TAIWANESE JOURNAL OF MATHEMATICS
Vol. 16, No. 3, pp. 1017-1026, June 2012
This paper is available online at http://journal.taiwanmathsoc.org.tw

DIFFERENTIAL SUBORDINATION FOR FUNCTIONS ASSOCIATED WITH THE LEMNISCATE OF BERNOULLI

Rosihan M. Ali, Nak Eun Cho*, V. Ravichandran and S. Sivaprasad Kumar

Dedicated to Professor H. M. Srivastava on the Occasion of his Seventieth Birth Anniversary

Abstract. Conditions on β are determined so that $1+\beta zp'(z)$ subordinated to $\sqrt{1+z}$ implies p is subordinated to $\sqrt{1+z}$. Analogous results are also obtained involving the expressions $1+\beta zp'(z)/p(z)$ and $1+\beta zp'(z)/p^2(z)$. These results are applied to obtain sufficient conditions for normalized analytic functions f to satisfy the condition $|\,(zf'(z)/f(z))^2-1|<1$.

1. Introduction

Let $\mathcal{A}$ denote the class of analytic functions in the unit disk $\mathbb{D}:=\{z\in\mathbb{C}:\ |z|<1\}$ normalized by the conditions $f(0)=0$ and $f'(0)=1$. Let $\mathcal{SL}$ be the class of functions defined by

$$\mathcal{SL}:=\left\{f\in\mathcal{A}:\left|\left(\frac{zf'(z)}{f(z)}\right)^2-1\right|<1\right\}\quad(z\in\mathbb{D}).$$

Thus a function $f\in\mathcal{SL}$ if $zf'(z)/f(z)$ lies in the region bounded by the right-half of the lemniscate of Bernoulli given by $|w^2-1|<1$. Since this region is contained in the right-half plane, functions in $\mathcal{SL}$ are starlike functions, and in particular univalent. A starlike function is characterized by the condition $Re zf'(z)/f(z)>0$ in $\mathbb{D}$. For two functions f and g analytic in $\mathbb{D}$, the function f is said to be *subordinate* to g, written $f(z)\prec g(z)\quad(z\in\mathbb{D})$, if there exists a function w analytic in $\mathbb{D}$ with $w(0)=0$ and $|w(z)|<1$ such that $f(z)=g(w(z))$. In particular, if the function g is univalent in $\mathbb{D}$, then $f(z)\prec g(z)$ is equivalent to $f(0)=g(0)$ and $f(\mathbb{D})\subset g(\mathbb{D})$. In terms of subordination, the class $\mathcal{SL}$ consists of normalized analytic functions f satisfying $zf'(z)/f(z)\prec\sqrt{1+z}$. This class $\mathcal{SL}$ was introduced by Sokó l and Stankiewicz

Received May 10, 2011, accepted May 20, 2011.
Communicated by H. M. Srivastava.
2010 *Mathematics Subject Classification*: Primary 30C45; Secondary 30C80.
Key words and phrases: Starlike functions, Lemniscate of Bernoulli, Differential subordination, Differential superordination, Best subordinant, Best dominant.
*Corresponding author.

[23]. Paprocki and Sokó l[14] discussed a more general class $\mathcal{S}^*(a,b)$ consisting of normalized analytic functions f satisfying $|[zf'(z)/f(z)]^a - b| < b$, $b \geq \frac{1}{2}$, $a \geq 1$. Sokó l and Stankiewicz [23] determined the radius of convexity for functions in the class $\mathcal{SL}$. They also obtained structural formula, as well as growth and distortion theorems for these functions. Estimates for the first few coefficients of functions in $\mathcal{SL}$ were obtained in [24]. Recently, Sokó l [25] determined various radii for functions belonging to the class $\mathcal{SL}$; these include the radii of convexity, starlikeness and strong starlikeness of order α. Recently the $\mathcal{SL}$-radii for certain well-known classes of functions including the Janowski starlike functions were obtained in [1]. General radii problems were also recently considered in [2] wherein certain radii results for the class $\mathcal{SL}$ were obtained as special cases.

The class of *Janowski starlike functions* [7], denoted by $S^*[A,B]$, consists of functions $f \in \mathcal{A}$ satisfying the subordination

$$\frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz}, \quad (-1 \leq B < A \leq 1).$$

Silverman [20], Obradovic and Tuneski [11] and several others (see [9, 10, 12, 16, 18]) have studied properties of functions defined in terms of the quotient $(1 + zf''(z)/f'(z))/(zf'(z)/f(z))$. In fact, Silverman [20] derived the order of starlikeness for functions in the class G_b defined by

$$G_b := \left\{ f \in \mathcal{A} : \left| \frac{1+zf''(z)/f'(z)}{zf'(z)/f(z)} - 1 \right| < b,\ 0 < b \leq 1,\ z \in \mathbb{D} \right\}.$$

Obradovic and Tuneski [11] have improved the result of Silverman [20] by showing $G_b \subset S^*[0,-b] \subset S^*(2/(1+\sqrt{1+8b}))$. Later Tuneski [26] obtained conditions for the inclusion $G_b \subset S^*[A,B]$ to hold. Letting $zf'(z)/f(z) =: p(z)$, then $G_b \subset S^*[A,B]$ becomes a special case of the differential chain

$$1 + \beta\frac{zp'(z)}{p(z)^2} \prec \frac{1+Dz}{1+Ez} \Rightarrow p(z) \prec \frac{1+Az}{1+Bz}. \tag{1.1}$$

Similarly, for $f \in \mathcal{A}$ and $0 \leq \alpha < 1$, Frasin and Darus [5] showed that

$$\frac{(zf(z))''}{f'(z)} - \frac{2zf'(z)}{f(z)} \prec \frac{(1-\alpha)z}{2-\alpha} \Rightarrow \left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| < 1 - \alpha.$$

Again by writing $\frac{z^2f'(z)}{(f(z))^2}$ as $p(z)$, the above implication is a particular case of

$$1 + \beta\frac{zp'(z)}{p(z)} \prec \frac{1+Dz}{1+Ez} \Rightarrow p(z) \prec \frac{1+Az}{1+Bz}. \tag{1.2}$$

Li and Owa [13] showed that $f(z) \in S^*$ if $f(z) \in \mathcal{A}$ satisfies

$$\text{Re}\left\{ \frac{zf'(z)}{f(z)} \left(\alpha\frac{zf''(z)}{f'(z)} + 1 \right) \right\} > -\frac{\alpha}{2}, \quad z \in \mathbb{D}$$

for some α ($\alpha \geq 0$). Related results may also be found in the works of [15, 17, 21, 22].

The implications (1.1) and (1.2) have been considered in [3]. All the results discussed above led us to consider differential implications with the superordinate function $(1 + Az)/(1 + Bz)$ replaced by the superordinate function $\sqrt{1+z}$ that maps $\mathbb{D}$ onto the right-half of the lemniscate of Bernoulli. Additionally, applications of our results will yield sufficient conditions for functions $f \in \mathcal{A}$ to belong to the class $\mathcal{SL}$.

The following results will be required.

Lemma 1.1. [8, Corollary 3.4h.1, p. 135]. *Let q be univalent in $\mathbb{D}$, and let φ be analytic in a domain containing $q(\mathbb{D})$. Let $zq'(z)\varphi(q(z))$ be starlike. If p is analytic in $\mathbb{D}$, $p(0) = q(0)$ and satisfies*

$$zp'(z)\varphi(p(z)) \prec zq'(z)\varphi(q(z)),$$

then $p(z) \prec q(z)$, and q is the best dominant.

A more general version of the above lemma is the following:

Lemma 1.2. [8, Theorem 3.4h, p. 132]. *Let q be univalent in the unit disk $\mathbb{D}$ and ϑ and φ be analytic in a domain D containing $q(\mathbb{D})$ with $\varphi(w) \neq 0$ when $w \in q(\mathbb{D})$. Set $Q(z) = zq'(z)\varphi(q(z))$, $h(z) = \vartheta(q(z)) + Q(z)$. Suppose that*

(1) *either h is convex, or Q is starlike univalent in $\mathbb{D}$, and*

(2) $\operatorname{Re} \frac{zh'(z)}{Q(z)} > 0$ *for $z \in \mathbb{D}$.*

If p is analytic in $\mathbb{D}$, $p(0) = q(0)$ and satisfies

$$\vartheta(p(z)) + zp'(z)\varphi(p(z)) \prec \vartheta(q(z)) + zq'(z)\varphi(q(z)),$$

then $p(z) \prec q(z)$, and q is the best dominant.

2. Main Results

We first determine a lower bound for β so that $1 + \beta zp'(z) \prec \sqrt{1+z}$ implies $p(z) \prec \sqrt{1+z}$.

Lemma 2.1. *Let p be an analytic function on $\mathbb{D}$ and $p(0) = 1$. Let $\beta_0 = 2\sqrt{2}$ $(\sqrt{2} - 1) \approx 1.17$. If the function p satisfies the subordination*

$$1 + \beta zp'(z) \prec \sqrt{1+z} \quad (\beta \geq \beta_0),$$

then p also satisfies the subordination

$$p(z) \prec \sqrt{1+z}.$$

The lower bound β_0 is best possible.

Proof. Define the function $q : \mathbb{D} \to \mathbb{C}$ by $q(z) = \sqrt{1+z}$ with $q(0) = 1$. Since $q(\mathbb{D}) = \{w : |w^2 - 1| < 1\}$ is the right-half of the lemniscate of Bernoulli, $q(\mathbb{D})$ is a convex set and hence q is a convex function. This shows that the function $zq'(z)$ is starlike with respect to 0. By Lemma 1.1, it follows that the subordination

$$1 + \beta z p'(z) \prec 1 + \beta z q'(z)$$

implies $p(z) \prec q(z)$. In light of this differential chain, the result is proved if it could be shown that

$$q(z) = \sqrt{1+z} \prec 1 + \beta z q'(z) = 1 + \frac{\beta z}{2\sqrt{1+z}} =: h(z).$$

Since $q^{-1}(w) = w^2 - 1$, it follows that

$$q^{-1}(h(z)) = \left(2 + \frac{\beta z}{2\sqrt{1+z}}\right) \frac{\beta z}{2\sqrt{1+z}}.$$

For $z = e^{it}$, $t \in [-\pi, \pi]$, clearly

$$|q^{-1}(h(z))| = |q^{-1}(h(e^{it}))| = \frac{\beta}{2\sqrt{2\cos\frac{t}{2}}} \left| 2 + \frac{\beta e^{i\frac{3t}{4}}}{2\sqrt{2\cos\frac{t}{2}}} \right|.$$

A calculation shows that the minimum of the above expression is attained at $t = 0$. Hence

$$|q^{-1}(h(e^{it}))| \geq \frac{\beta}{2\sqrt{2}} \left(2 + \frac{\beta}{2\sqrt{2}}\right) = \left(1 + \frac{\beta}{2\sqrt{2}}\right)^2 - 1 \geq 1$$

provided $\beta \geq 2\sqrt{2}(\sqrt{2} - 1)$. Hence $q^{-1}(h(\mathbb{D})) \supset \mathbb{D}$ or $h(\mathbb{D}) \supset q(\mathbb{D})$. This shows that $q(z) \prec h(z)$, and completes the proof. ∎

Theorem 2.2. *Let* $\beta_0 = 2\sqrt{2}(\sqrt{2} - 1) \approx 1.17$ *and* $f \in \mathcal{A}$.

(1) *If* f *satisfies the subordination*

$$1 + \beta \frac{zf'(z)}{f(z)} \left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right) \prec \sqrt{1+z} \quad (\beta \geq \beta_0),$$

then $f \in \mathcal{SL}$.

(2) *If* $1 + \beta z f''(z) \prec \sqrt{1+z}$ $\quad (\beta \geq \beta_0)$, *then* $f'(z) \prec \sqrt{1+z}$.

Proof. Define the function $p : \mathbb{D} \to \mathbb{C}$ by

$$p(z) = \frac{zf'(z)}{f(z)}.$$

Then p is analytic in $\mathbb{D}$ and $p(0) = 1$. A calculation shows that

$$zp'(z) = \frac{zf'(z)}{f(z)} \left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right).$$

Applying Lemma 2.1 to this function p yields the first part of the theorem. The second part follows by taking $p(z) = f'(z)$ in Lemma 2.1. ∎

Lemma 2.3. *Let* $\beta_0 = 4(\sqrt{2} - 1) \approx 1.65$. *If*

$$1 + \frac{\beta z p'(z)}{p(z)} \prec \sqrt{1+z} \quad (\beta \geq \beta_0),$$

then

$$p(z) \prec \sqrt{1+z}.$$

The lower bound β_0 *is best possible.*

Proof. Let q be the convex function given by $q(z) = \sqrt{1+z}$, and consider the subordination

$$1 + \frac{\beta z p'(z)}{p(z)} \prec 1 + \frac{\beta z q'(z)}{q(z)}.$$

A calculation shows that

$$\frac{\beta z q'(z)}{q(z)} = \frac{\beta z}{2(1+z)}$$

is convex in $\mathbb{D}$ (and hence starlike). Thus, in view of Lemma 1.1, it follows that $p(z) \prec q(z)$. To complete the proof, it is left to show that

$$q(z) = \sqrt{1+z} \prec 1 + \frac{\beta z q'(z)}{q(z)} = 1 + \frac{\beta z}{2(1+z)} =: h(z).$$

Since $h(\mathbb{D}) = \{w : Re w < 1 + \beta/4\}$, and $q(\mathbb{D}) = \{w : |w^2 - 1| < 1\} \subset \{w : Re w < \sqrt{2}\}$, it follows that $q(\mathbb{D}) \subset h(\mathbb{D})$ if $\sqrt{2} \leq 1 + \beta/4$. Thus $q(z) \prec h(z)$ for $\beta \geq 4(\sqrt{2} - 1)$, and this completes the proof. ∎

Theorem 2.4. *Let* $\beta_0 = 4(\sqrt{2} - 1) \approx 1.65$ *and* $f \in \mathcal{A}$.

(1) *If* f *satisfies*

$$1 + \beta\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right) \prec \sqrt{1+z} \quad (\beta \geq \beta_0),$$

then $f \in \mathcal{SL}$.

(2) *If* f *satisfies*

$$1 + \beta\left(\frac{(zf(z))''}{f'(z)} - \frac{2zf'(z)}{f(z)}\right) \prec \sqrt{1+z} \quad (\beta \geq \beta_0),$$

then

$$\frac{z^2 f'(z)}{f^2(z)} \prec \sqrt{1+z}.$$

Proof. The results follows from Lemma 2.3 by taking $p(z) = \frac{zf'(z)}{f(z)}$ and $p(z) = \frac{z^2 f'(z)}{f^2(z)}$ respectively. ∎

Lemma 2.5. *Let* $\beta_0 = 4\sqrt{2}(\sqrt{2}-1) \approx 2.34$. *If*

$$1 + \frac{\beta z p'(z)}{p^2(z)} \prec \sqrt{1+z} \quad (\beta \geq \beta_0),$$

then

$$p(z) \prec \sqrt{1+z}.$$

The lower bound β_0 *is best possible.*

Proof. With q being the convex function $q(z) = \sqrt{1+z}$, consider the function Q defined by

$$Q(z) := \frac{zq'(z)}{q^2(z)} = \frac{z}{2(1+z)^{\frac{3}{2}}}.$$

Since

$$Re\frac{1+(1-2\alpha)z}{1-z} > \alpha \quad (0 \leq \alpha < 1),$$

it follows that

$$Re\frac{zQ'(z)}{Q(z)} = Re\frac{2-z}{2(1+z)} > \frac{1}{4} > 0.$$

Thus the function Q is starlike and Lemma 1.1 shows that the subordination

$$1 + \frac{\beta z p'(z)}{p^2(z)} \prec 1 + \frac{\beta z q'(z)}{q^2(z)}$$

implies $p(z) \prec q(z)$. We next show that

$$q(z) = \sqrt{1+z} \prec 1 + \frac{\beta z q'(z)}{q^2(z)} = 1 + \frac{\beta z}{2(1+z)^{\frac{3}{2}}} =: h(z).$$

Since $q^{-1}(w) = w^2 - 1$, then

$$q^{-1}(h(z)) = \left(2 + \frac{\beta z}{2(1+z)^{\frac{3}{2}}}\right)\frac{\beta z}{2(1+z)^{\frac{3}{2}}}.$$

Thus with $z = e^{it}$, $t \in [-\pi, \pi]$, yields

$$|q^{-1}(h(z))| = |q^{-1}(h(e^{it}))| = \frac{\beta}{2(2\cos\frac{t}{2})^{\frac{3}{2}}}\left|2 + \frac{\beta e^{i\frac{t}{4}}}{2(2\cos\frac{t}{2})^{\frac{3}{2}}}\right|.$$

A computation shows that the minimum of the above expression is attained at $t = 0$. Hence

$$|q^{-1}(h(e^{it}))| \geq \frac{\beta}{4\sqrt{2}}\left(2 + \frac{\beta}{4\sqrt{2}}\right) = \left(1 + \frac{\beta}{4\sqrt{2}}\right)^2 - 1 \geq 1$$

for $\beta \geq 4\sqrt{2}(\sqrt{2}-1)$. Hence $q(z) \prec h(z)$. ∎

By taking $p(z) = \frac{zf'(z)}{f(z)}$ in Lemma 2.5, we obtain the following theorem.

Theorem 2.6. *Let* $\beta_0 = 4\sqrt{2}(\sqrt{2}-1) \approx 2.34$ *and* $f \in \mathcal{A}$*. Then* $f \in \mathcal{SL}$ *if*

$$1-\beta+\beta\frac{1+\frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}} \prec \sqrt{1+z} \quad (\beta \geq \beta_0).$$

Lemma 2.7. *Let* $0 < \alpha \leq 1$*. If* $p \in \mathcal{A}$ *satisfies*

$$(1-\alpha)p(z)+\alpha p^2(z)+\alpha z p'(z) \prec \sqrt{1+z},$$

then $p(z) \prec \sqrt{1+z}$*.*

Proof. Define the function q by $q(z) = \sqrt{1+z}$. We first show that $p(z) \prec q(z)$ if p satisfies

$$(1-\alpha)p(z)+\alpha p^2(z)+\alpha z p'(z) \prec (1-\alpha)q(z)+\alpha q^2(z)+\alpha z q'(z).$$

For this purpose, let the functions ϑ and φ be defined by $\vartheta(w) := (1-\alpha)w+\alpha w^2$ and $\varphi(w) := \alpha$. Clearly the functions ϑ and φ are analytic in $\mathbb{C}$ and $\varphi(w) \neq 0$. Also let Q and h be the functions defined by

$$Q(z) := zq'(z)\varphi(q(z)) = \alpha z q'(z)$$

and

$$h(z) := \vartheta(q(z)) + Q(z) = (1-\alpha)q(z)+\alpha q^2(z)+\alpha z q'(z).$$

Since q is convex, the function $zq'(z)$ is starlike, and therefore Q is starlike univalent in $\mathbb{D}$. In view of the fact that $Re q(z) > 0$, it follows that

$$Re\frac{zh'(z)}{Q(z)} = \frac{1}{\alpha}Re\left[(1-\alpha)+2\alpha q(z)+\alpha\left(1+\frac{zq''(z)}{q'(z)}\right)\right] > 0 \quad (z \in \mathbb{D})$$

for $0 < \alpha \leq 1$. By Lemma 1.2, it follows that $p \prec q = \sqrt{1+z}$. To complete the proof, we seek conditions on α so that $q(z) \prec h(z)$, or equivalently $|[h(e^{it})]^2-1| \geq 1$ for all $t \in [-\pi, \pi]$. Now

$$h(z) = \frac{\alpha z + 2(1-\alpha)(1+z)+2\alpha(1+z)^{3/2}}{2\sqrt{1+z}},$$

and a calculation shows that $|[h(e^{it})]^2-1|$ attains its minimum at $t = 0$. Thus $|[h(e^{it})]^2-1| \geq |(h(1))^2-1| > 1$ if $h(1) = \frac{8-3\sqrt{2}}{4}\alpha+\sqrt{2} > \sqrt{2}$ and this holds for $\alpha > 0$. Hence we conclude that $(1-\alpha)p(z)+\alpha p^2(z)+\alpha z p'(z) \prec \sqrt{1+z}$ implies $p(z) \prec \sqrt{1+z}$. ■

Theorem 2.8. *If* $f \in \mathcal{A}$ *satisfies*

$$\frac{zf'(z)}{f(z)}\left(1+\alpha\frac{zf''(z)}{f'(z)}\right) \prec \sqrt{1+z} \quad (0 < \alpha \leq 1),$$

then $\frac{zf'(z)}{f(z)} \prec \sqrt{1+z}$*, or equivalently* $f \in \mathcal{SL}$*.*

Proof. With $p(z) = \frac{zf'(z)}{f(z)}$, a computation shows that

$$p(z) + \frac{zp'(z)}{p(z)} = 1 + \frac{zf''(z)}{f'(z)}.$$

Evidently

$$\frac{zf'(z)}{f(z)}\left(1+\alpha\frac{zf''(z)}{f'(z)}\right) = \frac{zf'(z)}{f(z)}+\alpha\frac{z^2f''(z)}{f(z)} = (1-\alpha)p(z) + \alpha p^2(z) + \alpha zp'(z).$$

The result now follows from Lemma 2.7. ∎

Acknowledgments

The work presented here was supported in part by grants from Universiti Sains Malaysia, National Research Foundation of Korea (No. 2011-0007037) and University of Delhi.

References

1. R. M. Ali, N. K. Jain and V. Ravichandran, *Radii of starlikeness associated with the lemniscate of Bernoulli and the left-half plane*, preprint.
2. R. M. Ali, N. E. Cho, N. K. Jain and V. Ravichandran, *Radii of starlikeness and convexity of functions defined by subordination with fixed second coefficient*, preprint.
3. R. M. Ali, V. Ravichandran and N. Seenivasagan, Sufficient conditions for Janowski starlikeness, *Int. J. Math. Math. Sci.*, **2007**, Art. ID 62925, 7 pp.
4. R. M. Ali, V. Ravichandran and N. Seenivasagan, On Bernardi's integral operator and the Briot-Bouquet differential subordination, *J. Math. Anal. Appl.*, **324** (2006), 663-668.
5. B. A. Frasin and M. Darus, On certain analytic univalent functions, *Int. J. Math. Math. Sci.*, **25(5)** (2001), 305-310.
6. A. W. Goodman, *Univalent Functions*, Vols. 1 & 2, Polygonal Publ. House, Washington, New Jersey, 1983.
7. W. Janowski, Some extremal problems for certain families of analytic functions I, *Ann. Polon. Math.*, **28** (1973), 297-326.
8. S. S. Miller and P. T. Mocanu, *Differential Subordination, Theory and Application*, Marcel Dekker, Inc., New York, Basel, 2000.
9. M. Nunokawa, S. Owa, H. Saitoh, A. Ikeda and N. Koike, Some results for strongly starlike functions, *J. Math. Anal. Appl.*, **212(1)** (1997), 98-106.
10. M. Nunokawa, S. Owa, H. Saitoh and N. Takahashi, On a strongly starlikeness criteria, *Bull. Inst. Math. Acad. Sinica*, **31(3)** (2003), 195-199.
11. M. Obradovic and N. Tuneski, On the starlike criteria defined by Silverman, *Zeszyty Nauk. Politech. Rzeszowskiej Mat.*, **24** (2000), 59-64.

12. M. Obradović and S. Owa, On some criterions for starlikeness of order α, *Rend. Mat. Appl.* (7), **8(2)** (1988), 283-289.

13. J.-L. Li and S. Owa, Sufficient Conditions for Starlikeness, *Indian J. Pure Appl. Math.*, **33(3)** (2002), 313-318.

14. E. Paprocki and J. Sokół, The extremal problems in some subclass of strongly starlike functions, *Zeszyty Nauk. Politech. Rzeszowskiej Mat.*, **20** (1996), 89-94.

15. V. Ravichandran, C. Selvaraj and R. Rajalaksmi, Sufficient conditions for starlike functions of order α, *JIPAM. J. Inequal. Pure Appl. Math.*, **3(5)** (2002), Article 81, 6 pp. (electronic).

16. V. Ravichandran and M. Darus, On a criteria for starlikeness, *Int. Math. J.*, **4(2)** (2003), 119-125.

17. V. Ravichandran, Certain applications of first order differential subordination, *Far East J. Math. Sci.* $($*FJMS*$)$, **12(1)** (2004), 41-51.

18. V. Ravichandran, M. Darus and N. Seenivasagan, On a criteria for strong starlikeness, *Aust. J. Math. Anal. Appl.*, **2(1)** (2005), Art. 6, 12 pp.

19. T. N. Shanmugam and V. Ravichandran, Certain properties of uniformly convex functions, in: *Computational methods and function theory* $($*Penang*$)$, 319-324, World Sci. Publ., River Edge, NJ. 1994.

20. H. Silverman, Convex and starlike criteria, *Int. J. Math. Math. Sci.*, **22(1)** (1999), 75-79.

21. S. Singh and S. Gupta, First order differential subordinations and starlikeness of analytic maps in the unit disc, *Kyungpook Math. J.*, **45(3)** (2005), 395-404.

22. S. Singh and S. Gupta, A differential subordination and starlikeness of analytic functions, *Appl. Math. Lett.*, **19(7)** (2006), 618-627.

23. J. Sokół and J. Stankiewicz, Radius of convexity of some subclasses of strongly starlike functions, *Zeszyty Nauk. Politech. Rzeszowskiej Mat.*, **19** (1996), 101-105.

24. J. Sokól, Coefficient estimates in a class of strongly starlike functions, *Kyungpook Math. J.*, **49(2)** (2009), 349-353.

25. J. Sokól, Radius problems in the class $\mathcal{SL}$, *Appl. Math. Comput.*, **214(2)** (2009), 569-573.

26. N. Tuneski, On the quotient of the representations of convexity and starlikeness, *Math. Nachr.*, **248/249** (2003), 200-203.

Rosihan M. Ali
School of Mathematical Sciences
Universiti Sains Malaysia
11800 USM, Penang
Malaysia
E-mail: rosihan@cs.usm.my

Nak Eun Cho
Department of Applied Mathematics
Pukyong National University
Busan 608-737
South Korea
E-mail: necho@pknu.ac.kr

V. Ravichandran
Department of Mathematics
University of Delhi
Delhi-110 007
India
E-mail: vravi68@gmail.com
vravi@maths.du.ac.in

S. Sivaprasad Kumar
Department of Applied Mathematics
Delhi Technological University
Bawana Road, Delhi-110042
India
E-mail: spkumar@dce.ac.in