n) - \beta_n(1-\beta_n)g(\|x_n - Tx_n\|),
\end{aligned}$$

where $g : [0,\infty) \to [0,\infty)$ is a continuous, strictly increasing and convex function with $g(0) = 0$. So, we have

$$\beta_n(1-\beta_n)g(\|x_n - Tx_n\|) \leq \phi(x_n, u) - \phi(T_n x_n, u). \tag{3.1}$$

Let $\{\|x_{n_k} - Tx_{n_k}\|\}$ be any subsequence of $\{\|x_n - Tx_n\|\}$. Since $\{x_{n_k}\}$ is bounded, there exists a subsequence $\{x_{n'_j}\}$ of $\{x_{n_k}\}$ such that

$$\lim_{j\to\infty}\phi(x_{n'_j}, u) = \limsup_{k\to\infty}\phi(x_{n_k}, u) = a.$$

Using properties (P2) and (P3) of ϕ, we have

$$\begin{aligned}
&\phi(x_{n'_j}, u)\\
&= \phi(x_{n'_j}, T_{n'_j}x_{n'_j}) + \phi(T_{n'_j}x_{n'_j}, u) + 2\langle x_{n'_j} - T_{n'_j}x_{n'_j}, JT_{n'_j}x_{n'_j} - Ju\rangle\\
&\leq \phi(T_{n'_j}x_{n'_j}, u) + \|x_{n'_j}\|\|Jx_{n'_j} - JT_{n'_j}x_{n'_j}\| + \|x_{n'_j} - T_{n'_j}x_{n'_j}\|\|T_{n'_j}x_{n'_j}\|\\
&\quad +2\|x_{n'_j} - T_{n'_j}x_{n'_j}\|\|JT_{n'_j}x_{n'_j} - Ju\|.
\end{aligned} \tag{3.2}$$

Since $\lim_{n\to\infty}\|x_n - T_n x_n\| = 0$ and E is a uniformly smooth, we have

$$\lim_{n\to\infty}\|Jx_n - JT_n x_n\| = 0.$$

It follows from (3.2) that

$$a = \liminf_{j\to\infty}\phi(x_{n'_j}, u) \leq \liminf_{j\to\infty}\phi(T_{n'_j}x_{n'_j}, u).$$

On the other hand, since $\phi(T_n x_n, u) \leq \phi(x_n, u)$, we have

$$\limsup_{j\to\infty} \phi(T_{n'j} x_{n'_j}, u) \leq \limsup_{j\to\infty} \phi(x_{n'_j}, u) = a.$$

Hence

$$\lim_{j\to\infty} \phi(x_{n'_j}, u) = \lim_{j\to\infty} \phi(T_{n'_j} x_{n'_j}, u) = a.$$

Since $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$, it follows from (3.1) that $\lim_{n\to\infty} g(\|x_{n'_j} - Tx_{n'_j}\|) = 0$. By properties of the function g, we have $\lim_{j\to\infty} \|x_{n'_j} - Tx_{n'_j}\| = 0$ and hence $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. ∎

Lemma 3.2. *Let C be a subset of a uniformly smooth and uniformly convex Banach space E and let S and T be generalized nonexpansive mappings from C into E with $F(S)\cap F(T) \neq \emptyset$. Let $\{\beta_n\} \subset (0,1)$ satisfy $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$. For $n \in \mathbb{N}$, define the mapping T_n from C into E by*

$$T_n x = \beta_n Sx + (1-\beta_n)Tx$$

for all $x \in C$. Then, $\{T_n\}$ is a countable family of generalized nonexpansive mappings satisfying the NST-condition with $\mathcal{T} = \{S, T\}$.

Proof. First, we can easily show that $\bigcap_{n=1}^{\infty} F(T_n) = F(\mathcal{T})$ and T_n are generalized nonexpansive mappings for all $n \in \mathbb{N}$. Indeed, note that

$$F(\mathcal{T}) = F(S) \cap F(T) \subset \bigcap_{n=1}^{\infty} F(T_n).$$

Let $u \in F(S) \cap F(T)$. We obtain that for any $x \in C$,

$$\begin{aligned}\phi(T_n x, u) &= \phi(\beta_n Sx + (1-\beta_n)Tx, u)\\ &= \|\beta_n Sx + (1-\beta_n)Tx\|^2 - 2\langle \beta_n Sx + (1-\beta_n)Tx, Ju\rangle + \|u\|^2\\ &\leq \beta_n\|Sx\|^2 + (1-\beta_n)\|Tx\|^2 - 2\beta_n\langle Sx, Ju\rangle - 2(1-\beta_n)\langle Tx, Ju\rangle + \|u\|^2\\ &= \beta_n\phi(Sx, u) + (1-\beta_n)\phi(Tx, u)\\ &\leq \beta_n\phi(x, u) + (1-\beta_n)\phi(x, u)\\ &= \phi(x, u).\end{aligned}$$

Then, for $v \in F(T_n)$ we have

$$\begin{aligned}\phi(v, u) &= \phi(T_n v, u)\\ &= \phi(\beta_n Sv + (1-\beta_n)Tv, u)\\ &= \|\beta_n Sv + (1-\beta_n)Tv\|^2 - 2\langle \beta_n Sv + (1-\beta_n)Tv, Ju\rangle + \|u\|^2\end{aligned}$$

$$\leq \beta_n\|Sv\|^2 + (1-\beta_n)\|Tv\|^2 - 2\beta_n\langle Sx, Ju\rangle - 2(1-\beta_n)\langle Tv, Ju\rangle + \|u\|^2$$
$$= \beta_n\phi(Sv,u) + (1-\beta_n)\phi(Tv,u)$$
$$\leq \beta_n\phi(v,u) + (1-\beta_n)\phi(v,u)$$
$$= \phi(v,u),$$

that is,

$$\phi(\beta_n Sv + (1-\beta_n)Tv, u) = \beta_n\phi(Sv,u) + (1-\beta_n)\phi(Tv,u) = \phi(v,u).$$

By Lemma 2.9, we have $Sv = Tv$. This implies that $v = T_n v = Sv = Tv$. So $F(T_n) \subset F(S) \cap F(T)$ for all $n \in \mathbb{N}$. Hence $\bigcap_{n=1}^{\infty} F(T_n) = F(\mathcal{T})$. Next, we show that $\{T_n\}$ satisfies the NST-condition with $\{S, T\}$. Let $\{x_n\}$ be a bounded sequence in C such that $\lim_{n\to\infty}\|x_n - T_n x_n\| = 0$. By Lemma 2.8, we have that for $u \in \bigcap_{n=1}^{\infty} F(T_n)$,

$$\begin{aligned}
&\phi(T_n x_n, u)\\
&= \phi(\beta_n Sx_n + (1-\beta_n)Tx_n, u)\\
&= \|\beta_n Sx_n + (1-\beta_n)Tx_n\|^2 - 2\langle \beta_n Sx_n + (1-\beta_n)Tx_n, Ju\rangle + \|u\|^2\\
&\leq \beta_n\|Sx_n\|^2 + (1-\beta_n)\|Tx_n\|^2 - 2\beta_n\langle Sx_n, u\rangle - 2(1-\beta_n)\langle Tx_n, u\rangle + \|u\|^2\\
&\quad - \beta_n(1-\beta_n)g(\|Sx_n - Tx_n\|)\\
&= \beta_n\phi(Sx_n,u) + (1-\beta_n)\phi(Tx_n,u) - \beta_n(1-\beta_n)g(\|Sx_n - Tx_n\|)\\
&\leq \beta_n\phi(x_n,u) + (1-\beta_n)\phi(x_n,u) - \beta_n(1-\beta_n)g(\|Sx_n - Tx_n\|)\\
&= \phi(x_n,u) - \beta_n(1-\beta_n)g(\|Sx_n - Tx_n\|),
\end{aligned}$$

where $g : [0,\infty) \to [0,\infty)$ is a continuous, strictly increasing and convex function with $g(0) = 0$. So, we have

$$\beta_n(1-\beta_n)g(\|Sx_n - Tx_n\|) \leq \phi(x_n,u) - \phi(T_n x_n, u). \tag{3.3}$$

Let $\{\|Sx_{n_k} - Tx_{n_k}\|\}$ be any subsequence of $\{\|Sx_n - Tx_n\|\}$. Since $\{x_{n_k}\}$ is bounded, there exists a subsequence $\{x_{n'_j}\}$ of $\{x_{n_k}\}$ such that

$$\lim_{j\to\infty}\phi(x_{n'_j},u) = \limsup_{k\to\infty}\phi(x_{n_k},u) = a.$$

Using properties (P2) and (P3) of ϕ, we have

$$\begin{aligned}
&\phi(x_{n'_j},u)\\
&= \phi(x_{n'_j}, T_{n'_j}x_{n'_j}) + \phi(T_{n'_j}x_{n'_j},u) + 2\langle x_{n'_j} - T_{n'_j}x_{n'_j}, JT_{n'_j}x_{n'_j} - Ju\rangle\\
&\leq \phi(T_{n'_j}x_{n'_j},u) + \|x_{n'_j}\|\|Jx_{n'_j} - JT_{n'_j}x_{n'_j}\| + \|T_{n'_j}x_{n'_j} - x_{n'_j}\|\|T_{n'_j}x_{n'_j}\|\\
&\quad +2\|x_{n'_j} - T_{n'_j}x_{n'_j}\|\|JT_{n'_j}x_{n'_j} - Ju\|.
\end{aligned} \tag{3.4}$$

Since $\lim_{n\to\infty} \|x_n - T_n x_n\| = 0$ and E is uniformly smooth, we have $\lim_{n\to\infty} \|Jx_n - JT_n x_n\| = 0$. It follows from (3.4) that

$$a = \liminf_{j\to\infty} \phi(x_{n'_j}, u) \le \liminf_{j\to\infty} \phi(T_{n'_j} x_{n'_j}, u).$$

On the other hand, since $\phi(T_n x_n, u) \le \phi(x_n, u)$, we have

$$\limsup_{j\to\infty} \phi(T_{n'_j} x_{n'_j}, u) \le \limsup_{j\to\infty} \phi(x_{n'_j}, u) = a.$$

It follows that

$$\lim_{j\to\infty} \phi(x_{n'_j}, u) = \lim_{j\to\infty} \phi(T_{n'_j} x_{n'_j}, u) = a.$$

Since $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$, it follows from (3.3) that

$$\lim_{n\to\infty} g(\|Sx_{n'_j} - Tx_{n'_j}\|) = 0.$$

By properties of the function g, we have $\lim_{j\to\infty} \|Sx_{n'_j} - Tx_{n'_j}\| = 0$ and hence $\lim_{n\to\infty} \|Sx_n - Tx_n\| = 0$. Since

$$\|x_n - Sx_n\| \le \|x_n - T_n x_n\| + \|T_n x_n - Sx_n\| = \|x_n - T_n x_n\| + (1-\beta_n)\|Sx_n - Tx_n\|,$$

we obtain $\lim_{n\to\infty} \|x_n - Sx_n\| = 0$. Similarly, we have $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. ∎

4. Strong Convergence Theorems

In this section, we prove a strong convergence theorem for a family of non-self generalized nonexpansive mappings in a Banach space by using the monotone hybrid method. Before proving it, we give the following lemma for non-self generalized nonexpansive mappings in a Banach space.

Lemma 4.1. *Let E be a smooth, strictly convex, and reflexive Banach space and let C be a closed subset of E such that JC is closed and convex. Let T be a generalized nonexpansive mapping from C into E. Then F(T) is closed and $JF(T)$ is closed and convex.*

Proof. First, let us show that $JF(T)$ is closed. Let $\{x_n^*\} \subset JF(T)$ such that $x_n^* \to x^*$ for some $x^* \in E^*$. Since JC is closed, we have $x^* \in JC$. Since E is smooth, strictly convex and reflexive, $J : E \to E^*$ is one-to-one and onto. Then, there exist $x \in C$ and $\{x_n\} \subset F(T)$ such that $x^* = Jx$ and $x_n^* = Jx_n$ for all $n \in \mathbb{N}$. Since T is generalized nonexpansive and $x_n \in F(T)$, we have that

$$\begin{aligned}\phi(Tx, x) &= \|Tx\|^2 - 2\langle Tx, Jx\rangle + \|Jx\|^2 \\ &= \|Tx\|^2 - 2\langle Tx, x^*\rangle + \|x^*\|^2 \\ &= \lim_{n\to\infty} (\|Tx\|^2 - 2\langle Tx, x_n^*\rangle + \|x_n^*\|^2)\end{aligned}$$

$$
\begin{aligned}
&= \lim_{n\to\infty} (\|Tx\|^2 - 2\langle Tx, Jx_n\rangle + \|Jx_n\|^2) \\
&= \lim_{n\to\infty} \phi(Tx, x_n) \\
&\leq \lim_{n\to\infty} \phi(x, x_n) \\
&= \lim_{n\to\infty} (\|x\|^2 - 2\langle x, x_n^*\rangle + \|x_n^*\|^2) \\
&= \lim_{n\to\infty} (\|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2) \\
&= \|x\|^2 - 2\langle x, Jx\rangle + \|x\|^2 \\
&= \phi(x, x) = 0.
\end{aligned}
$$

Thus we have $\phi(Tx, x) = 0$. Since E is strictly convex, we have $x = Tx$. This implies $x^* = Jx \in JF(T)$. Next, we show that $JF(T)$ is convex. Let $x^*, y^* \in JF(T)$ and let $\alpha, \beta \in (0, 1)$ with $\alpha + \beta = 1$. Then we have $x, y \in F(T)$ such that $x^* = Jx$ and $y^* = Jy$. Since $x, y \in F(T)$ and T is generalized nonexpansive, we have that

$$
\begin{aligned}
&\phi(TJ^{-1}(\alpha Jx + \beta Jy), J^{-1}(\alpha Jx + \beta Jy)) \\
&= \|TJ^{-1}(\alpha Jx + \beta Jy)\|^2 - 2\langle TJ^{-1}(\alpha Jx + \beta Jy), \alpha Jx + \beta Jy\rangle \\
&\quad + \|J^{-1}(\alpha Jx + \beta Jy)\|^2 + \alpha\|x\|^2 + \beta\|y\|^2 - (\alpha\|x\|^2 + \beta\|y\|^2) \\
&= \alpha\phi(TJ^{-1}(\alpha Jx + \beta Jy), x) + \beta\phi(TJ^{-1}(\alpha Jx + \beta Jy), y) \\
&\quad + \|\alpha Jx + \beta Jy\|^2 - (\alpha\|x\|^2 + \beta\|y\|^2) \\
&\leq \alpha\phi(J^{-1}(\alpha Jx + \beta Jy), x) + \beta\phi(J^{-1}(\alpha Jx + \beta Jy), y) \\
&\quad + \|\alpha Jx + \beta Jy\|^2 - (\alpha\|x\|^2 + \beta\|y\|^2) \\
&= \alpha\big\{\|\alpha Jx + \beta Jy\|^2 - 2\langle J^{-1}(\alpha Jx + \beta Jy), Jx\rangle + \|x\|^2\big\} \\
&\quad + \beta\big\{\|\alpha Jx + \beta Jy\|^2 - 2\langle J^{-1}(\alpha Jx + \beta Jy), Jy\rangle + \|y\|^2\big\} \\
&\quad + \|\alpha Jx + \beta Jy\|^2 - (\alpha\|x\|^2 + \beta\|y\|^2) \\
&= 2\|\alpha Jx + \beta Jy\|^2 - 2\langle J^{-1}(\alpha Jx + \beta Jy), \alpha Jx + \beta Jy\rangle \\
&= 2\|\alpha Jx + \beta Jy\|^2 - 2\|\alpha Jx + \beta Jy\|^2 = 0.
\end{aligned}
$$

Then we have $TJ^{-1}(\alpha Jx+\beta Jy) = J^{-1}(\alpha Jx+\beta Jy)$ and hence $J^{-1}(\alpha Jx+\beta Jy) \in F(T)$. This implies that $\alpha Jx + \beta Jy \in JF(T)$. Therefore $JF(T)$ is convex. Since $JF(T)$ is closed and convex, $JF(T)$ is weakly closed. Furthermore, since J is norm to weak continuous, $F(T)$ is closed. This completes the proof. ■

Using Lemma 2.4 and Lemma 4.1, we have the following lemma.

Lemma 4.2. *Let E be a smooth, strictly convex, and reflexive Banach space and let C be a closed subset of E such that JC is closed and convex. Let T be a generalized nonexpansive mapping from C into E. Then F(T) is a sunny generalized nonexpansive retract of E.*

Recall that an operator T in a Banach space is called *closed*, if $x_n \to x$ and $Tx_n \to y$, then $Tx = y$.

Theorem 4.3. *Let E be a uniformly smooth and uniformly convex Banach space and let C be a nonempty closed subset of E such that JC is closed and convex. Let $\{T_n\}$ be a countable family of generalized nonexpansive mappings from C into E and let $\mathcal{T}$ be a family of closed generalized nonexpansive mappings from C into E such that $\bigcap_{n=1}^{\infty} F(T_n) = F(\mathcal{T}) \neq \emptyset$. Suppose that $\{T_n\}$ satisfies the NST-condition with $\mathcal{T}$. Let $\{x_n\}$ be the sequence generated by*

$$\begin{cases} x_1 = x \in C, \ C_0 = Q_0 = C, \\ u_n = \alpha_n x_n + (1-\alpha_n) T_n x_n, \\ C_n = \{z \in C_{n-1} \cap Q_{n-1} : \phi(u_n, z) \leq \phi(x_n, z)\}, \\ Q_n = \{z \in C_{n-1} \cap Q_{n-1} : \langle x - x_n, Jx_n - Jz \rangle \geq 0\}, \\ x_{n+1} = R_{C_n \cap Q_n} x \end{cases}$$

for all $n \in \mathbb{N}$, where J is the duality mapping on E and $\{\alpha_n\} \subset [0,1]$ satisfies $\liminf_{n\to\infty}(1-\alpha_n) > 0$. Then, $\{x_n\}$ converges strongly to $R_{F(\mathcal{T})}x$, where $R_{F(\mathcal{T})}$ is the sunny generalized nonexpansive retraction from E onto $F(\mathcal{T})$.

Proof. We first show that JC_n and JQ_n are closed and convex for each $n \in \mathbb{N}$. From the definitions of C_n and Q_n, it is obvious that JC_n is closed and JQ_n is closed and convex for each $n \in \mathbb{N}$. Next, we prove that JC_n is convex. Since $\phi(u_n, z) \leq \phi(x_n, z)$ is equivalent to

$$0 \leq \|x_n\|^2 - \|u_n\|^2 - 2\langle x_n - u_n, Jz \rangle,$$

JC_n is convex. Since J is one-to-one, we have $J(C_n \cap Q_n) = JC_n \cap JQ_n$ is a closed and convex for all $n \in \mathbb{N}$. By Lemma 2.4, we obtain that $C_n \cap Q_n$ is a sunny generalized nonexpansive retract of E. It is obvious that $F(\mathcal{T}) \subset C = C_0 \cap Q_0$. Next, we show that $F(\mathcal{T}) \subset C_n \cap Q_n$ for all $n \in \mathbb{N}$. Suppose that $F(\mathcal{T}) \subset C_{k-1} \cap Q_{k-1}$ for some $k \in \mathbb{N}$. Let $u \in F(\mathcal{T})$. Since T_n are generalized nonexpansive mappings for all $n \in \mathbb{N}$, we have

$$\begin{aligned} \phi(u_k, u) &= \phi(\alpha_k x_k + (1-\alpha_k) T_k x_k, u) \\ &= \|\alpha_k x_k + (1-\alpha_k) T_k x_k\|^2 - 2\langle \alpha_k x_k + (1-\alpha_k) T_k x_k, Ju \rangle + \|u\|^2 \\ &\leq \alpha_k \|x_k\|^2 + (1-\alpha_k)\|T_k x_k\|^2 - 2\alpha_n \langle x_k, Ju \rangle - 2(1-\alpha_k)\langle T_k x_k, Ju \rangle + \|u\|^2 \\ &= \alpha_k \phi(x_k, u) + (1-\alpha_k)\phi(T_k x_k, u) \\ &\leq \alpha_k \phi(x_k, u) + (1-\alpha_k)\phi(x_k, u) \\ &= \phi(x_k, u). \end{aligned}$$

This implies that $F(\mathcal{T}) \subset C_k$. From $x_k = R_{C_{k-1} \cap Q_{k-1}} x$, by Lemma 2.3 (i) we have

$$\langle x - x_k, Jx_k - Jz \rangle \geq 0, \qquad \forall z \in C_{k-1} \cap Q_{k-1}.$$

Since $F(\mathcal{T}) \subset C_{k-1} \cap Q_{k-1}$, we have

$$\langle x - x_k, Jx_k - Jz \rangle \geq 0, \qquad \forall z \in F(\mathcal{T}).$$

This together with definition of Q_k implies that $F(\mathcal{T}) \subset Q_k$ and hence $F(\mathcal{T}) \subset C_k \cap Q_k$. By induction, we obtain $F(\mathcal{T}) \subset C_n \cap Q_n$ for all $n \in \mathbb{N}$. This implies that $\{x_n\}$ is well defined. From definition of Q_n, we have $x_n = R_{Q_n}x$. From $x_{n+1} = R_{C_n \cap Q_n}x \in C_n \cap Q_n \subset Q_n$, we have

$$\phi(x, x_n) \leq \phi(x, x_{n+1}), \qquad \forall n \geq 0.$$

Therefore, $\{\phi(x, x_n)\}$ is nondecreasing. It follows from Lemma 2.3 (ii) and $x_n = R_{Q_n}x$ that

$$\phi(x, x_n) = \phi(x, R_{Q_n}x) \leq \phi(x, u) - \phi(R_{Q_n}x, u) \leq \phi(x, u)$$

for all $u \in F(\mathcal{T}) \subset Q_n$. Therefore, $\{\phi(x, x_n)\}$ is bounded. Moreover, by definition of ϕ, we know that $\{x_n\}$ is bounded. So, the limit of $\{\phi(x, x_n)\}$ exists. From $x_n = R_{Q_n}x$, we have that for any positive integer k

$$\phi(x_n, x_{n+k}) = \phi(R_{Q_n}x, x_{n+k}) \leq \phi(x, x_{n+k}) - \phi(x, R_{Q_n}x) = \phi(x, x_{n+k}) - \phi(x, x_n).$$

This implies that $\lim_{n\to\infty} \phi(x_n, x_{n+k}) = 0$. Using Lemma 2.7, we have that, for $m, n \in \mathbb{N}$ with $m > n$,

$$g(\|x_n - x_m\|) \leq \phi(x_n, x_m) \leq \phi(x, x_m) - \phi(x, x_n),$$

where $g : [0, \infty) \to [0, \infty)$ is a continuous, strictly increasing and convex function with $g(0) = 0$. Then the property of the function g yields that $\{x_n\}$ is a Cauchy sequence in C. So there exists $w \in C$ such that $x_n \to w$. In view of $x_{n+1} = R_{C_n \cap Q_n}x \in C_n$ and definition of C_n, we also have

$$\phi(u_n, x_{n+1}) \leq \phi(x_n, x_{n+1}).$$

It follows that $\lim_{n\to\infty} \phi(u_n, x_{n+1}) = \lim_{n\to\infty} \phi(x_n, x_{n+1}) = 0$. Since E is uniformly convex and smooth, we have from Lemma 2.6 that

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = \lim_{n\to\infty} \|u_n - x_{n+1}\| = 0.$$

So, we have $\lim_{n\to\infty} \|x_n - u_n\| = 0$. On the other hand, we have

$$\begin{aligned}
\|x_{n+1} - u_n\| &= \|x_{n+1} - \alpha_n x_n - (1 - \alpha_n)T_n x_n\| \\
&= \|\alpha_n(x_{n+1} - x_n) + (1 - \alpha_n)(x_{n+1} - T_n x_n)\| \\
&= \|(1 - \alpha_n)(x_{n+1} - T_n x_n) - \alpha_n(x_n - x_{n+1})\| \\
&\geq (1 - \alpha_n)\|x_{n+1} - T_n x_n\| - \alpha_n\|x_n - x_{n+1}\|.
\end{aligned}$$

It follows that

$$\|x_{n+1} - T_n x_n\| \leq \frac{1}{1-\alpha_n}\big(\|x_{n+1} - u_n\| + \alpha_n \|x_n - x_{n+1}\|\big).$$

Since $\liminf_{n\to\infty}(1-\alpha_n) > 0$, we obtain that $\lim_{n\to\infty} \|x_{n+1} - T_n x_n\| = 0$. From

$$\|x_n - T_n x_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - T_n x_n\|,$$

we have

$$\lim_{n\to\infty} \|x_n - T_n x_n\| = 0.$$

Since $\{T_n\}$ satisfies the NST-condition with $\mathcal{T}$, we have that

$$\lim_{n\to\infty} \|x_n - T x_n\| = 0, \ \forall T \in \mathcal{T}.$$

Since $x_n \to w$ and T is closed, it follows that w is a fixed point of T. From Lemma 2.3 (ii), we have

$$\phi(x, R_{F(\mathcal{T})}x) \leq \phi(x, R_{F(\mathcal{T})}x) + \phi(R_{F(\mathcal{T})}x, w) \leq \phi(x, w).$$

Since $x_{n+1} = R_{C_n \cap Q_n} x$ and $w \in F(\mathcal{T}) \subset C_n \cap Q_n$, we get from Lemma 2.3 (ii) that

$$\phi(x, x_{n+1}) \leq \phi(x, x_{n+1}) + \phi(x_{n+1}, R_{F(\mathcal{T})}x) \leq \phi(x, R_{F(\mathcal{T})}x).$$

Since $x_n \to w$, it follows that $\phi(x, w) \leq \phi(x, R_{F(\mathcal{T})}x)$. Hence $\phi(x, w) = \phi(x, R_{F(\mathcal{T})}x)$. Therefore, it follows from the uniqueness of the $R_{F(\mathcal{T})}x$ that $w = R_{F(\mathcal{T})}x$. This completes the proof. ∎

5. Deduced Results

In this section, using Theorem 4.3, we obtain some new strong convergence theorems for a generalized nonexpansive mapping and two generalized nonexpansive mappings in a Banach space.

Theorem 5.1. *Let E be a uniformly convex and uniformly smooth Banach space and let C be a nonempty closed subset of E such that JC is closed and convex. Let T be a closed generalized nonexpansive mapping of C into E such that $F(T) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by*

$$\begin{cases} x_1 = x \in C, \ C_0 = Q_0 = C, \\ u_n = \alpha_n x_n + (1-\alpha_n) T x_n, \\ C_n = \{z \in C_{n-1} \cap Q_{n-1} : \phi(u_n, z) \leq \phi(x_n, z)\}, \\ Q_n = \{z \in C_{n-1} \cap Q_{n-1} : \langle x - x_n, Jx_n - Jz \rangle \geq 0\}, \\ x_{n+1} = R_{C_n \cap Q_n} x \end{cases}$$

for all $n \in \mathbb{N}$, where J is the duality mapping on E and $\{\alpha_n\} \subset [0,1]$ satisfies $\liminf_{n\to\infty}(1-\alpha_n) > 0$. Then, $\{x_n\}$ converges strongly to $R_{F(T)}x$, where $R_{F(T)}$ is the sunny generalized nonexpansive retraction from E onto $F(T)$.

Proof. Put $T_n = T$ for all $n \in \mathbb{N}$. It obvious that $\{T_n\}$ satisfies the NST-condition with T. So, we obtain the desired result from Theorem 4.3. ■

Theorem 5.2. *Let E be a uniformly convex and uniformly smooth Banach space and let C be a nonempty closed subset of E such that JC is closed and convex. Let T be a closed generalized nonexpansive mapping of C into E such that $F(T) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by*

$$\begin{cases} x_1 = x \in C, \; C_0 = Q_0 = C, \\ u_n = \alpha_n x_n + (1-\alpha_n)(\beta_n x_n + (1-\beta_n)Tx_n), \\ C_n = \{z \in C_{n-1} \cap Q_{n-1} : \phi(u_n, z) \leq \phi(x_n, z)\}, \\ Q_n = \{z \in C_{n-1} \cap Q_{n-1} : \langle x - x_n, Jx_n - Jz \rangle \geq 0\}, \\ x_{n+1} = R_{C_n \cap Q_n} x \end{cases}$$

for all $n \in \mathbb{N}$, where J is the duality mapping on E and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ satisfying $\liminf_{n\to\infty}(1-\alpha_n) > 0$ and $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$. Then, $\{x_n\}$ converges strongly to $R_{F(T)}x$, where $R_{F(T)}$ is the sunny generalized nonexpansive retraction from E onto $F(T)$.

Proof. Define $T_n x = \beta_n x + (1-\beta_n)Tx$ for all $n \in \mathbb{N}$ and $x \in C$. By Lemma 3.1, we know that $\{T_n\}$ satisfies the NST-condition with T. So, we obtain the desired result from Theorem 4.3. ■

Theorem 5.3. *Let E be a uniformly convex and uniformly smooth Banach space and let C be a nonempty closed subset of E such that JC is closed and convex. Let S and T be closed generalized nonexpansive mappings of C into E such that $F(S) \cap F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1 = x \in C, \; C_0 = Q_0 = C, \\ u_n = \alpha_n x_n + (1-\alpha_n)(\beta_n Sx_n + (1-\beta_n)Tx_n), \\ C_n = \{z \in C_{n-1} \cap Q_{n-1} : \phi(u_n, z) \leq \phi(x_n, z)\}, \\ Q_n = \{z \in C_{n-1} \cap Q_{n-1} : \langle x - x_n, Jx_n - Jz \rangle \geq 0\}, \\ x_{n+1} = R_{C_n \cap Q_n} x \end{cases}$$

for all $n \in \mathbb{N}$, where J is the duality mapping on E and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ satisfying $\liminf_{n\to\infty}(1-\alpha_n) > 0$ and $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$. Then, $\{x_n\}$ converges strongly to $R_{F(S)\cap F(T)}x$, where $R_{F(S)\cap F(T)}$ is the sunny generalized nonexpansive retraction from E onto $F(S) \cap F(T)$.

Proof. Define $T_n x = \beta_n Sx + (1-\beta_n)Tx$ for all $n \in \mathbb{N}$ and $x \in C$. By Lemma 3.2, we know that $\{T_n\}$ satisfies the NST-condition with $\mathcal{T} = \{S, T\}$. So, we obtain the desired result from Theorem 4.3. ■

Using our results, we finally get results in Hilbert spaces. In a Hilbert space, we know that $\phi(x,y) = \|x-y\|^2$ for all $x,y \in H$ and $J = I$, where I is an identity mapping and every nonexpansive mapping with a fixed point is generalized nonexpansive. As direct consequences of Lemma 3.1 and Lemma 3.2, we get the following two lemmas obtained by Takahashi, Takeuchi and Kubota [30]

Lemma 5.4. ([30, Lemma 2.1]). *Let C be a closed and convex subset of a Hilbert space H and let T be a nonexpansive mapping from C into H with $F(T) \neq \emptyset$. Let $\{\beta_n\} \subset [0,1]$ satisfy $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$. For $n \in \mathbb{N}$, define the mapping T_n of C into itself by*

$$T_n x = \beta_n x + (1-\beta_n)Tx,$$

for all $x \in C$. Then, $\{T_n\}$ is a countable family nonexpansive mappings satisfying the NST-condition with T.

Lemma 5.5. ([30, Lemma 2.3]). *Let C be a closed and convex subset of a Hilbert space H and let S and T be nonexpansive mappings from C into H with $F(S) \cap F(T) \neq \emptyset$. Let $\{\beta_n\} \subset [0,1]$ satisfying $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$. For $n \in \mathbb{N}$, define the mapping T_n of C into itself by*

$$T_n x = \beta_n Sx + (1-\beta_n)Tx$$

for all $x \in C$. Then, $\{T_n\}$ is a countable family of nonexpansive mappings satisfying the NST-condition with $\{S,T\}$.

We can also get the following new result for a countable family of nonexpansive non-self mappings in a Hilbert space by using Theorem 4.3.

Theorem 5.6. *Let H be a Hilbert space and let C be a nonempty closed and convex subset of H. Let $\{T_n\}$ and $\mathcal{T}$ be families of nonexpansive mappings of C into H such that $\bigcap_{n=1}^{\infty} F(T_n) = F(\mathcal{T}) \neq \emptyset$. Suppose that $\{T_n\}$ satisfies the NST-condition with $\mathcal{T}$. Let $\{x_n\}$ be the sequence generated by*

$$\begin{cases} x_1 = x \in C, \ C_0 = Q_0 = C, \\ u_n = \alpha_n x_n + (1-\alpha_n)T_n x_n, \\ C_n = \{z \in C_{n-1} \cap Q_{n-1} : \|z-u_n\| \leq \|z-x_n\|\}, \\ Q_n = \{z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x \end{cases}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [0,1]$ satisfies $\liminf_{n\to\infty}(1-\alpha_n) > 0$. Then, $\{x_n\}$ converges strongly to $P_{F(\mathcal{T})}x$, where $P_{F(\mathcal{T})}$ is the metric projection from C onto $F(\mathcal{T})$.

Proof. In a Hilbert space, we know that $\phi(x, y) = \|x - y\|^2$ for all $x, y \in H$ and $J = I$, where I is an identity mapping and a nonexpansive mapping $T : C \to H$ with a fixed point is also a generalized nonexpansive mapping. By using Theorem 4.3, we are able to obtain the desired conclusion. ∎

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REFERENCES

1. C. Byrne, A unified treatment of some iterative algorithms in signal processing and image reconstruction, *Inverse Problems*, **20** (2004), 103-120.
2. B. Halpern, Fixed points of nonexpanding maps, *Bull. Amer. Math. Soc.*, **73** (1967), 957-961.
3. T. Ibaraki and W. Takahashi, A new projection and convergence theorems for the projections in Banach spaces, *J. Approx. Theory*, **149** (2007), 1-14.
4. T. Ibaraki and W. Takahashi, Block iterative methods for finite family of generalized nonexpansive mappings in Banach spaces, *Numer. Funct. Anal. Optim.*, **29** (2008), 362-375.
5. T. Ibaraki and W. Takahashi, *Generalized nonexpansive mappings and a proximal-type algorithm in Banach spaces*, Contemp. Math., **513**, Amer. Math. Soc., Providence, RI, 2010, pp. 169-180.
6. W. Inthakon, S. Dhompongsa and W. Takahashi, Strong convergence theorems for maximal monotone operators and generalized nonexpansive mappings in Banach spaces, *J. Nonlinear Convex Anal.*, **11** (2010), 45-63.
7. G. Inoue, W. Takahashi and K. Zembayashi, Strong convergence theorems by hybrid methods for maximal monotone operator and relatively nonexpansive mappings in Banach spaces, *J. Convex Anal.*, **16** (2009), 791-806.
8. S. Kamimura and W. Takahashi, Approximating solutions of maximal monotone operators in Hilbert spaces, *J. Approx. Theory*, **106** (2000), 226-240.
9. S. Kamimura and W. Takahashi, Strong convergence of proximal-type algorithm in a Banach space, *SIAM J. Optim.*, **13** (2002), 938-945.
10. S. Kamimura, F. Kohsaka and W. Takahashi, Weak and strong convergence theorems for maximal monotone operators in a Banach space, *Set-valued Anal.*, **12** (2004), 417-429.
11. F. Kohsaka and W. Takahashi, Strong convergence of an iterative sequence for maximal monotone operators in a Banach space, *Abstr. Appl. Anal.*, **2004** (2004), 239-249.
12. F. Kohsaka and W. Takahashi, *Block iterative methods for a finite family of relatively nonexpansive mappings in Banach spaces*, Fixed Point Theory Appl., Vol. 2007, Article ID 21972, 18 pages, 2007.

13. F. Kohsaka and W. Takahashi, Generalized nonexpansive retractions and a proximal-type algorithm in Banach spaces, *J. Nonlinear Convex Anal.*, **8** (2007), 197-209.

14. F. Kohsaka and W. Takahashi, Existence and approximation of fixed points of firmly nonexpansive type mappings in Banach space, *SIAM J. Optim.*, **19** (2008), 824-835.

15. W. R. Mann, Mean Vauled methods in iteration, *Proc. Amer. Math. Soc.*, **4** (1953), 506-510.

16. C. Martinez-Yanes and H. K. Xu, Strong convergence of the CQ method for fixed point iteration processes, *Nonlinear Anal.*, **64** (2006), 2400-2411.

17. S. Matsushita and W. Takahashi, Weak and strong convergence theorems for relatively nonexpansive mappings in Banach space, *Fixed Point Theory Appl.*, **2004** (2004), 37-47.

18. S. Matsushita and W. Takahashi, A strong convergence theorem for relatively nonexpansive mappings in a Banach space, *J. Approx. Theory*, **134** (2005), 257-266.

19. K. Nakajo, K. Shimoji and W. Takahashi, Strong convergence theorems to common fixed points of families of nonexpansive mappings in Banach spaces, *J. Nonlinear Convex Anal.*, **8** (2007), 11-34.

20. K. Nakajo and W. Takahashi, Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups, *J. Math. Anal. Appl.*, **279** (2003), 372-379.

21. X. Qin and Y. Su, Strong convergence of monotone hybrid method for fixed point iteration processes, *J. Syst. Sci. and Complexity*, **21** (2008), 474-482.

22. X. Qin and Y. Su, Strong convergence theorems for relatively nonexpansive mappings in a Banach space, *Nonlinear Anal.*, **67** (2007), 1958-1965.

23. R. T. Rockafellar, On the maximality of sums of nonlinear monotone operators, *Trans. Amer. Math. Soc.*, **194** (1970), 75-88.

24. R. T. Rockafellar, Monotone operators and the proximal point algorithm, *SIAM J. Control Oprim.*, **14** (1976), 877-898.

25. M. I. Sezan and H. Stark, Applications of convex projection theory to image recovery in tomography and related areas, in: *Image Recovery Theory and Applications*, Academic Press, Orlando, 1987, pp. 415-562.

26. M. V. Solodov and B. F. Svaiter, Forcing strong convergence of proximal point iterations in a Hilbert space, *Math. Program.*, **87** (2000), 189-202.

27. Y. Su, D. Wang and M. Shang, *Strong convergence of monotone hybrid algorithm for hemi-relatively nonexpansive mappings*, Fixed Point Theory Appl., Vol. 2008, Article ID 284613, 8 pages, 2008.

28. W. Takahashi, *Nonlinear Functional Analysis - Fixed Point Theory and its Applications*, Yokohama Publishers Inc, Yokohama, 2000.

29. W. Takahashi, *Convex Analysis and Application of Fixed Points*, Yokohama Publishers inc, Yokohama, 2000, (Japanese).

30. W. Takahashi, Y. Takeuchi and R. Kubota, Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces, *J. Math. Anal. Appl.*, **341** (2008), 276-286.

31. C. Zalinescu, On uniformly convex functions, *J. Math. Anal. Appl.*, **95** (1983), 344-374.

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