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CODERIVATIVE AND MONOTONICITY OF CONTINUOUS MAPPINGS

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Abstract. Sufficient conditions for a norm-to-weak* continuous mapping $f : X \to X^*$ being monotone or submonotone are established by its Fréchet and normal coderivatives, where X is an Asplund space with its dual space X^*. Under some additional assumptions, they are also necessary conditions. Among other things, we obtain a criterion for the monotonicity of continuous mappings which extends the following classical result: a differentiable mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ is monotone if and only if for each $x \in \mathbb{R}^n$ the Jacobian matrix $\nabla F(x)$ is positive semi-definite; see [22, Proposition 12.3]. As a by-product, sufficient conditions for a function being convex or approximately convex are given.

1. INTRODUCTION

Monotonicity plays a remarkable role in studying algorithm theory, operator theory, variational inequality, and many important mappings in variational analysis such as gradient and subgradient mappings, solution mappings, ect.; see [2, 18, 20, 21, 22].

A classical result on characterizing the monotonicity reads as follows: a differentiable mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ is monotone if and only if for each $x \in \mathbb{R}^n$ the Jacobian matrix $\nabla F(x)$ is positive semi-definite; see [22, Proposition 12.3]. This criterion is useful in checking the monotonicity of differentiable functions. Thus, it is natural to hope that such a criterion is still valid for wider classes of mappings with the Jacobian matrix $\nabla F(x)$ being replaced by some kind of generalized differentiation.

The *coderivative* of set-valued mappings originated by Mordukhovich [11] is a kind of generalized differentiation which has been well recognized as a convenient

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tool to study many important issues in variational analysis and optimization; see [11-15, 19, 22] and the references therein.

Our first aim is to *extend the result cited above to the class of continuous mappings by using coderivativies in place of the Jacobian matrix.*

A concept closely related to monotonicity is that of submonotonicity introduced by Spingarn [23]. It is well-known that a proper lower semicontinuous extended-real-valued function is convex if and only if its subdifferential is monotone. Like that, recently, Daniilidis et.al. [8] and Ngai, Penot [17] independently proved that the submonotonicity of subdifferentials of a proper lower semicontinuous extended-real-valued function can characterize the approximate convexity of the function under consideration. Due to the analogy between the monotonicity and the submonotonicity, it is reasonable to pose the question about recognizing the submonotonicity of a mapping by its coderivatives.

The second aim is to *answer partially this question.*

In this paper we establish sufficient conditions for the monotonicity and submonotonicity of norm-to-weak* continuous mappings $f : X \to X^*$ from an Asplund space X into its dual space X^* via the Fréchet and normal coderivatives. Under some additional assumptions, they are also necessary conditions. Among other things, we obtain a criterion for the monotonicity of continuous mappings which extends the classical result cited above in this direction. Besides, sufficient conditions for a function being convex or approximately convex are given.

The rest of the paper is organized as follows. Section 2 collects some definitions and results which are used in the sequel. Section 3 is devoted to the results relating to the monotonicity. The sufficient conditions for the submonotonicity and for approximate convexity are presented in Section 4.

2. Preliminaries

Let X be a real Banach space with its dual topological space X^* and let Ω be a nonempty subset of X. Denote the weak-star topology in X^* (resp., the canonical pairing between X^* and X) by w^* (resp., $\langle x^*, x\rangle$). The closed unit ball of X is denoted by $\mathbb{B}_X$.

For a set-valued mapping $\Phi : X \rightrightarrows X^*$, the expression $\operatorname*{Lim\,sup}_{x\to\bar{x}} \Phi(x)$ stands for the sequential Kuratowski-Painlevé upper limit of Φ with respect to the norm topology of X and the weak* topology of X^*, i.e.,

$$\operatorname*{Lim\,sup}_{x\to\bar{x}} \Phi(x) = \{x^* \in X^* \mid \exists \text{ sequences } x_k \to \bar{x},\ x_k^* \xrightarrow{w^*} x^*, \\ \text{with } x_k^* \in \Phi(x_k) \text{ for all } k = 1, 2, \ldots\}.$$

Normal cones to sets, coderivatives of set-valued mappings, and subdifferentials of extended-real-valued functions are defined [12] as follows.

The set of Fréchet ε-normals to Ω at $x \in \Omega$ is given by

$$\widehat{N}_\varepsilon(x;\Omega) = \left\{ x^* \in X^* \mid \limsup_{u \xrightarrow{\Omega} x} \frac{\langle x^*, u - x\rangle}{\|u - x\|} \le \varepsilon \right\},$$

where $u \xrightarrow{\Omega} x$ means $u \to x$ with $u \in \Omega$. If $x \notin \Omega$, we put $\widehat{N}_\varepsilon(x;\Omega) = \emptyset$ for all $\varepsilon \ge 0$. For $\varepsilon = 0$, the set $\widehat{N}_0(x;\Omega)$ is called the *Fréchet normal cone* to Ω at x and is denoted by $\widehat{N}(x;\Omega)$. The *normal cone* to Ω at $\bar{x} \in \Omega$ in the sense of Mordukhovich is the set $N(\bar{x};\Omega)$ defined by

$$N(\bar{x};\Omega) = \operatorname*{Lim\,sup}_{x \to \bar{x}, \varepsilon \downarrow 0} \widehat{N}_\varepsilon(x;\Omega).$$

If $\bar{x} \notin \Omega$, $N(\bar{x};\Omega) = \emptyset$ by convention.

For any $(\bar{x}, \bar{y}) \in \operatorname{gph}\Phi$, the set-valued mapping $D^*\Phi(\bar{x},\bar{y}) : Y^* \rightrightarrows X^*$ defined by

$$D^*\Phi(\bar{x},\bar{y})(y^*) = \left\{ x^* \in X^* \mid (x^*, -y^*) \in N((\bar{x},\bar{y}); \operatorname{gph}\Phi) \right\} \tag{2.1}$$

is said to be the *normal coderivative* (called also the *limiting coderivative* and the *coderivative in the sense of Mordukhovich*) of Φ at $(\bar{x},\bar{y})$. We put $D^*\Phi(\bar{x},\bar{y})(y^*) = \emptyset$ for any $y^* \in Y^*$ if $(\bar{x},\bar{y}) \notin \operatorname{gph}\Phi$. The *Fréchet coderivative* $\widehat{D}^*\Phi(\bar{x},\bar{y}) : Y^* \rightrightarrows X^*$ of Φ at $(\bar{x},\bar{y})$ is defined similarly, provided that $N((\bar{x},\bar{y}); \operatorname{gph}\Phi)$ in (2.1) is replaced by $\widehat{N}((\bar{x},\bar{y}); \operatorname{gph}\Phi)$. If Φ is a single-valued and $\bar{y} = \Phi(\bar{x})$, it is customary to write $D^*\Phi(\bar{x})$ for $D^*\Phi(\bar{x},\bar{y})$ and $\widehat{D}^*\Phi(\bar{x})$ for $\widehat{D}^*\Phi(\bar{x},\bar{y})$. If $\Phi : X \to Y$ is *strictly differentiable* at $\bar{x}$ with the derivative $\nabla\Phi(\bar{x})$, that is $\nabla\Phi(\bar{x}) : X \to Y$ is a continuous linear operator and

$$\lim_{x \to \bar{x}, u \to \bar{x}} \frac{\Phi(x) - \Phi(u) - \nabla\Phi(\bar{x})(x - u)}{\|x - u\|} = 0,$$

then

$$D^*\Phi(\bar{x})(y^*) = \widehat{D}^*\Phi(\bar{x})(y^*) = \left\{ (\nabla\Phi(\bar{x}))^* y^* \right\} \quad \forall y^* \in Y^* \tag{2.2}$$

(see [12, Theorem 1.38]). It is well known that the second equality in (2.2) is valid if Φ is Fréchet differentiable at $\bar{x}$. Formula (2.2) and this fact show that the normal coderivative (resp., the Fréchet coderivative) of set-valued mappings is a natural extension of the adjoint operator of the strict derivative (resp., the Fréchet derivative) of single-valued maps.

Let $\varphi : X \to \bar{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ be a proper function from a Banach space X into $\bar{\mathbb{R}}$. As usual, φ is said to be lower semicontinuous (l.s.c.) at $\bar{x} \in \operatorname{dom}\varphi$ provided that $\liminf_{x \to \bar{x}} \varphi(x) \ge \varphi(\bar{x})$; φ is said to be lower semicontinuous if it is l.s.c. at any

$\bar{x} \in \operatorname{dom} \varphi$; where $\operatorname{dom} \varphi := \{x \in X \mid \varphi(x) \in \mathbb{R}\}$. The *limiting subdifferential* of φ at $\bar{x} \in \operatorname{dom} \varphi$ is the set

$$\partial\varphi(\bar{x}) := \{x^* \in X^* \mid (x^*, -1) \in N((\bar{x}, \varphi(\bar{x})); \text{epi}\varphi)\},$$

where $\operatorname{epi} \varphi := \{(x, \mu) \in X \times \mathbb{R} \mid \mu \geq \varphi(x)\}$. The *Fréchet subdifferential* of φ at $\bar{x} \in \operatorname{dom} \varphi$ is defined by

$$\widehat{\partial}\varphi(\bar{x}) := \{x^* \in X^* \mid (x^*, -1) \in \widehat{N}((\bar{x}, \varphi(\bar{x})); \text{epi}\varphi)\}.$$

If $\bar{x} \notin \text{dom}\varphi$ then $\partial\varphi(\bar{x}) = \widehat{\partial}\varphi(\bar{x}) = \emptyset$ by convention.

It is not hard to see that $\widehat{\partial}\varphi(\bar{x}) \subset \partial\varphi(\bar{x})$ and $\widehat{\partial}\varphi(\bar{x})$ is a closed convex set (may be empty). If φ is strictly differentiable at $\bar{x}$, then $\partial\varphi(\bar{x}) = \widehat{\partial}\varphi(\bar{x}) = \{\nabla\varphi(\bar{x})\}$; see [12]. In particular, for C^1-functions, i.e., continuously differentiable functions, one has $\partial\varphi(\bar{x}) = \widehat{\partial}\varphi(\bar{x}) = \{\nabla\varphi(\bar{x})\}$.

The following statement, which is a special case of the *approximate mean value theorem* established by Mordukhovich and Shao [15], plays a crucial role in proving our main results.

Theorem 2.1. (see [12, Theorem 3.49] or [15, Theorem 8.2]). *Let $\varphi : X \to \bar{\mathbb{R}}$ be a proper l.s.c. function from an Asplund space X into $\bar{\mathbb{R}}$ finite at two given points $a \neq b$. Suppose $\varphi(a) = \varphi(b)$ and $\bar{x} \in (a, b)$ is a point at which φ attains its minimum on $[a, b]$. Then there are sequences $x_k \xrightarrow{\varphi} \bar{x}$ and $x_k^* \in \widehat{\partial}\varphi(x_k)$ satisfying* $\lim\limits_{k\to\infty} \langle x_k^*, b - a\rangle = 0$.

Note that the first mean value theorem of the approximate type was introduced by Zagrodny [24]. The reader interested in this subject should consult [1, 8, 12, 15, 24].

Let $\varphi : X \to \bar{\mathbb{R}}$ be a function finite at $\bar{x} \in X$ and let $\bar{y} \in \partial\varphi(\bar{x})$. The *normal second-order subdifferential* of φ at $\bar{x}$ relative to $\bar{y}$ is the set-valued mapping $\partial^2\varphi(\bar{x}, \bar{y}) : X^{**} \rightrightarrows X^*$ defined by

$$\partial^2\varphi(\bar{x}, \bar{y})(u) = (D^*\partial\varphi)(\bar{x}, \bar{y})(u)$$

for all $u \in X^{**}$; see [12]. For any $\bar{y} \in \widehat{\partial}\varphi(\bar{x})$, the set-valued mapping $\widehat{\partial}^2\varphi(\bar{x}, \bar{y}) : X^{**} \rightrightarrows X^*$ with the values

$$\widehat{\partial}^2\varphi(\bar{x}, \bar{y})(u) := (\widehat{D}^*\widehat{\partial}\varphi)(\bar{x}, \bar{y})(u) \quad (u \in X^{**})$$

is said to be the *Fréchet second-order subdifferential* of φ at $\bar{x}$ relative to $\bar{y}$. If $\partial\varphi(\bar{x})$ is singleton, then we write $\partial^2\varphi(\bar{x})$ for $\partial^2\varphi(\bar{x}, \bar{y})$. Similarly, we write $\widehat{\partial}^2\varphi(\bar{x})$ for $\widehat{\partial}^2\varphi(\bar{x}, \bar{y})$ if $\widehat{\partial}\varphi(\bar{x})$ is singleton. We refer the reader to [3, 4, 5, 10, 12, 14, 19, 22] for more information on the second-order subdifferentials and their applications.

3. Coderivative and Monotonicity

We recall from [18] that a set-valued map $T : X \rightrightarrows X^*$ is said to be a *monotone operator*, if $\langle x^* - y^*, x - y\rangle \geq 0$ for all $x, y \in X,\ x^* \in T(x),\ y^* \in T(y)$; T is said to be *maximal monotone* if T is a monotone operator and $\mathrm{gph}T$ is not a proper subset of the graph of any other monotone operator. The reader is referred to [2, 18, 21, 22] for many results on monotone operators and their applications.

Our first result is stated as follows.

Theorem 3.1. *Let X be an Asplund space with its dual space X^* and let $f : X \to X^*$ be a norm-to-weak* continuous mapping. Consider the two properties:*

(i) *For any $x \in X$, one has*

$$\langle u^*, u\rangle \geq 0 \quad \forall u \in X,\ u^* \in \widehat{D}^* f(x)(u). \tag{3.1}$$

(ii) *f is a monotone operator.*

Then one has $(i) \Rightarrow (ii)$. The implication $(ii) \Rightarrow (i)$ is also valid if f is locally Lipschitz or X is a Hilbert space.

As can be seen from the following example, removing the continuity of f may make the implication $(i) \Rightarrow (ii)$ invalid.

Example 3.2. Let us consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by setting $f(x) = 0$ if $x \in \mathbb{Q}$ and $f(x) = 1$ otherwise. Note that f is discontinuous and non-monotone. By the definitions of the Fréchet normal cone and the Fréchet coderivative, $\widehat{N}((x, f(x)); \mathrm{gph}f) = \{0\} \times \mathbb{R}$ and $\widehat{D}^* f(x)(u) = \{0\}$ for all $x, u \in \mathbb{R}$. Hence (3.1) is satisfied.

To prove Theorem 3.1, we need the following lemma which was given in [4]. For sake of completeness we will provide the proof of this result.

Lemma 3.3. *If $T : X \rightrightarrows X$ is a maximal monotone operator from a Hilbert space X into itself, then for every point $(\bar{x}, \bar{y}) \in \mathrm{gph}T$, it holds*

$$\langle z, u\rangle \geq 0 \quad \textit{whenever } u \in \widehat{D}^* T(\bar{x}, \bar{y})(z).$$

Proof. We follow the scheme given in [19, Theorem 2.1]. Take any $(\bar{x}, \bar{y}) \in \mathrm{gph}T$. Consider the linear function $J : X \times X \to X \times X$ with $J(x, y) = (y + x, y - x)$ for all $(x, y) \in X \times X$. Let $S : X \rightrightarrows X$ be the set-valued mapping defined by $\mathrm{gph}S = J(\mathrm{gph}T)$. Put $(\bar{z}, \bar{w}) = J(\bar{x}, \bar{y})$. Since $(\bar{x}, \bar{y}) \in \mathrm{gph}T$, it follows that $(\bar{z}, \bar{w}) \in \mathrm{gph}S$. Note that Proposition 12.11 in [22] is also valid if $\mathbb{R}^n$ is replaced by a Hilbert space X. Since T is a maximal monotone operator, by [22, Proposition 12.11], S is a single-valued mapping from all of X into itself that is Lipschitzian with constant 1. Hence, by [13, Proposition 3.5],

$$\widehat{D}^* S(\tilde{x})(\tilde{y}) = \widehat{\partial}\langle \tilde{y}, S\rangle(\tilde{x}) \quad \forall \tilde{x}, \tilde{y} \in X. \tag{3.2}$$

Observe that $\text{gph}T = J^{-1}(\text{gph}S)$ and J is strictly differentiable at $(\bar{x}, \bar{y})$ with its derivative $\nabla J((\bar{x}, \bar{y})) = J$ being surjective. It is not hard to see that

$$\nabla J((\bar{x}, \bar{y}))^*(x, y) = (x - y, x + y) \quad \forall\ (x, y) \in X \times X.$$

Hence, by [12, Corollary 1.15],

$$(z, -u) \in \widehat{N}((\bar{x}, \bar{y}); \text{gph}T) \Longleftrightarrow (z - u, -z - u) \in \widehat{N}((\bar{z}, \bar{w}); \text{gph}S).$$

This means that

$$z \in \widehat{D}^*T(\bar{x}, \bar{y})(u) \Longleftrightarrow z - u \in \widehat{D}^*S(\bar{z}, \bar{w})(z + u).$$

For each $\tilde{y} \in X$, due to the fact that S is Lipschitzian with constant 1, the mapping $\langle \tilde{y}, S\rangle$ is Lipschitzian with constant $\|\tilde{y}\|$. Hence $\|\tilde{u}\| \le \|\tilde{y}\|$ for all $\tilde{u} \in \widehat{\partial}\langle \tilde{y}, S\rangle(\bar{z})$. By (3.2), $\|\tilde{u}\| \le \|\tilde{y}\|$ whenever $\tilde{u} \in \widehat{D}^*S(\bar{z}, \bar{w})(\tilde{y})$. Applying this to $\tilde{u} = z - u$ and $\tilde{y} = z + u$ in the case of an arbitrary pair (u, z) with $z \in \widehat{D}^*T(\bar{x}, \bar{y})(u)$, we obtain $\|z - u\| \le \|z + u\|$. Hence $\langle z, u\rangle = 4^{-1}(\|z + u\|^2 - \|z - u\|^2) \ge 0$. ■

Lemma 3.4. (see [6, Theorem 2.5]). *Let X be a Hilbert space let $f : X \to X$ be a norm-to-weak continuous monotone mapping. Then f is maximal monotone.*

Proof of Theorem 3.1. $(i) \Rightarrow (ii)$: The proof is based on the scheme given in [4]. On the contrary, suppose that one could find a norm-to-weak* continuous nonmonotone mapping $f : X \to X^*$ satisfying condition (3.1). Then there exist $a, b \in X$ such that

$$\langle f(b) - f(a), b - a\rangle < 0. \tag{3.3}$$

Put $\psi(x) = \langle f(a) - f(b), x\rangle + \langle f(x), b - a\rangle$. Since $\psi(a) = \psi(b)$ and ψ is continuous, there exists $\bar{x} \in (a, b)$ satisfying $\psi(\bar{x}) = \min\limits_{x\in[a,b]} \psi(x)$ or $\psi(\bar{x}) = \max\limits_{x\in[a,b]} \psi(x)$.

Case 1. $\psi(\bar{x}) = \min\limits_{x\in[a,b]} \psi(x)$. By Theorem 2.1, there are sequences $x_k \to \bar{x}$ and $x_k^* \in \widehat{\partial}\psi(x_k)$ satisfying

$$\lim_{k\to\infty} \langle x_k^*, b - a\rangle = \psi(b) - \psi(a) = 0.$$

Since

$$x_k^* \in \widehat{\partial}\psi(x_k) = f(a) - f(b) + \widehat{\partial}\langle f(\cdot), b - a\rangle(x_k)$$

and

$$\widehat{\partial}\langle f(\cdot), b - a\rangle(x_k) \subset \widehat{D}^*f(x_k)(b - a) \quad \forall k,$$

it follows that

$$f(b) - f(a) + x_k^* \in \widehat{D}^* f(x_k)(b-a) \quad \forall k.$$

According to (3.1),

$$\langle f(b) - f(a) + x_k^*, b-a \rangle \geq 0 \quad \forall k.$$

Taking the limits as $k \to \infty$ yields $\langle f(b) - f(a), b-a \rangle \geq 0$. This contradicts (3.3).

Case 2. $\psi(\bar{x}) = \max\limits_{x \in [a,b]} \psi(x)$. Using the same arguments as in case 1 for $-\psi$, we obtain $\langle f(a) - f(b), a-b \rangle \geq 0$, which contradicts (3.3).

$(ii) \Rightarrow (i)$: (a) Suppose that f is monotone and locally Lipschitz. If (i) is invalid, then one could find $x_0 \in X$, $u_0 \in X$ and $u_0^* \in \widehat{D}^* f(x_0)(u_0)$ such that $\langle u_0^*, u_0 \rangle < 0$. Since $u_0^* \in \widehat{D}^* f(x_0)(u_0)$, it holds

$$\limsup_{x \to x_0} \frac{\langle u_0^*, x - x_0 \rangle - \langle f(x) - f(x_0), u_0 \rangle}{\|x - x_0\| + \|f(x) - f(x_0)\|} \leq 0. \tag{3.4}$$

Let $\ell > 0$ be a Lipschitz constant of f on some neighborhood of x_0. We have

$$\begin{aligned}
&\limsup_{x \to x_0} \frac{\langle u_0^*, x - x_0 \rangle - \langle f(x) - f(x_0), u_0 \rangle}{\|x - x_0\| + \|f(x) - f(x_0)\|} \\
&\geq \limsup_{n \to \infty} \frac{\langle u_0^*, -n^{-1}u_0 \rangle - \langle f(x_0 - n^{-1}u_0) - f(x_0), u_0 \rangle}{n^{-1}\|u_0\| + \|f(x_0 - n^{-1}u_0) - f(x_0)\|} \\
&\geq \limsup_{n \to \infty} \frac{\langle u_0^*, -n^{-1}u_0 \rangle - \langle f(x_0 - n^{-1}u_0) - f(x_0), u_0 \rangle}{n^{-1}\|u_0\| + \ell\| - n^{-1}u_0\|} \\
&\geq \limsup_{n \to \infty} \frac{\langle u_0^*, -n^{-1}u_0 \rangle}{n^{-1}\|u_0\| + \ell\| - n^{-1}u_0\|} = \frac{-\langle u_0^*, u_0 \rangle}{(1+\ell)\|u_0\|} > 0.
\end{aligned}$$

This contradicts (3.4). Hence (i) is valid.

(b) Suppose that X is a Hilbert space and (ii) holds. Since f is a monotone mapping continuous with respect to the norm topology of X and the weak-star topology of X^*, by Lemma 3.4, f is maximal monotone. According to Lemma 3.3, (i) is valid. The proof is complete. ∎

Corollary 3.5. ([22, Proposition 12.3]). *A differentiable mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ is monotone if and only if for each $x \in \mathbb{R}^n$ the Jacobian matrix $\nabla F(x)$ is positive semi-definite.*

Proof. Since F is differentiable, by [12, Theorem 1.38], $\widehat{D}^* F(x)(u) = \{\nabla F(x)^* u\}$ for all $u \in \mathbb{R}^n$. Hence (3.1) amounts to the fact that for each $x \in \mathbb{R}^n$ the Jacobian matrix $\nabla F(x)$ is positive semi-definite. By Theorem 3.1, we obtain the desired conclusion. This finishes the proof. ∎

Remark 3.6. Since $\widehat{D}^*f(x)(u) \subset D^*f(x)(u)$ for all $x \in X$ and $u \in X$, the implication $(i) \Rightarrow (ii)$ in Theorem 3.1 holds if the Fréchet coderivative is replaced by the normal coderivative.

Corollary 3.7. *Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous mapping. Then the following properties are equivalent:*

(i) *For any $x \in \mathbb{R}^n$, one has $\langle u^*, u\rangle \geq 0 \quad \forall u \in \mathbb{R}^n,\ u^* \in D^*f(x)(u)$.*

(ii) *For any $x \in \mathbb{R}^n$, one has $\langle u^*, u\rangle \geq 0 \quad \forall u \in \mathbb{R}^n,\ u^* \in \widehat{D}^*f(x)(u)$.*

(iii) *f is a monotone operator.*

Proof. By Theorem 3.1, we have $(ii) \Leftrightarrow (iii)$. According to Remark 4.2, the implication $(i) \Rightarrow (iii)$ holds. The validity of the reverse implication is due to [19, Theorem 2.1]. This finishes the proof. ■

Corollary 3.8. *Let $\varphi : X \to \mathbb{R}$ be a Fréchet differentiable function from an Asplund space X to $\mathbb{R}$. Suppose that the derivative $\nabla\varphi : X \to X^*$ is norm-to-weak* continuous. Consider the following properties:*

(i) *For every $x \in X$, $\langle u^*, u\rangle \geq 0$ for all $u \in X$ and $u^* \in \partial^2\varphi(x)(u)$.*

(ii) *For every $x \in X$, $\langle u^*, u\rangle \geq 0$ for all $u \in X$ and $u^* \in \widehat{\partial}^2\varphi(x)(u)$.*

(iii) *The function φ is convex.*

Then $(ii) \Rightarrow (iii)$. When $\nabla\varphi$ is locally Lipschitz or X is a Hilbert space, the implication $(iii) \Rightarrow (ii)$ is also valid. If in addition that the derivative $\nabla\varphi : X \to X^*$ is norm-to-norm continuous, then $(i) \Rightarrow (ii) \Rightarrow (iii)$. In the case where X is finite-dimensional, one has $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$.

Proof. Suppose that $\varphi : X \to \mathbb{R}$ is a Fréchet differentiable function and the derivative $\nabla\varphi : X \to X^*$ is norm-to-weak* continuous. Put $f(x) = \nabla\varphi(x)$ for all $x \in X$. Then $f : X \to X^*$ is a norm-to-weak* continuous mapping. Since $\varphi : X \to \mathbb{R}$ is continuous, by [12, Theorem 3.56], f is monotone if and only if φ is convex. Observe that $\widehat{\partial}^2\varphi(x)(u) = \widehat{D}^*\nabla\varphi(x)(u)$ for all $x \in X$ and $u \in X$. Hence (ii) amounts to the fact that for every $x \in X$ (3.1) holds for the mapping $f := \nabla\varphi$. According to Theorem 3.1, if (ii) is valid then f is monotone; and thus φ is convex. Suppose now that either f is locally Lipschitz or X is a Hilbert space. If φ is convex then f is monotone. By Theorem 3.1, (ii) is valid. Note that if the derivative $\nabla\varphi : X \to X^*$ is norm-to-norm continuous, then $\partial\varphi(x) = \widehat{\partial}\varphi(x) = \{\nabla\varphi(x)\}$ and thus $\widehat{\partial}^2\varphi(x)(u) \subset \partial^2\varphi(x)(u)$ for all $x \in X$ and $u \in X$. Hence $(i) \Rightarrow (ii) \Rightarrow (iii)$. In the case where X is finite-dimensional, by using Corollary 3.7, we obtain that $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$. ■

Let $T : X \rightrightarrows X$ be a set-valued mapping from a Hilbert space X into itself. Recall from [22] that T is said to be strong monotone if there exists $\sigma > 0$ such that $T - \sigma I$ is monotone. T is said to be hypomonotone if there exists $\sigma > 0$ such that $T + \sigma I$ is monotone. Here I is the identity mapping on X.

Proposition 3.9. *Let X be a Hilbert space and $f : X \to X$ a norm-to-weak continuous mapping. Consider the following properties:*

(i) *f is a strongly monotone operator;*

(ii) *there exists $\sigma > 0$ such that for each $x \in X$,*

$$\langle u^*, u\rangle \geq \sigma\|u\|^2 \quad \forall u \in X,\ u^* \in \widehat{D}^* f(x)(u);$$

(iii) *there exists $\sigma > 0$ such that for each $x \in X$,*

$$\langle u^*, u\rangle \geq -\sigma\|u\|^2 \quad \forall u \in X,\ u^* \in \widehat{D}^* f(x)(u);$$

(iv) *f is a hypomonotone operator.*

Then one has $(i) \Leftrightarrow (ii) \Rightarrow (iii) \Leftrightarrow (iv)$.

Proof. The implication $(ii) \Rightarrow (iii)$ is trivial. Our task now is to prove that $(i) \Leftrightarrow (ii)$ ($(iii) \Leftrightarrow (iv)$ is proved similarly). By the definition of the strong monotonicity, the property (i) is valid if and only if there exists $\sigma > 0$ such that $g := f - \sigma I$ is monotone. According to Theorem 3.1, the latter is equivalent to the fact that for each $x \in X$,

$$\langle u^*, u\rangle \geq 0 \quad \forall u \in X,\ u^* \in \widehat{D}^* g(x)(u). \tag{3.5}$$

By [12, Theorem 1.62],

$$\widehat{D}^* g(x)(u) = \widehat{D}^* f(x)(u) - \sigma u \quad \forall x \in X,\ u \in X.$$

Combining this fact with (3.5), we obtain the desired conclusion. ∎

Corollary 3.10. *Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous mapping. Consider the following properties:*

(i) *f is a strongly monotone operator;*

(ii) *there exists $\sigma > 0$ such that for each $x \in \mathbb{R}^n$,*

$$\langle u^*, u\rangle \geq \sigma\|u\|^2 \quad \forall u \in \mathbb{R}^n,\ u^* \in D^* f(x)(u);$$

(iii) *there exists $\sigma > 0$ such that for each $x \in \mathbb{R}^n$,*

$$\langle u^*, u\rangle \geq \sigma\|u\|^2 \quad \forall u \in \mathbb{R}^n,\ u^* \in \widehat{D}^* f(x)(u);$$

(iv) *there exists $\sigma > 0$ such that for each $x \in \mathbb{R}^n$,*

$$\langle u^*, u\rangle \geq -\sigma\|u\|^2 \quad \forall u \in \mathbb{R}^n,\ u^* \in \widehat{D}^* f(x)(u);$$

(v) *there exists* $\sigma > 0$ *such that for each* $x \in \mathbb{R}^n$*,*

$$\langle u^*, u\rangle \geq -\sigma\|u\|^2 \quad \forall u \in \mathbb{R}^n,\ u^* \in D^*f(x)(u);$$

(vi) f *is a hypomonotone operator.*

Then one has $(i) \Leftrightarrow (ii) \Leftrightarrow (iii) \Rightarrow (iv) \Leftrightarrow (v) \Leftrightarrow (vi)$.

Proof. Follows Corollary 3.7 and the scheme of the proof of Corollary 3.9. ■

4. Coderivative and Submonotonicity

Recall [8] that a set-valued operator $T : X \rightrightarrows X^*$ from a Banach space X into its dual space X^* is said to be submonotone at $x_0 \in X$ if for each $\varepsilon > 0$ there exists $\rho > 0$ such that $\langle x_1^* - x_2^*, x_1 - x_2\rangle \geq -\varepsilon\|x_1 - x_2\|$ for all $x_i \in x_0 + \rho\mathbb{B}_X$ and all $x_i^* \in T(x_i)$ $(i = 1, 2)$. This concept, which is known also as the approximate monotonicity [17], was introduced by Spingarn [23] who called it the *strict submonotonicity*. Here we follow Daniilidis et.al. [8] in using the terminology the "submonotonicity." The reader interested in submonotone operators should consult [7, 8, 9, 17, 23].

We now present the sufficient condition for the submonotonicity of a continuous mapping via the Fréchet coderivative.

Theorem 4.1. *Let* X *be an Asplund space with its dual space* X^* *and let* $f : X \to X^*$ *be a mapping norm-to-weak** *continuous around* $x_0 \in X$*. Suppose that for any* $\varepsilon > 0$ *there exists* $\rho > 0$ *such that for every* $x \in x_0 + \rho\mathbb{B}_X$*, one has*

$$\langle u^*, u\rangle \geq -\varepsilon\|u\| \quad \forall u \in \rho\mathbb{B}_X,\ u^* \in \widehat{D}^*f(x)(u). \tag{4.1}$$

Then f *is submonotone at* x_0*.*

Proof. Suppose that (4.1) is valid but f is nonsubmonotone at x_0. Then there exists $\varepsilon > 0$ such that for each $\rho > 0$,

$$\langle f(b) - f(a), b - a\rangle < -\varepsilon\|b - a\| \tag{4.2}$$

for some $a, b \in x_0 + 2^{-1}\rho\mathbb{B}_X$. Let us choose $\rho > 0$ and $a, b \in x_0 + 2^{-1}\rho\mathbb{B}_X$ such that (4.1) and (4.2) are satisfied. Consider the function $\psi(x) := \langle f(a) - f(b), x\rangle + \langle f(x), b - a\rangle$ for all $x \in X$. We have $\psi(a) = \psi(b)$ and ψ is continuous. Thus, there exists $\bar{x} \in (a, b)$ satisfying $\psi(\bar{x}) = \min\limits_{x\in[a,b]} \psi(x)$ or $\psi(\bar{x}) = \max\limits_{x\in[a,b]} \psi(x)$.

Case 1. $\psi(\bar{x}) = \min\limits_{x\in[a,b]} \psi(x)$. As in the proof of Theorem 3.1, we can find sequences $x_k \to \bar{x}$ and $x_k^* \in \widehat{\partial}\psi(x_k)$ such that

$$\lim_{k\to\infty} \langle x_k^*, b - a\rangle = 0 \quad \text{and} \quad f(b) - f(a) + x_k^* \in \widehat{D}^*f(x_k)(b - a).$$

Note that $\|b - a\| \le \rho$ and $\|\bar{x} - x_0\| < \rho$. By virtual of (4.1),

$$\langle f(b) - f(a) + x_k^*, b - a\rangle \ge -\varepsilon\|b - a\|$$

for k sufficiently large. Letting $k \to \infty$ in both side of this inequality, we obtain

$$\langle f(b) - f(a), b - a\rangle \ge -\varepsilon\|b - a\|,$$

which contradicts (4.2).

Case 2. $\psi(\bar{x}) = \max_{x\in[a,b]} \psi(x)$. Using the same arguments as in case 1 for $-\psi$, we arrive at a contradiction. This finishes the proof. ■

Remark 4.2. Since $\widehat{D}^*f(x)(u) \subset D^*f(x)(u)$ for all $x \in X$ and $u \in X$, the conclusion of Theorem 4.1 is also valid if the Fréchet coderivative is replaced by the normal coderivative.

Recall [16] that a function $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ is said to be approximately convex at $x_0 \in X$, if for any $\varepsilon > 0$ there exists $\rho > 0$ such that for all $x_1, x_2 \in x_0 + \rho\mathbb{B}_X$ and $t \in (0, 1)$, one has

$$\varphi((1 - t)x_1 + tx_2) \le (1 - t)\varphi(x_1) + t\varphi(x_2) + \varepsilon t(1 - t)\|x_1 - x_2\|.$$

Corollary 4.3. *Let $\varphi : X \to \mathbb{R}$ be a Fréchet differentiable function from an Asplund space X to $\mathbb{R}$. Suppose that the derivative $\nabla\varphi : X \to X^*$ is norm-to-weak* continuous around $x_0 \in X$. Then φ is approximately convex at x_0 if for any $\varepsilon > 0$ there exists $\rho > 0$ such that*

(4.3) $\langle u^*, u\rangle \ge -\varepsilon\|u\|$ *for all* $u \in \rho\mathbb{B}_X, u^* \in \widehat{\partial}^2\varphi(x)(u)$ *with* $x \in x_0 + \rho\mathbb{B}_X$.

Proof. Suppose that $\varphi : X \to \mathbb{R}$ is a Fréchet differentiable function with its the derivative $\nabla\varphi$ being continuous around $x_0 \in X$ with respect to the norm topology of X and the weak* topology of X^*, and suppose that (4.3) is valid. Put $f(x) = \nabla\varphi(x)$ for all $x \in X$. Then $f : X \to X^*$ is a mapping continuous around $x_0 \in X$ with respect to the norm topology of X and the weak* topology of X^*. By the definition of the Fréchet second-order subdifferential,

$$\widehat{\partial}^2\varphi(x)(u) = \widehat{D}^*\nabla\varphi(x)(u) \quad \forall\, x \in X,\ u \in X.$$

Hence (4.3) implies that (4.1) is valid for the mapping $f = \nabla\varphi$. We have already shown that the mapping f satisfies all the assumptions of Theorem 4.1. According to Theorem 4.1, f is submonotone at x_0. Since the function $\varphi : X \to \mathbb{R}$ is continuous and its derivative $f(\cdot) = \nabla\varphi(\cdot)$ is submonotone at x_0, by [8, Theorem 2], φ is an approximately convex function at x_0. The proof is complete. ■

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