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FLAT ϕ CURVATURE FLOW OF CONVEX SETS

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Abstract. For an arbitrary initial compact and convex subset K_0 of $\mathbb{R}^n$, and for an arbitrary norm ϕ on $\mathbb{R}^n$, we construct a flat ϕ curvature flow $K(t)$ such that $K(t)$ is compact and convex throughout the evolution. Previously and using similar methods, R. McCann had shown that flat ϕ curvature flow in the plane preserves convex, balanced sets. More recently, G. Bellettini, V. Caselles, A. Chambolle, and M. Novaga showed that flat ϕ curvature flow in $\mathbb{R}^n$ preserves compact, convex sets. We also establish a new Hölder continuity estimate for the flow. Flat ϕ curvature flows, introduced by F. Almgren, J. Taylor, and L. Wang, model motion by ϕ-weighted mean curvature. Under certain regularity assumptions, they coincide with smooth ϕ-weighted mean curvature flows given by partial differential equations as long as the smooth flows exist.

1. Introduction

Motion by weighted mean curvature is a model for the time evolution of solids in which the normal velocity at a boundary point is given by the ϕ-weighted mean curvature at that point (cf. [26]). This dynamical process generalizes ordinary motion by mean curvature by accounting for anisotropic surface tension, as when a crystal melts or relaxes. Here, ϕ is a surface energy density function, whose values model the preferred directions of motion for a non-equilibrium crystal evolving due to its surface tension. When the unit ball for ϕ is a sphere, so that no directions are preferred, ϕ is isotropic. When $\phi = \phi_E$, the Euclidean norm, ϕ-weighted mean curvature is ordinary mean curvature. More generally, we can consider evolutions where the surface normal velocity is given by

$$v = M\left(wmc_\phi + \Omega\right), \tag{1}$$

where wmc_ϕ is the ϕ-weighted mean curvature, Ω is a bulk quantity such as under-cooling below the freezing temperature of a planar interface, and M (the "mobility")

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measures the direction-dependent response of the interface to surface tension and bulk driving forces. When $\Omega = 0$, the resulting motion is motion by ϕ-weighted mean curvature with mobility M, or simply motion by ϕ-weighted mean curvature if M is identically one on unit vectors. See [25, 26, 17], and [18] for detailed discussions of the physical motivation.

Curvature-driven flow has been the subject of extensive research by many authors in the last three decades. In this paper, we are interested in the ϕ-weighted mean curvature flow $K(t)$ of a compact, convex subset $K(0) = K_0 \subset \mathbb{R}^n$. In particular, we will take $\Omega = 0$ throughout this paper.

Most of the work has been in the case where $M = \phi_E$, so that mobility is identically one for unit vectors $w \in \mathbb{R}^n$ (see (4) and Remark 3). For the special case when $\phi = \phi_E$ and $n = 2$, M. E. Gage and R. S. Hamilton [14, 15] showed that the evolution remains convex. Also in an isotropic setting, G. Huisken [19] and later L. C. Evans and J. Spruck [12] showed the same result for all $n \geq 2$, using somewhat different methods. Whereas M. E. Gage, R. S. Hamilton, and G. Huisken used classical models, L. C. Evans and J. Spruck used a viscosity approach, based on previous work [24] of S. Osher and J. A. Sethian and conducted independent of related work [10] by Y. G. Chen, Y. Giga, and S. Goto, to construct weak solutions which can exist in the presence of singularities and topological changes. The flows they constructed agree with classical flows so long as the latter exist.

In [2], F. Almgren, J. Taylor, and L. Wang introduced *flat ϕ curvature flow*, a variational time-stepping scheme for ϕ-weighted mean curvature flow, set in the context of the integral and rectifiable currents of geometric measure theory (cf. [13]). Their main result ([2], Theorem 4.5) is an existence and Hölder continuity theorem for these flows $K(t)$, for general norms ϕ (in fact, they do not require that ϕ be an even function) and $n \geq 3$. They also showed that, under additional restrictions on K_0 and ϕ, flat ϕ curvature flows agree with smooth ϕ-weighted mean curvature flows given by partial differential equations, so long as the latter flows remain smooth ([2] § 7).

Their strategy, which we emulate here, was to construct discrete flows, which approximate motion by ϕ-weighted mean curvature, and extract a limit "flat" flow (named after H. Whitney's flat norm, $|\ |_b$) as the time step tends to zero. The discrete flows involved successive minimization steps, in which a surface plus bulk energy (4) is minimized at each step. (4) was chosen so that, by design, the resulting limit flow will agree with smooth ϕ-weighted mean curvature flows when ϕ and K_0 are sufficiently regular.

Their main existence and Hölder continuity theorem ([2], Theorem 4.5) relies on a critical density ratio estimate ([2] § 3.4) which involves the ratio $(n-1)/(n-2)$ throughout, and so the results hold for $n \geq 3$. The author's paper [8] completed the analysis for the case $n = 2$. S. Luckhaus and T. Sturzenhecker [21] used a scheme similar to that of F. Almgren, J. Taylor, and L. Wang, for the special case of ordinary mean curvature flow in $\mathbb{R}^n$. In the plane, R. J. McCann [23] showed that, for any norm ϕ on $\mathbb{R}^2$, any convex and balanced set ($K_0 = -K_0$) remains convex and balanced as it evolves by flat ϕ curvature flow.

More recently, G. Bellettini, V. Caselles, A. Chambolle, and M. Novaga ([4] and [9]; also, see [5]) showed that flat ϕ curvature flow with mobility $M = \phi$ preserves convexity in $\mathbb{R}^n$, for each norm ϕ on $\mathbb{R}^n$, and for any $n \geq 2$. Mathematically, this is a convenient choice of mobility function for the following reason. The Wulff shape W_ϕ for ϕ is the unit ball of its polar ϕ°, defined by

$$\phi^\circ(y) = \sup_{x \in \mathbb{R}^n} \{\langle y, x\rangle : \phi(x) \leq 1\}, \tag{2}$$

where $\langle\ \rangle$ denotes the usual inner product on $\mathbb{R}^n$. Among all sets with a specified volume, the one minimizing the surface energy (3) is (up to translation and scaling) W_ϕ. The boundary of a round ball evolving by ordinary mean curvature (so that $\phi = \phi_E$ and $M = \phi_E$ as well) will shrink homothetically until it vanishes. More generally, the boundary of a Wulff shape W_ϕ evolving by ϕ-weighted mean curvature with mobility M will shrink homothetically until it vanishes – but only if M is a multiple of ϕ (see, for example, [27]). By choosing $M = \phi$, G. Bellettini, V. Caselles, A. Chambolle, and M. Novaga ensured that the bulk and surface energy driving forces would be maximally compatible, so that the flow would tend toward the Wulff shape for ϕ in a strong sense (also, see [5]). This allowed them to prove useful comparison and uniqueness results.

In this paper, we work in $\mathbb{R}^n$ for any $n \geq 2$, using an arbitrary norm ϕ on $\mathbb{R}^n$, and using an isotropic mobility, $M = \phi_E$, as in [2]. For any initial compact, convex set $K_0 \subset \mathbb{R}^n$, we construct a flat ϕ curvature flow $K(t)$ which is compact and convex for all $t \geq 0$.

In Section 2, we introduce flat ϕ curvature flows and establish some key results about convex sets and E-minimizers. Theorems 8 and 9 give volume estimates for discrete flat ϕ curvature flows $K_j(t)$. Using Cantor's diagonal process, we use the discrete flows to define a limit flow $K(t)$ for all non-negative dyadic rational times t. Because the $K(t)$'s are defined as limits of different sequences of sets, the flow $K(t)$ as constructed could potentially be pathologically discontinuous in t. Theorem 10 gives a crucial Hölder estimate which rules out such behavior. Finally, we establish the existence and Hölder continuity of the flat ϕ curvature flow $K(t)$ for all $t \geq 0$ (Theorem 11). The convexity of K_0 allows us to give different arguments from those in [2] to establish our main Hölder inequalities, which improve upon those in [2] for the case we consider.

It follows from ([2] § 7) that, when ϕ and ∂K_0 are sufficiently smooth, and when ϕ is elliptic, these flat ϕ curvature flows agree with smooth ϕ-weighted mean curvature flows, until the latter develop singularities.

2. Flat ϕ Curvature Flows

We will measure volume and surface area in $\mathbb{R}^n$ (for $n \geq 2$) with n-dimensional Lebesgue measure $\mathcal{L}^n$ and $(n-1)$-dimensional Hausdorff measure $\mathcal{H}^{n-1}$, respectively. For $x \in \mathbb{R}$, $\lfloor x \rfloor$ is the floor function, defined to be the greatest integer $w \leq x$, and $\lceil x \rceil$ is the ceiling function, defined to be the least integer y such that $y \geq x$.

For each $K \subset \mathbb{R}^n$, we let χ_K be the characteristic function of K, defined by $\chi_K(x) = 1$ for $x \in K$ and $\chi_K(x) = 0$ otherwise. For $A, B \subset \mathbb{R}^n$, $A \triangle B = (A \setminus B) \cup (B \setminus A)$ denotes the symmetric difference of A and B. We write $A =_n B$ (i.e., "A is $\mathcal{L}^n$ almost equal to B") provided $\mathcal{L}^n(A \triangle B) = 0$. Similarly, we write $A \subset_n B$ provided $\mathcal{L}^n(A \setminus B) = 0$. For $K \subset \mathbb{R}^n$, we let $\operatorname{conv} K$ denote the convex hull of K.

Let $\mathcal{C}$ denote the collection of all bounded, $\mathcal{L}^n$ measurable subsets $K \subset \mathbb{R}^n$ for which $\chi_K \in BV(\mathbb{R}^n)$. Whenever $K \in \mathcal{C}$, we let ∂K (the *reduced boundary* of K) be the set of all points x in $\mathbb{R}^n$ at which K has a measure-theoretic exterior unit normal $n_K(x)$ in the sense of Federer ([13], § 4.5.5). Each set $K \in \mathcal{C}$ has finite perimeter, given by $P(K) = \mathcal{H}^{n-1}(\partial K) < \infty$. We say that a sequence $K_1, K_2, K_3, \ldots$ of $\mathcal{L}^n$ measurable subsets of $\mathbb{R}^n$ *converges in volume* to a set $K \subset \mathbb{R}^n$ provided $\mathcal{L}^n(K \triangle K_i) \to 0$ as $i \to \infty$. Some excellent references that treat sets of finite perimeter and functions of bounded variation in detail are [1], [6], [11], [16], [20], and [22].

Throughout this paper, we will let ϕ denote an arbitrary norm on $\mathbb{R}^n$, and we will set $\phi_0 = \inf\{\phi(w) : |w| = 1\}$ and $\phi^0 = \sup\{\phi(w) : |w| = 1\}$. An easy continuity-compactness argument shows that these extrema are attained and that $0 < \phi_0 \leq \phi^0 < \infty$. We define the *surface energy* of $K \in \mathcal{C}$ to be its ϕ-weighted perimeter, and we write

$$SE(\partial K) = \int_{x \epsilon \partial K} \phi(n_K(x)) \, d\mathcal{H}^{n-1}x. \tag{3}$$

We note that this surface energy functional is lower semicontinuous with respect to convergence in volume ([13], Theorem 5.1.5).

The following lemma, a consequence of Jensen's Inequality (cf. [13] § 2.4.19), asserts that the surface energy of a planar region does not exceed that of any other region having the same boundary.

Lemma 1. (Half-space comparisons [2], § 3.1.9; cf. [7] Theorem 17, where a proof is given). *Suppose $K \in \mathcal{C}$ and $\lambda : \mathbb{R}^n \to R$ is linear with $\|\lambda\| = 1$. Then for every r in $\mathbb{R}$ we have $SE(\partial(K \cap \{x : \lambda(x) < r\})) \leq SE(\partial K)$.*

We will now show that, if a set $K \in \mathcal{C}$ contains a convex set L, then it has at least as much surface energy as L.

Proposition 2. (Convex body comparisons). *Suppose $K \in \mathcal{C}$ and L is a convex set such that $L \subset_n K$. Then $SE(\partial L) \leq SE(\partial K)$.*

Proof. If H is any closed half-space in $\mathbb{R}^n$ containing L, then we may write $H = \{x : \lambda(x) \leq r\}$ and $H^\circ = \{x : \lambda(x) < r\}$, for some λ and r as in Lemma 1. Since $K \cap H$ and $K \cap H^\circ$ differ by a set having $\mathcal{L}^n$ measure 0, their surface energies are equal, and so we use Lemma 1 to estimate $SE(\partial(K \cap H)) = SE(\partial(K \cap H^\circ)) \leq SE(\partial K)$. Intersecting K with each closed half-space containing L gives L (up to a set of $\mathcal{L}^n$ measure 0), without increasing surface energy, and so $SE(\partial L) \leq SE(\partial K)$. ■

If $K_0 \in \mathcal{C}$ we let $\rho(x) = \mathrm{dist}(\mathrm{x}, \partial \mathrm{K}_0)$ denote the ordinary Euclidean distance function to ∂K_0. We define the *signed distance* function $\rho_\pm(x)$ to be $-\rho(x)$ inside K_0 and $\rho(x)$ outside K_0. If K_0 is convex, then $\rho_\pm(x)$ is a convex function of x ([23], Lemma 4.2).

For K_0 and K in $\mathcal{C}$, with $\mathcal{L}^n(K_0) > 0$, and for any $\Delta t > 0$ we define the energy (cf. [2] § 2.6)

$$E(K_0, K, \Delta t) = SE(\partial K) + \frac{1}{\Delta t} \int_{x \in K_o \triangle K} \mathrm{dist}(\mathrm{x}, \partial \mathrm{K}_0)\, \mathrm{d}\mathcal{L}^\mathrm{n}\mathrm{x}. \tag{4}$$

If $\mathcal{L}^n(K_0) > 0$, we say that K is an *E-minimizer for K_0 over Δt* provided

$$E(K_0, K, \Delta t) = \inf\{E(K_0, L, \Delta t) : L \in \mathcal{C}\}.$$

If $K_0 =_n \emptyset$, we take $K = \emptyset$ to be the E-minimizer for K_0.

Remark 3. (Mobilities in flatϕ curvature flows). *If we wanted to use a norm M other than the Euclidean norm as the mobility function, we would replace* dist $(x, \partial K_0)$ *in (4) by*

$$\mathrm{dist}_M(x, \partial K_0) = \inf_{y \in \partial K_0} M^\circ(x - y),$$

where M° is the polar of M (see (2)).

If $\mathcal{L}^n(K \triangle L) = 0$, then $SE(\partial K) = SE(\partial L)$, and $K_0 \triangle K =_n K_0 \triangle L$, so $E(K_0, K, \Delta t) = E(K_0, L, \Delta t)$. Therefore, we do not distinguish between sets with differ by $\mathcal{L}^n$ measure 0. Thus, any convex subset of $\mathbb{R}^n$ with empty interior is equivalent to the empty set. Also, since the boundary of a bounded convex subset of $\mathbb{R}^n$ with non-empty interior has $\mathcal{L}^n$ measure zero, we may and will without loss of generality regard such a set as being closed, hence compact.

Proposition 4. (E-minimizers starting from a compact, convex set). *Suppose K_0 is a compact, convex subset of $\mathbb{R}^n$. If K is an E-minimizer for K_0 over Δt, then $K \subset_n K_0$.*

Proof. If $K_0 =_n \emptyset$, the result is immediate, so we'll assume $\mathcal{L}^n(K_0) > 0$. If $\mathcal{L}^n(K \setminus K_0) > 0$, then we can find a closed half-space H which contains K_0 and for which $\mathcal{L}^n(K \setminus H) > 0$. Let $L = K \cap H$. Then $SE(\partial L) \leq SE(\partial K)$ by Proposition 2. Also, $K_0 \triangle L \subset K_0 \triangle K$, with $\mathcal{L}^n((K_0 \triangle K) \setminus (K_0 \triangle L)) > 0$, so

$$E(K_0, L, \Delta t) = SE(\partial L) + \frac{1}{\Delta t}\int_{K_0 \triangle L} \rho d\mathcal{L}^n < SE(\partial K) + \frac{1}{\Delta t}\int_{K_0 \triangle K} \rho d\mathcal{L}^n$$
$$= E(K_0, K, \Delta t),$$

which contradicts the E-minimality of K. ■

The proof of Proposition 4 can be easily modified to show that, for any $K_0 \in \mathcal{C}$, if an E-minimizer K exists then it must satisfy $K \subset_n \operatorname{conv}(\mathrm{K}_0)$. A standard compactness-lower semicontinuity argument then shows that, for any $K_0 \in \mathcal{C}$, and for any $\Delta t > 0$, there is an E-minimizer $K \in \mathcal{C}$. This E-minimizer may be empty, and it need not be unique. However, if $\mathcal{L}^n(K_0) > 0$ then for sufficiently small values of Δt we will have $\mathcal{L}^n(K) > 0$ as well, because of [2] Theorem 4.4.

When K_0 is convex, there exists an E-minimizer which is also convex (though possibly empty). This was first shown in the plane for convex, balanced sets by R. McCann ([23], Theorem 4.1), and more recently for general convex sets for each $n \geq 2$ by V. Caselles and A. Chambolle ([9] § 5.5).

Lemma 5. (Convexity of E-minimizers (cf. [23] and [9])). *If K_0 is a compact, convex subset of $\mathbb{R}^n$, and $\Delta t > 0$, then there exists a compact, convex set $K \in \mathcal{C}$ such that K is an E-minimizer for K_0 over Δt.*

We say that $K_j(\cdot)$, for $j = 1, 2, 3, \ldots$, are *discrete flat ϕ curvature flows* (cf. [2] § 2.6) of the initial set $K_0 \subset \mathbb{R}^n$ if for each $j \geq 1$ we have $K_j(0) = K_0$ and $\Delta t_j = 1/2^j$, and for each $t \geq 0$ we have

$$K_j(t) = K_j(\lceil t/\Delta t_j \rceil \Delta t_j), \tag{5}$$

where $K_j(k\Delta t_j)$ is an E-minimizer for $K_j((k-1)\Delta t_j)$ over Δt_j for each $k = 1, 2, 3, \ldots$.

We say that $K(\cdot) : [0, \infty) \to \mathcal{C}$ is a *flat ϕ curvature flow* (cf. [2] § 2.6) if there exists a sequence $\{K_j(\cdot)\}_{j=1,2,3,\ldots}$ of discrete flat flows, and a subsequence $s(1), s(2), s(3), \ldots$ of $1, 2, 3, \ldots$ such that

$$\mathcal{L}^n\left(K(t) \triangle K_{s(i)}(t)\right) \to 0$$

locally uniformly for all t.

Remark 6. *In [2], the term flat ϕ curvature flow is used to refer instead to the evolution of the boundaries $\partial K(t)$ (modeled in the essentially equivalent language of currents).*

3. EXISTENCE AND HÖLDER CONTINUITY OF FLAT ϕ CURVATURE FLOWS STARTING FROM COMPACT CONVEX SETS

We will now restrict our attention to the case when K_0 is a compact, convex subset of $\mathbb{R}^n$. For each $j \geq 1$, repeated application of Lemma 5 guarantees the existence of discrete flat ϕ curvature flows $K_j(t)$, such that $K_j(t)$ is compact and convex for all t.

Next, we will define $K(t)$ for non-negative dyadic rational times t ($t = p/2^q$, where p and q are non-negative integers). Let $\{t_k\}_{k=1,2,3,\ldots}$ denote the set of all such times. By Proposition 4, $K_j(t) \subset_n K_0$ for each j and t. By standard compactness theorems (cf. [2] § 3.1.5), there exist a bounded set $Q_1 \subset \mathbb{R}^n$ and a subsequence $j(1,1), j(1,2), j(1,3), \ldots$ of $1, 2, 3, \ldots$ such that $\mathcal{L}^n\left(Q_1 \triangle K_{j(1,i)}(t_1)\right) \to 0$ as $i \to \infty$. Q_1 must be convex since convergence in volume preserves convexity. Since its boundary has $\mathcal{L}^n$ measure zero, as noted above we may replace Q_1 by its closure, so that it is compact.

Similarly, there exist a compact, convex set $Q_2 \subset \mathbb{R}^n$ and a subsequence $\{j(2,i)\}$ of $\{j(1,i)\}$ such that $\mathcal{L}^n\left(Q_2 \triangle K_{j(2,i)}(t_2)\right) \to 0$ as $i \to \infty$. Since $\{j(2,i)\}$ is a subsequence of $\{j(1,i)\}$, we also have $\mathcal{L}^n\left(Q_1 \triangle K_{j(2,i)}(t_1)\right) \to 0$ as $i \to \infty$. Continuing in this fashion, for each $m \geq 2$ there exist a compact, convex set $Q_m \subset \mathbb{R}^n$ and a subsequence $\{j(m,i)\}$ of $\{j(m-1,i)\}$ such that

$$\mathcal{L}^n\left(Q_k \triangle K_{j(m,i)}(t_k)\right) \to 0 \text{ as } i \to \infty \tag{6}$$

whenever $k \leq m$.

For each $i \geq 1$, we set $s(i) = j(i,i)$. Then, for any positive integer k, since $\{s(i)\}$ is a subsequence of $\{j(k,i)\}$ for each $i \geq k$, we have $\mathcal{L}^n\left(Q_k \triangle K_{s(i)}(t_k)\right) \to 0$ as $i \to \infty$. Whenever $t = t_k$ for some k, we define

$$K(t) = Q_k, \tag{7}$$

so that $K(t)$ is defined, compact, and convex for a dense set of times. This will give us control of the flow once we establish a continuity estimate for the discrete flows.

Lemma 7. ***([2], Proposition 4.1, Step 1)*** *If $f : [0,R] \to [0,S]$, then*

$$\int_0^R f(z)\,dz \leq \sqrt{2S}\left(\int_0^R z f(z)\,dz\right)^{1/2}.$$

Theorem 8. (A volume inequality for **E**-minimizers of compact, convex sets). *Suppose K_0 is a compact, convex subset of $\mathbb{R}^n$. If K is an E-minimizer for K_0 over Δt, then*

$$\mathcal{L}^n(K_0 \triangle K) \leq \left(\frac{2}{\phi_0}\right)^{1/2} (SE(\partial K_0))^{1/2} \, (SE(\partial K_0) - SE(\partial K))^{1/2} \, \Delta t^{1/2}$$

Proof. For each $x \in \mathbb{R}^n$ let $\rho(x) = \mathrm{dist}(\mathrm{x}, \partial \mathrm{K}_0)$. If K_0 has empty interior, then $K =_n \emptyset$, and the result is immediate. Suppose, then, that K_0 has a positive inradius, r, so that $\rho(x) \leq r$ for each $x \in K_0$. For each $0 < z < r$, let $A(z) = \mathcal{H}^{n-1}\left((K_0 \setminus K) \cap \rho^{-1}\{z\}\right)$. Then

(8)

$$A(z) \leq \mathcal{H}^{n-1}\left(K_0 \cap \rho^{-1}\{z\}\right) = \mathcal{H}^{n-1}\left(\partial\left(K_0 \cap \{x : \rho_\pm(x) \leq -z\}\right)\right) \leq \mathcal{H}^{n-1}(\partial K_0),$$

since the convexity of $\rho_\pm$ ([23], Lemma 4.2) implies that its sublevel sets are convex. Hence, $K_0 \cap \{x : \rho_\pm(x) \leq -z\}$ is a convex subset of K_0, and so the last inequality in (8) follows from Proposition 2. Using Proposition 4 and then Federer's coarea formula ([13] § 3.2), we calculate

$$\mathcal{L}^n(K_0 \triangle K) = \mathcal{L}^n(K_0 \setminus K) = \mathcal{L}^n\left((K_0 \setminus K) \cap \rho^{-1}(0, r)\right) = \int_0^r A(z)\, dz.$$

Applying Lemma 7 with f, R, and S there replaced respectively by A, r, and $\mathcal{H}^{n-1}(\partial K_0)$ we get

$$\mathcal{L}^n(K_0 \triangle K) \leq \sqrt{2\mathcal{H}^{n-1}(\partial K_0)} \left(\int_0^r zA(z)\, dz\right)^{1/2}. \tag{9}$$

Using the coarea formula again, we get

$$\int_0^r zA(z)\, dz = \int_{K_0 \setminus K} \rho\, d\mathcal{L}^n. \tag{10}$$

The E-minimality of K implies $E(K_0, K, \Delta t) \leq E(K_0, K_0, \Delta t)$, so

$$SE(\partial K) + \frac{1}{\Delta t} \int_{K_0 \setminus K} \rho\, d\mathcal{L}^n \leq SE(\partial K_0),$$

which implies

$$\int_{K_0 \setminus K} \rho\, d\mathcal{L}^n \leq \left(SE(\partial K_0) - SE(\partial K)\right)\, \Delta t. \tag{11}$$

Combining (9), (10), and (11) gives

$$\mathcal{L}^n(K_0 \triangle K) \leq \sqrt{2\mathcal{H}^{n-1}(\partial K_0)}\, \left(SE(\partial K_0) - SE(\partial K)\right)^{1/2}\, \Delta t^{1/2}, \tag{12}$$

and the result follows since $SE(\partial K_0) \geq \phi_0 \mathcal{H}^{n-1}(\partial K_0)$. ■

Theorem 9. (Hȯlder inequality for discrete flat ϕ curvature flows of a compact, convex set). *Suppose K_0 is a compact, convex subset of $\mathbb{R}^n$, and that $K_j(t)$, for $j = 1, 2, 3, \ldots,$ are discrete flat ϕ curvature flows of the initial set K_0. Then for* *each positive integer j, and for non-negative integers a and b with $0 \le a < b$, the quantity $\mathcal{L}^n\left(K_j\left(a\Delta t_j\right) \triangle K_j\left(b\Delta t_j\right)\right)$ is bounded above by*

$$\left(\frac{2}{\phi_0}\right)^{1/2} \left(SE\left(\partial K_j\left(a\Delta t_j\right)\right)\right)^{1/2} \left(SE\left(\partial K_j\left(a\Delta t_j\right)\right) - SE\left(\partial K_j\left(b\Delta t_j\right)\right)\right)^{1/2} \left(\left(b-a\right)\Delta t_j\right)^{1/2}.$$

Proof. For each m such that $a \le m \le b$, let $L_m = K_j\left(m\Delta t_j\right)$. For each m such that $a \le m \le b-1$, we let $\Delta SE_{m,m+1} = SE\left(\partial L_m\right) - SE\left(\partial L_{m+1}\right)$, which must be non-negative since L_{m+1} is an E-minimizer for L_m over Δt_j. For each $1 \le i \le b-a$ we apply Theorem 8 with $K_0, K,$ and Δt there replaced by L_{a+i-1}, L_{a+i}, and Δt_j respectively and then use Cauchy's Inequality:

$$\begin{aligned}
&\mathcal{L}^n\left(K_j\left(a\Delta t_j\right) \triangle K_j\left(b\Delta t_j\right)\right) \\
&\le \sum_{i=1}^{b-a} \mathcal{L}^n\left(L_{a+i-1} \triangle L_{a+i}\right) \\
&\le \left(\frac{2}{\phi_0}\right)^{1/2} \sum_{i=1}^{b-a} \left(SE\left(\partial L_{a+i-1}\right)\right)^{1/2} \left(\Delta SE_{a+i-1,\, a+i}\right)^{1/2} \Delta t_j^{1/2} \\
&\le \left(\frac{2}{\phi_0}\right)^{1/2} \left(SE\left(\partial L_a\right)\right)^{1/2} \sum_{i=1}^{b-a} \left(\Delta SE_{a+i-1,\, a+i}\right)^{1/2} \Delta t_j^{1/2} \\
&\le \left(\frac{2}{\phi_0}\right)^{1/2} \left(SE\left(\partial L_a\right)\right)^{1/2} \left(SE\left(\partial L_a\right) - SE\left(\partial L_b\right)\right)^{1/2} \left(\left(b-a\right)\Delta t_j\right)^{1/2}. \quad \blacksquare
\end{aligned}$$

Theorem 10. (A Hölder estimate for flat ϕ curvature flows at dyadic rational times). *Suppose K_0 is a compact, convex subset of $\mathbb{R}^n$, that $K_j\left(t\right)$, for $j = 1, 2, 3, \ldots,$ are discrete flat ϕ curvature flows of the initial set K_0, and that $K\left(t\right)$ is defined for non-negative dyadic rational times t according to (7). For any two such times s and t with $0 \le s < t < \infty$, we have*

$$\mathcal{L}^n\left(K\left(s\right) \triangle K\left(t\right)\right) \le \left(\frac{2}{\phi_0}\right)^{1/2} SE\left(\partial K_0\right)\left(t-s\right)^{1/2}. \tag{13}$$

Proof. For any $j \ge 1$, we have

$$\begin{aligned}
&\mathcal{L}^n\left(K\left(s\right) \triangle K\left(t\right)\right) \\
&\le \mathcal{L}^n\left(K\left(s\right) \triangle K_j\left(s\right)\right) + \mathcal{L}^n\left(K_j\left(s\right) \triangle K_j\left(t\right)\right) + \mathcal{L}^n\left(K_j\left(t\right) \triangle K\left(t\right)\right).
\end{aligned} \tag{14}$$

Since s and t are dyadic, we may write $s = a_s/2^{b_s}$ and $t = a_t/2^{b_t}$, where a_s, a_t, b_s, and b_t are non-negative integers. Fix $\epsilon > 0$. Since s and t are dyadic, we may fix j so large that the first and third summands of (14) are each bounded above by $\epsilon/2$, and so that $j > b_s$ and $j > b_t$. It follows that $s = \left(a_s \cdot 2^{j-b_s}\right)\Delta t_j$ and $t = \left(a_t \cdot 2^{j-b_t}\right)\Delta t_j$, so that in particular s and t are each integer multiples of Δt_j. We now apply Theorem 9 with $a = a_s \cdot 2^{j-b_s}$ and $b = a_t \cdot 2^{j-b_t}$:

$$\begin{aligned}
&\mathcal{L}^n\left(K_j(s) \triangle K_j(t)\right) \\
&\leq \left(\frac{2}{\phi_0}\right)^{1/2} \left(SE\left(\partial K_j(s)\right)\right)^{1/2} \left(SE\left(\partial K_j(s)\right) - SE\left(\partial K_j(t)\right)\right)^{1/2} (t-s)^{1/2} \\
&\leq \left(\frac{2}{\phi_0}\right)^{1/2} SE\left(\partial K_j(s)\right)(t-s)^{1/2}.
\end{aligned}$$

Because $SE\left(\partial K_j(s)\right) \leq SE\left(\partial K_j(0)\right) = SE\left(\partial K_0\right)$, we deduce (13) since ϵ was arbitrary. ■

We can now give our main theorem.

Theorem 11. (Existence and Hölder continuity of flatϕ curvature flows of convex sets). *Suppose K_0 is a compact, convex subset of $\mathbb{R}^n$. Then there exists a flat ϕ curvature flow $K(t)$, where $K(0) = K_0$ and $K(t)$ is a compact, convex (possibly empty) set for each $t \geq 0$. Moreover, for any two real numbers s and t with $0 \leq s < t < \infty$, we have*

$$\mathcal{L}^n\left(K(s) \triangle K(t)\right) \leq \left(\frac{2}{\phi_0}\right)^{1/2} SE\left(\partial K_0\right)(t-s)^{1/2}. \tag{15}$$

Proof. $K(t)$ is defined for all non-negative dyadic rational numbers t according to (7). If $t \geq 0$ is not dyadic, we can define $K(t)$ as the limit of the sequence $\left\{K\left(\lfloor 2^m t\rfloor / 2^m\right)\right\}$, as $m \to \infty$: because of compactness, and because of Theorem 10, the sequence must converge in volume to a unique bounded set $L \subset \mathbb{R}^n$ (unique up to $\mathcal{L}^n$ measure zero); moreover, L must be convex since convergence in volume preserves convexity. As before, since the boundary of L has $\mathcal{L}^n$ measure zero we will replace it by its closure, if necessary, so that it is compact. In this way, we uniquely extend $K(t)$ to all $t \geq 0$. Since we may approximate s and t with dyadic rationals arbitrarily closely, inequality (15) follows from Theorem 10. That $K(t)$ is a flat ϕ curvature flow follows from this uniform estimate. ■

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