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ON A CLASS OF OPERATORS FROM WEIGHTED BERGMAN SPACES TO SOME SPACES OF ANALYTIC FUNCTIONS

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Abstract. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$, $H(\mathbb{D})$ be the space of all analytic functions on $\mathbb{D}$, φ be an analytic self-map of $\mathbb{D}$ and $u \in H(\mathbb{D})$. Define operators by $DW_{\varphi,u}f = (u \cdot f \circ \varphi)'$ and $W_{\varphi,u}Df = (u \cdot f' \circ \varphi)$ for $f \in H(\mathbb{D})$. In this paper we characterize bounded operators $DW_{\varphi,u}$ and $W_{\varphi,u}D$ from weighted Bergman space to Zygmund-type space, Bloch-type space and Bers-type space on the open unit disk. We also give some sufficient and necessary conditions for these operators to be compact operators in terms of inducing maps φ and u.

1. INTRODUCTION

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$, $H(\mathbb{D})$ be the space of all analytic functions in $\mathbb{D}$ and $dA(z)$ be the area measure on $\mathbb{D}$. For $p \in [1, \infty)$ and $\alpha \in [-1, \infty)$, the weighted Bergman space $A^p_\alpha(\mathbb{D}) = A^p_\alpha$ consists of those functions f analytic in $\mathbb{D}$ such that

$$\|f\|^p_{A^p_\alpha} = \int_{\mathbb{D}} |f(z)|^p (1 - |z|^2)^\alpha dA(z) < \infty.$$

The weighted Bergman space A^p_α with the norm $\|\cdot\|_{A^p_\alpha}$ is a Banach space.

For $\beta \in [0, \infty)$ the n-th weighted-tpye space consists of all $f \in H(\mathbb{D})$ such that

$$\sup_{z\in\mathbb{D}} (1 - |z|^2)^\beta |f^{(n)}(z)| < \infty,$$

where $n \in \mathbb{N}_0$. For $n = 0$ the space is called the Bers-type space and is denoted by $\mathcal{A}^\infty_\beta$. With the norm

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$$\|f\|_{\mathcal{A}_\beta^\infty}=\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|f(z)|$$

$\mathcal{A}_\beta^\infty$ is a Banach space. For $n=1$ it is called the Bloch-type space $\mathcal{B}_\beta^\infty$. Under the norm

$$\|f\|_{\mathcal{B}_\beta^\infty}=|f(0)|+\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|f'(z)|$$

$\mathcal{B}_\beta^\infty$ becomes a Banach space. Define

$$\mathcal{B}_{\beta,0}^\infty=\{f\in\mathcal{B}_\beta^\infty : f(0)=0\}.$$

Then $\mathcal{B}_\beta^\infty$ is a closed subspace of $\mathcal{B}_\beta^\infty$. For Bers-type spaces and Bloch-type spaces on the unit disk or the unit ball and some operators on them, see, e.g., [1, 5, 6, 11, 13, 15, 18, 19] and the references therein.

When $n=2$, we call the space the Zygmund-type space on the unit disk and denote it by $\mathcal{Z}_\beta^\infty$. If $\beta=0$, it is called the Zygmund space. Recall that the Zygmund-type space consists of all $f\in H(\mathbb{D})$ such that

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|f''(z)|<\infty.$$

A natural norm on the Zygmund-type space can be defined as follows

$$\|f\|_{\mathcal{Z}_\beta^\infty}=|f(0)|+|f'(0)|+\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|f''(z)|.$$

With this norm the Zygmund-type space becomes a Banach space. We define

$$\mathcal{Z}_{\beta,0}^\infty=\{f\in\mathcal{Z}_\beta^\infty : f(0)=f'(0)=0\}.$$

Obviously, $\mathcal{Z}_{\beta,0}^\infty$ is a closed subspace of $\mathcal{Z}_\beta^\infty$. For Zygmund-type space on the unit disk or unit ball and some operators on it can be found, e.g., in [8, 9, 10, 14].

Let $\varphi:\mathbb{D}\to\mathbb{D}$ be an analytic self-map of $\mathbb{D}$ and $u\in H(\mathbb{D})$. For $f\in H(\mathbb{D})$, the weighted composition operator $W_{\varphi,u}$ is defined by

$$W_{\varphi,u}f(z)=u(z)f(\varphi(z)),\quad z\in\mathbb{D}.$$

By using the weighted composition operator $W_{\varphi,u}$, for $f\in H(\mathbb{D})$ we define the following two operators:

$$DW_{\varphi,u}f(z)=(u\cdot f\circ\varphi)'(z),\quad z\in\mathbb{D}$$

and

$$W_{\varphi,u}Df(z)=(u\cdot f'\circ\varphi)(z),\quad z\in\mathbb{D}.$$

If $u\equiv 1$ on $\mathbb{D}$, $W_{\varphi,1}=C_\varphi$ is called the composition operator. When $\varphi(z)=z$, $W_{z,u}=M_u$ is the multiplication operator. During the past few decades, weighted composition operators have been studied extensively on spaces of analytic functions on the unit disk or the unit ball. For some recent results on these operators, see [2, 4, 9, 11, 13, 15, 18, 19].

If $u \equiv 1$ on $\mathbb{D}$, then $DW_{\varphi,1} = DC_\varphi$ and $W_{\varphi,1}D = C_\varphi D$ are called the products of differentiation and composition. Hibschweiler and Portnoy [7] considered the behavior of the differentiation on the range of the composition operator on Hardy or weighted Bergman spaces on the unit disk. Recently, Ohno [12] has studied the products of differentiation and composition on Bloch and little Bloch spaces on the unit disk. A natural problem is to investigate bounded or compact operators $DW_{\varphi,u}$ and $W_{\varphi,u}D$ between two given spaces of analytic functions in terms of inducing symbols φ and u. Here we characterize bounded operators $DW_{\varphi,u}$ and $W_{\varphi,u}D$ from weighted Bergman space to Zygmund-type space, Bloch-type space and Bers-type space on the open unit disk. We also give some sufficient and necessary conditions for these operators to be compact operators.

Throughout this paper, constants are denoted by C, they are positive and may differ from one occurrence to the other. The notation $a \asymp b$ means that there is a positive constant C such that $a/C \le b \le Ca$.

2. Auxiliary Results

Here we quote and show several auxiliary results, which will be used in the proofs of main theorems.

The following lemma is also right for the bounded operator $W_{\varphi,u}D : A^p_\alpha \to \mathcal{Z}^\infty_\beta$. Since the proof is standard, it is omitted (see, e.g., Proposition 3.11 in [3]).

Lemma 2.1. *Suppose φ is an analytic self-map of $\mathbb{D}$, $u \in H(\mathbb{D})$ and the operator $DW_{\varphi,u} : A^p_\alpha \to \mathcal{Z}^\infty_\beta$ is bounded, then the operator $DW_{\varphi,u} : A^p_\alpha \to \mathcal{Z}^\infty_\beta$ is compact if and only if for bounded sequence $(f_n)_{n\in\mathbb{N}}$ in A^p_α such that $f_n \to 0$ uniformly on every compact subset of $\mathbb{D}$ as $n \to \infty$ it follows that*

$$\lim_{n\to\infty} \|DW_{\varphi,u} f_n\|_{\mathcal{Z}^\infty_\beta} = 0.$$

Lemma 2.2. ([17, Theorem 3.1]). *Suppose $z \in \mathbb{D}$, $\alpha > -1$, then for $t \ge 0$ we have*

$$\frac{1}{(1-|z|^2)^t} \asymp \int_{\mathbb{D}} \frac{(1-|w|^2)^\alpha}{|1-z\overline{w}|^{\alpha+2+t}} dA(w).$$

Lemma 2.3. *Suppose $p \ge 1$, $\alpha > -1$ and $w \in \mathbb{D}$, then for $t \ge 0$ the function*

$$f_{w,t}(z) = \frac{(1-|w|^2)^{t+\frac{\alpha+2}{p}}}{(1-z\overline{w})^{t+\frac{2\alpha+4}{p}}}$$

belongs to A^p_α *and* $\|f_{w,t}\|_{A^p_\alpha} \asymp 1$.

Proof. By an easy calculation and Lemma 2.2, we have

$$\begin{aligned}\|f_w\|^p_{A^p_\alpha} &= \int_{\mathbb{D}} |f_w(z)|^p (1-|z|^2)^\alpha dA(z)\\ &= (1-|w|^2)^{\alpha+2+pt} \int_{\mathbb{D}} \frac{(1-|z|^2)^\alpha}{|1-z\overline{w}|^{2\alpha+4+pt}} dA(z)\\ &\asymp 1.\end{aligned}$$

Lemma 2.4. ([17, Proposition 2.5]). *Suppose* $p \geq 1$, $\alpha > -1$, *then*

$$f(z) = (\alpha+1)\int_{\mathbb{D}} \frac{(1-|w|^2)^\alpha}{(1-z\overline{w})^{\alpha+2}} f(w)dA(w)$$

for $f \in A^p_\alpha$ *and* $z \in \mathbb{D}$.

When $n = 0$, the following lemma was proved in [16, Theorem 2.1]. Here by using the Jensen's inequality we prove it for each $n \in \mathbb{N}$.

Lemma 2.5. *Suppose* $p \geq 1$, $\alpha > -1$, *then there exists a positive constant* C *independent of* f *such that*

$$|f^{(n)}(z)| \leq C\frac{\|f\|_{A^p_\alpha}}{(1-|z|^2)^{n+\frac{\alpha+2}{p}}}. \tag{1}$$

Proof. For $f \in A^p_\alpha$, Lemma 2.4 shows that

$$f(z) = (\alpha+1)\int_{\mathbb{D}} \frac{(1-|w|^2)^\alpha}{(1-z\overline{w})^{\alpha+2}} f(w)dA(w) \tag{2}$$

for all $z \in \mathbb{D}$. Differentiating in (2) under the integral sign n times, we have

$$f^{(n)}(z) = C_\alpha \int_{\mathbb{D}} \frac{\overline{w}^n(1-|w|^2)^\alpha f(w)}{(1-z\overline{w})^{\alpha+n+2}} dA(w).$$

Then

$$|f^{(n)}(z)| \leq C_\alpha \int_{\mathbb{D}} \frac{(1-|w|^2)^\alpha |f(w)|}{|1-z\overline{w}|^{\alpha+n+2}} dA(w). \tag{3}$$

By Lemma 2.2, this implies

$$\frac{1}{(1-|z|^2)^n} \asymp \int_{\mathbb{D}} \frac{(1-|w|^2)^\alpha}{|1-z\overline{w}|^{\alpha+n+2}} dA(w). \tag{4}$$

From (4) and applying Jensen's inequality in (3) and an elementary inequality, it follows that

$$(5)\qquad \begin{aligned}(1-|z|^2)^{np}|f^{(n)}(z)|^p &\le C_\alpha^p(1-|z|^2)^n\int_{\mathbb{D}}\frac{(1-|w|^2)^\alpha|f(w)|^p}{|1-z\overline{w}|^{\alpha+n+2}}dA(w)\\ &\le C^p\frac{\|f\|_{A_\alpha^p}^p}{(1-|z|^2)^{\alpha+2}},\end{aligned}$$

where $C=2^{(\alpha+n+2)/p}C_\alpha$. From (5), we obtain the desired result.

3. The Operators $DW_{\varphi,u}:A_\alpha^p\to\mathcal{Z}_\beta^\infty$ and $W_{\varphi,u}D:A_\alpha^p\to\mathcal{Z}_\beta^\infty$

We first give the conditions for $DW_{\varphi,u}:A_\alpha^p\to\mathcal{Z}_\beta^\infty$ and $W_{\varphi,u}D:A_\alpha^p\to\mathcal{Z}_\beta^\infty$ to be bounded operators.

Theorem 3.1 *Suppose φ is an analytic self-map of $\mathbb{D}$ and $u\in H(\mathbb{D})$, then the operator $DW_{\varphi,u}:A_\alpha^p\to\mathcal{Z}_\beta^\infty$ is bounded if and only if the following conditions are satisfied:*

$$(i)\qquad \sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}|u'''(z)|<\infty;$$

$$(ii)\qquad \sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{1+\frac{\alpha+2}{p}}}\Big|3u''(z)\varphi'(z)+3u'(z)\varphi''(z)+u(z)\varphi'''(z)\Big|<\infty;$$

$$(iii)\qquad \sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{2+\frac{\alpha+2}{p}}}\Big|u'(z)\varphi'(z)^2+u(z)\varphi'(z)\varphi''(z)\Big|<\infty;$$

$$(iv)\qquad \sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{3+\frac{\alpha+2}{p}}}|u(z)||\varphi'(z)|^3<\infty.$$

Moreover, if the operator $DW_{\varphi,u}:A_\alpha^p\to\mathcal{Z}_{\beta,0}^\infty$ is bounded, then

$$(6)\qquad \begin{aligned}&\|DW_{\varphi,u}\|_{A_\alpha^p\to\mathcal{Z}_{\beta,0}^\infty}\\ &\asymp\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{1+\frac{\alpha+2}{p}}}\Big|3u''(z)\varphi'(z)+3u'(z)\varphi''(z)+u(z)\varphi'''(z)\Big|\\ &+\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{2+\frac{\alpha+2}{p}}}\Big|u'(z)\varphi'(z)^2+u(z)\varphi'(z)\varphi''(z)\Big|\\ &+\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{3+\frac{\alpha+2}{p}}}|u(z)||\varphi'(z)|^3+\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}|u'''(z)|.\end{aligned}$$

Proof. First suppose the operator $DW_{\varphi,u}:A_\alpha^p\to\mathcal{Z}_\beta^\infty$ is bounded. Take functions $f(z)=z$ and $f(z)=1$. Then

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\big|u'''(z)\varphi(z)+3u''(z)\varphi'(z)+3u'(z)\varphi''(z)+u(z)\varphi'''(z)\big| \\ \leq \|DW_{\varphi,u}z\|_{\mathcal{Z}_\beta^\infty} \leq C\|DW_{\varphi,u}\| \tag{7}$$

and

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|u'''(z)| \leq \|DW_{\varphi,u}1\|_{\mathcal{Z}_\beta^\infty} \leq C\|DW_{\varphi,u}\|. \tag{8}$$

By these facts and the boundedness of $\varphi(z)$,

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|3u''(z)\varphi'(z)+3u'(z)\varphi''(z)+u(z)\varphi'''(z)| \leq C\|DW_{\varphi,u}\|. \tag{9}$$

Taking functions $f(z)=z^2$ and $f(z)=z^3$, we have

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\big|u'''(z)\varphi(z)^2+6u''(z)\varphi'(z)\varphi(z) \\ +6u'(z)\varphi''(z)\varphi(z)+6u'(z)\varphi'(z)^2 \\ +6u(z)\varphi'(z)\varphi''(z)+2u(z)\varphi'''(z)\varphi(z)\big| \leq \|DW_{\varphi,u}z^2\|_{\mathcal{Z}_\beta^\infty} \leq C\|DW_{\varphi,u}\|, \tag{10}$$

and

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\big|u'''(z)\varphi(z)^3+9u''(z)\varphi'(z)\varphi(z)^2 \\ +18u'(z)\varphi(z)\varphi'(z)^2+6u(z)\varphi'(z)^3 \\ +18u(z)\varphi(z)\varphi'(z)\varphi''(z)+9u'(z)\varphi''(z)\varphi(z)^2+3u(z)\varphi(z)^2\varphi'''(z)\big| \\ \leq \|DW_{\varphi,u}z^3\|_{\mathcal{Z}_\beta^\infty} \leq C\|DW_{\varphi,u}\|. \tag{11}$$

By (8) and the boundedness of $\varphi(z)$,

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|u'''(z)||\varphi(z)|^2 \leq C\|DW_{\varphi,u}\|, \tag{12}$$

and

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|u'''(z)||\varphi(z)|^3 \leq C\|DW_{\varphi,u}\|. \tag{13}$$

From (9), (10), (12) and the boundedness of $\varphi(z)$, it follows that

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|u'(z)\varphi'(z)^2+u(z)\varphi'(z)\varphi''(z)| \leq C\|DW_{\varphi,u}\|. \tag{14}$$

Inequalities (9), (11), (13), (14) and the boundedness of $\varphi(z)$ imply that

$$\sup_{z\in\mathbb{D}}(1-|z|^2)|u(z)||\varphi'(z)|^3 \leq C\|DW_{\varphi,u}\|. \tag{15}$$

By Lemma 2.3 we know that for $w \in \mathbb{D}$ the function

$$f_w(z) = \frac{(1-|\varphi(w)|^2)^{\frac{\alpha+2}{p}}}{(1-z\overline{\varphi(w)})^{\frac{2\alpha+4}{p}}}$$

belongs to A^p_α and $\sup_{w\in\mathbb{D}} \|f_w\|_{A^p_\alpha} \le C$. Since $DW_{\varphi,u} : A^p_\alpha \to \mathcal{Z}^\infty_\beta$ is bounded,

$$\begin{aligned} C\|DW_{\varphi,u}\| &\ge \|DW_{\varphi,u}f_w\|_{\mathcal{Z}^\infty_\beta} \ge \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta \Big|u'''(z)f_w(\varphi(z)) \\ &\quad +3u''(z)f'_w(\varphi(z))\varphi'(z) + 3u'(z)f''_w(\varphi(z))\varphi'(z)^2 \\ &\quad +3u'(z)f'_w(\varphi(z))\varphi''(z) + 3u(z)f''_w(\varphi(z))\varphi'(z)\varphi''(z) \\ &\quad +u(z)f'''_w(\varphi(z))\varphi'(z)^3 + u(z)f'_w(\varphi(z))\varphi'''(z)\Big|. \end{aligned} \tag{16}$$

By the calculation and taking $z = w$, it follows that

$$\begin{aligned} C\|DW_{\varphi,u}\| &\ge (1-|z|^2)^\beta \Big| \frac{u'''(z)}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}} + 3l_1 \frac{\overline{\varphi(z)}u''(z)\varphi'(z)}{(1-|\varphi(z)|^2)^{1+\frac{\alpha+2}{p}}} \\ &\quad +3l_2 \frac{\overline{\varphi(z)}^2 u'(z)\varphi'(z)^2}{(1-|\varphi(z)|^2)^{2+\frac{\alpha+2}{p}}} + 3l_1 \frac{\overline{\varphi(z)}u'(z)\varphi''(z)}{(1-|\varphi(z)|^2)^{1+\frac{\alpha+2}{p}}} \\ &\quad +3l_2 \frac{\overline{\varphi(z)}^2 u(z)\varphi'(z)\varphi''(z)}{(1-|\varphi(z)|^2)^{2+\frac{\alpha+2}{p}}} + l_3 \frac{\overline{\varphi(z)}^3 u(z)\varphi'(z)^3}{(1-|\varphi(z)|^2)^{3+\frac{\alpha+2}{p}}} \\ &\quad +l_1 \frac{\overline{\varphi(z)}u(z)\varphi'''(z)}{(1-|\varphi(z)|^2)^{1+\frac{\alpha+2}{p}}} \Big|, \end{aligned} \tag{17}$$

where

$$l_1 = \frac{2\alpha+4}{p}, \quad l_2 = \frac{l_1(2\alpha+4+p)}{p} \text{ and } l_3 = \frac{l_1 l_2(2\alpha+4+2p)}{p}.$$

Then, by (17)

$$\begin{aligned} & l_1(1-|z|^2)^\beta \Big| 3\frac{\overline{\varphi(z)}u''(z)\varphi'(z)}{(1-|\varphi(z)|^2)^{1+\frac{\alpha+2}{p}}} + 3\frac{\overline{\varphi(z)}u'(z)\varphi''(z)}{(1-|\varphi(z)|^2)^{1+\frac{\alpha+2}{p}}} \\ & + \frac{\overline{\varphi(z)}u(z)\varphi'''(z)}{(1-|\varphi(z)|^2)^{1+\frac{\alpha+2}{p}}} \Big| \le C\|DW_{\varphi,u}\| + \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}|u'''(z)| \\ & +3l_2 \frac{(1-|z|^2)^\beta|\varphi(z)|^2}{(1-|\varphi(z)|^2)^{2+\frac{\alpha+2}{p}}} \times \Big|u'(z)\varphi'(z)^2 + u(z)\varphi'(z)\varphi''(z)\Big| \\ & +l_3 \frac{(1-|z|^2)^\beta|\varphi(z)|^3}{(1-|\varphi(z)|^2)^{3+\frac{\alpha+2}{p}}}|u(z)||\varphi'(z)|^3. \end{aligned} \tag{18}$$

For $w \in \mathbb{D}$, set

$$g_w(z) = \frac{2\alpha+4+3p}{\alpha+2}\frac{(1-|\varphi(w)|^2)^{\frac{\alpha+2}{p}}}{(1-z\overline{\varphi(w)})^{\frac{2\alpha+4}{p}}} + \frac{3(2\alpha+4+3p)}{2\alpha+4+2p}\frac{(1-|\varphi(w)|^2)^{2+\frac{\alpha+2}{p}}}{(1-z\overline{\varphi(w)})^{2+\frac{2\alpha+4}{p}}} + \frac{(1-|\varphi(w)|^2)^{3+\frac{\alpha+2}{p}}}{(1-z\overline{\varphi(w)})^{3+\frac{2\alpha+4}{p}}}.$$

It is easy to check that

$$g_w(\varphi(w)) = \frac{C}{(1-|\varphi(w)|)^{\frac{\alpha+2}{p}}},\ g'_w(\varphi(w)) = 0,\ g''_w(\varphi(w)) = 0 \text{ and } g'''_w(\varphi(w)) = 0.$$

By Lemma 2.3, we know that $g_w \in A^p_\alpha$ and $\sup_{w\in\mathbb{D}} \|g_w\|_{A^p_\alpha} < \infty$. From this, and since $DW_{\varphi,u} : A^p_\alpha \to \mathcal{Z}^\infty_\beta$ is bounded, we obtain

$$\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}|u'''(z)| \le \|DW_{\varphi,u}g_w\|_{\mathcal{Z}^\infty_\beta} \le C\|DW_{\varphi,u}\|. \tag{19}$$

For $w \in \mathbb{D}$, taking

$$h_w(z) = -\frac{(1-|\varphi(w)|^2)^{\frac{\alpha+2}{p}}}{(1-z\overline{\varphi(w)})^{\frac{2\alpha+4}{p}}} + 3\frac{(1-|\varphi(w)|^2)^{1+\frac{\alpha+2}{p}}}{(1-z\overline{\varphi(w)})^{1+\frac{2\alpha+4}{p}}} - 3\frac{(1-|\varphi(w)|^2)^{2+\frac{\alpha+2}{p}}}{(1-z\overline{\varphi(w)})^{2+\frac{2\alpha+4}{p}}} + \frac{(1-|\varphi(w)|^2)^{3+\frac{\alpha+2}{p}}}{(1-z\overline{\varphi(w)})^{3+\frac{2\alpha+4}{p}}},$$

we find

$$h_w(\varphi(w)) = 0, \quad h'_w(\varphi(w)) = 0, \quad h''_w(\varphi(w)) = 0,$$
$$h'''_w(\varphi(w)) = C\frac{\overline{\varphi(w)}^3}{(1-|\varphi(w)|)^{3+\frac{\alpha+2}{p}}},$$

and $h_w \in A^p_\alpha$ with $\sup_{w\in\mathbb{D}} \|h_w\|_{A^p_\alpha} < \infty$. Since $DW_{\varphi,u} : A^p_\alpha \to \mathcal{Z}^\infty_\beta$ is bounded,

$$\frac{(1-|z|^2)^\beta|\varphi(z)|^3}{(1-|\varphi(z)|^2)^{3+\frac{\alpha+2}{p}}}|u(z)||\varphi'(z)|^3 \le \|DW_{\varphi,u}h_w\|_{\mathcal{Z}^\infty_\beta} \le C\|DW_{\varphi,u}\|. \tag{20}$$

For $w \in \mathbb{D}$, define

$$k_w(z) = -\frac{2\alpha+4+3p}{2\alpha+4+2p}\frac{(1-|\varphi(w)|^2)^{\frac{\alpha+2}{p}}}{(1-z\overline{\varphi(w)})^{\frac{2\alpha+4}{p}}} + \frac{3\alpha+6+4p}{\alpha+2+p}\frac{(1-|\varphi(w)|^2)^{1+\frac{\alpha+2}{p}}}{(1-z\overline{\varphi(w)})^{1+\frac{2\alpha+4}{p}}} - \frac{6\alpha+12+7p}{2\alpha+4+2p}\frac{(1-|\varphi(w)|^2)^{2+\frac{\alpha+2}{p}}}{(1-z\overline{\varphi(w)})^{2+\frac{2\alpha+4}{p}}} + \frac{(1-|\varphi(w)|^2)^{3+\frac{\alpha+2}{p}}}{(1-z\overline{\varphi(w)})^{3+\frac{2\alpha+4}{p}}}.$$

Then

$$k_w(\varphi(w)) = 0, \quad k_w'(\varphi(w)) = 0, \quad k_w'''(\varphi(w)) = 0,$$
$$k_w''(\varphi(w)) = C \frac{\overline{\varphi(w)}^2}{(1-|\varphi(w)|)^{2+\frac{\alpha+2}{p}}},$$

and $k_w \in A^p_\alpha$ with $\sup_{w\in\mathbb{D}} \|k_w\|_{A^p_\alpha} < \infty$. The boundedness of $DW_{\varphi,u}$ implies that

$$\frac{(1-|z|^2)^\beta |\varphi(z)|^2}{(1-|\varphi(z)|^2)^{2+\frac{\alpha+2}{p}}} \left| u'(z)\varphi'(z)^2 + u(z)\varphi'(z)\varphi''(z) \right| \le C\|DW_{\varphi,u}\|. \tag{21}$$

Inequalities (18)-(21) imply that

$$\frac{(1-|z|^2)^\beta |\varphi(z)|}{(1-|\varphi(z)|^2)^{1+\frac{\alpha+2}{p}}} \left| 3u''(z)\varphi'(z) + 3u'(z)\varphi''(z) + u(z)\varphi'''(z) \right| \le C\|DW_{\varphi,u}\|, \tag{22}$$

i.e. we have

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta |\varphi(z)|}{(1-|\varphi(z)|^2)^{1+\frac{\alpha+2}{p}}} \left| 3u''(z)\varphi'(z) + 3u'(z)\varphi''(z) + u(z)\varphi'''(z) \right| \le C\|DW_{\varphi,u}\|.$$

Then for $\delta \in (0,1)$,

$$\sup_{\{z:|\varphi(z)|>\delta\}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{1+\frac{\alpha+2}{p}}} \left| 3u''(z)\varphi'(z) + 3u'(z)\varphi''(z) + u(z)\varphi'''(z) \right|$$
$$\le C\|DW_{\varphi,u}\|,$$

and by (9),

$$\sup_{\{z:|\varphi(z)|\le\delta\}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{1+\frac{\alpha+2}{p}}} \left| 3u''(z)\varphi'(z) + 3u'(z)\varphi''(z) + u(z)\varphi'''(z) \right|$$
$$\le C \sup_{\{z:|\varphi(z)|\le\delta\}} (1-|z|^2)^\beta \left| 3u''(z)\varphi'(z) + 3u'(z)\varphi''(z) + u(z)\varphi'''(z) \right| \le C\|DW_{\varphi,u}\|,$$

from which it follows that

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{1+\frac{\alpha+2}{p}}} \left| 3u''(z)\varphi'(z) + 3u'(z)\varphi''(z) + u(z)\varphi'''(z) \right| \le C\|DW_{\varphi,u}\|. \tag{23}$$

As the proof of inequality (23), by (15) and (20), we also have

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{3+\frac{\alpha+2}{p}}} |u(z)||\varphi'(z)|^3 \le C\|DW_{\varphi,u}\|. \tag{24}$$

Similarly, by (14) and (21),

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{2+\frac{\alpha+2}{p}}}\Big|u'(z)\varphi'(z)^2+u(z)\varphi'(z)\varphi''(z)\Big| \le C\|DW_{\varphi,u}\|. \tag{25}$$

Hence conditions (i)-(iv) hold.

Moreover, by (19), (23), (24) and (25),

$$\begin{aligned}
&\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}|u'''(z)| \\
&+\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{1+\frac{\alpha+2}{p}}}\Big|3u''(z)\varphi'(z)+3u'(z)\varphi''(z) \\
&+u(z)\varphi'''(z)\Big|+\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{2+\frac{\alpha+2}{p}}}\Big|u'(z)\varphi'(z)^2+u(z)\varphi'(z)\varphi''(z)\Big| \\
&+\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{3+\frac{\alpha+2}{p}}}|u(z)||\varphi'(z)|^3 \le C\|DW_{\varphi,u}\|.
\end{aligned} \tag{26}$$

By Lemma 2.5, for $f\in A^p_\alpha$ we have

$$\begin{aligned}
&\|DW_{\varphi,u}f\|_{\mathcal{Z}^\infty_\beta} \\
&=|(u\cdot f\circ\varphi)'(0)|+|(u\cdot f\circ\varphi)''(0)|+\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|(u\cdot f\circ\varphi)'''(z)| \\
&=|(u\cdot f\circ\varphi)'(0)|+|(u\cdot f\circ\varphi)''(0)|+\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\big|u'''(z)f(\varphi(z)) \\
&\quad+3u''(z)f'(\varphi(z))\varphi'(z)+3u'(z)f''(\varphi(z))\varphi'(z)^2+3u'(z)f'(\varphi(z))\varphi''(z) \\
&\quad+3u(z)f''(\varphi(z))\varphi'(z)\varphi''(z)+u(z)f'''(\varphi(z))\varphi'(z)^3+u(z)f'(\varphi(z))\varphi'''(z)\big| \\
&\le C\|f\|_{A^p_\alpha}\Big(1+\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{1+\frac{\alpha+2}{p}}}\Big|3u''(z)\varphi'(z) \\
&\quad+3u'(z)\varphi''(z)+u(z)\varphi'''(z)\Big| \\
&\quad+\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{2+\frac{\alpha+2}{p}}}\Big|u'(z)\varphi'(z)^2+u(z)\varphi'(z)\varphi''(z)\Big| \\
&\quad+\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{3+\frac{\alpha+2}{p}}}|u(z)||\varphi'(z)|^3+\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}|u'''(z)|\Big).
\end{aligned} \tag{27}$$

From conditions (i)-(iv) and (27), we conclude that $DW_{\varphi,u}:A^p_\alpha\to\mathcal{Z}^\infty_\beta$ is bounded. From (26) and (27), we also obtain the asymptotic relation (6).

Theorem 3.2 *Suppose φ is an analytic self-map of $\mathbb{D}$ and $u\in H(\mathbb{D})$, then the operator $W_{\varphi,u}D: A^p_\alpha\to\mathcal{Z}^\infty_\beta$ is bounded if and only if the following conditions are satisfied:*

$$(i)\qquad \sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{1+\frac{\alpha+2}{p}}}|u''(z)|<\infty;$$

$$(ii) \qquad \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{2+\frac{\alpha+2}{p}}}\big|2u'(z)\varphi'(z)+u(z)\varphi''(z)\big| < \infty;$$

$$(iii) \qquad \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{3+\frac{\alpha+2}{p}}}|u(z)| < \infty.$$

Moreover, if the operator $W_{\varphi,u}D : A^p_\alpha \to \mathcal{Z}^\infty_{\beta,0}$ *is bounded, then*

$$\begin{aligned} \|W_{\varphi,u}D\|_{A^p_\alpha\to\mathcal{Z}^\infty_{\beta,0}} &\asymp \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{1+\frac{\alpha+2}{p}}}|u''(z)| \\ &+ \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{3+\frac{\alpha+2}{p}}}|u(z)| \\ &+ \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{2+\frac{\alpha+2}{p}}}\big|2u'(z)\varphi'(z)+u(z)\varphi''(z)\big|. \end{aligned} \tag{28}$$

Proof. Suppose the operator $W_{\varphi,u}D : A^p_\alpha \to \mathcal{Z}^\infty_\beta$ is bounded. Taking functions $f(z) = z$ and $f(z) = z^2$, we have

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|u''(z)| \le \|W_{\varphi,u}Dz\|_{\mathcal{Z}^\infty_\beta} \le C\|W_{\varphi,u}D\| \tag{29}$$

and

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\big|u''(z)\varphi(z)+2u'(z)\varphi'(z)+u(z)\varphi''(z)\big| \le C\|W_{\varphi,u}D\|. \tag{30}$$

Combing these facts with the boundedness of $\varphi(z)$, we have

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|2u'(z)\varphi'(z)+u(z)\varphi''(z)| \le C\|W_{\varphi,u}D\|. \tag{31}$$

Let the function $f(z) = z^3$. Then

$$\begin{aligned} &\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\big|u''(z)\varphi(z)^2+2u(z)+4u'(z)\varphi'(z)\varphi(z)+2u(z)\varphi''(z)\varphi(z)\big| \\ &\le C\|W_{\varphi,u}D\|. \end{aligned} \tag{32}$$

By (29), (31) and (32),

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|u(z)| \le C\|W_{\varphi,u}D\|. \tag{33}$$

For $w \in \mathbb{D}$, by Lemma 2.3 we know that the function

$$f_w(z) = \frac{(1-|\varphi(w)|^2)^{\frac{\alpha+2}{p}}}{(1-z\overline{\varphi(w)})^{\frac{2\alpha+4}{p}}}$$

belongs to A^p_α and $\sup\limits_{w\in\mathbb{D}}\|f_w\|_{A^p_\alpha}\le C$. Since $W_{\varphi,u}D : A^p_\alpha\to\mathcal{Z}^\infty_\beta$ is bounded,

$$\begin{aligned}C\|W_{\varphi,u}D\| \ge \|W_{\varphi,u}Df_w\|_{\mathcal{Z}^\infty_\beta} \ge \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\big|u''(z)f'_w(\varphi(z))\\ +2u'(z)f''_w(\varphi(z))\varphi'(z)+u(z)f'''_w(\varphi(z))+u(z)f''_w(\varphi(z))\varphi''(z)\big|.\end{aligned}\tag{34}$$

Now write

$$l_1=\frac{2\alpha+4}{p},\quad l_2=\frac{l_1(2\alpha+4+p)}{p}\text{ and } l_3=\frac{l_1l_2(2\alpha+4+2p)}{p}.$$

Then

$$\begin{aligned}C\|W_{\varphi,u}D\| \ge (1-|z|^2)^\beta\bigg|l_1\frac{\overline{\varphi(z)}u''(z)}{(1-|\varphi(z)|^2)^{1+\frac{\alpha+2}{p}}}\\ +2l_2\frac{\overline{\varphi(z)^2}u'(z)\varphi'(z)}{(1-|\varphi(z)|^2)^{2+\frac{\alpha+2}{p}}}+l_3\frac{\overline{\varphi(z)}^3u(z)}{(1-|\varphi(z)|^2)^{3+\frac{\alpha+2}{p}}}\\ +l_2\frac{\overline{\varphi(z)}^2u(z)\varphi''(z)}{(1-|\varphi(z)|^2)^{2+\frac{\alpha+2}{p}}}\bigg|,\end{aligned}\tag{35}$$

In view of (35), we have

$$\begin{aligned}l_2\frac{(1-|z|^2)^\beta|\varphi(z)|^2}{(1-|\varphi(z)|^2)^{2+\frac{\alpha+2}{p}}}\Big|2u'(z)\varphi'(z)+u(z)\varphi''(z)\Big| \le C\|DW_{\varphi,u}\|\\ +\frac{(1-|z|^2)^\beta|u''(z)||\varphi(z)|}{(1-|\varphi(z)|^2)^{1+\frac{\alpha+2}{p}}}+l_3\frac{(1-|z|^2)^\beta|\varphi(z)|^3|u(z)|}{(1-|\varphi(z)|^2)^{3+\frac{\alpha+2}{p}}}.\end{aligned}\tag{36}$$

For $w\in\mathbb{D}$, set

$$\begin{aligned}g_w(z)=-\frac{(1-|\varphi(w)|^2)^{\frac{\alpha+2}{p}}}{(1-z\overline{\varphi(w)})^{\frac{2\alpha+4}{p}}}+3\frac{(1-|\varphi(w)|^2)^{1+\frac{\alpha+2}{p}}}{(1-z\overline{\varphi(w)})^{1+\frac{2\alpha+4}{p}}}-3\frac{(1-|\varphi(w)|^2)^{2+\frac{\alpha+2}{p}}}{(1-z\overline{\varphi(w)})^{2+\frac{2\alpha+4}{p}}}\\ +\frac{(1-|\varphi(w)|^2)^{3+\frac{\alpha+2}{p}}}{(1-z\overline{\varphi(w)})^{3+\frac{2\alpha+4}{p}}}.\end{aligned}$$

Then

$$g_w(\varphi(w))=0,\quad g'_w(\varphi(w))=0,\quad g''_w(\varphi(w))=0,$$

$$g'''_w(\varphi(w))=C\frac{\overline{\varphi(w)}^3}{(1-|\varphi(w)|)^{3+\frac{\alpha+2}{p}}},$$

and $g_w\in A^p_\alpha$ with $\sup_{w\in\mathbb{D}}\|g_w\|_{A^p_\alpha}<\infty$. Since $W_{\varphi,u}D : A^p_\alpha\to\mathcal{Z}^\infty_\beta$ is bounded,

$$\frac{(1-|z|^2)^\beta|\varphi(z)|^3}{(1-|\varphi(z)|^2)^{3+\frac{\alpha+2}{p}}}|u(z)| \le \|W_{\varphi,u}Dg_w\|_{\mathcal{Z}_\beta^\infty} \le C\|W_{\varphi,u}D\|. \tag{37}$$

For $w \in \mathbb{D}$, set

$$h_w(z) = \frac{2\alpha+4+3p}{\alpha+2}\frac{(1-|\varphi(w)|^2)^{\frac{\alpha+2}{p}}}{(1-z\overline{\varphi(w)})^{\frac{2\alpha+4}{p}}} + \frac{3(2\alpha+4+3p)}{2\alpha+4+2p}\frac{(1-|\varphi(w)|^2)^{2+\frac{\alpha+2}{p}}}{(1-z\overline{\varphi(w)})^{2+\frac{2\alpha+4}{p}}}$$
$$+\frac{(1-|\varphi(w)|^2)^{3+\frac{\alpha+2}{p}}}{(1-z\overline{\varphi(w)})^{3+\frac{2\alpha+4}{p}}}.$$

It is easy to see that

$$h_w(\varphi(w)) = 0,\ h'_w(\varphi(w)) = \frac{C\overline{\varphi(w)}}{(1-|\varphi(w)|)^{1+\frac{\alpha+2}{p}}},\ h''_w(\varphi(w)) = 0 \text{ and } h'''_w(\varphi(w)) = 0.$$

By Lemma 2.3, we know that $h_w \in A^p_\alpha$ and $\sup_{w\in\mathbb{D}} \|h_w\|_{A^p_\alpha} < \infty$. From this, and since $W_{\varphi,u}D : A^p_\alpha \to \mathcal{Z}^\infty_\beta$ is bounded, we obtain

$$\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{1+\frac{\alpha+2}{p}}}|u''(z)||\varphi(z)| \le \|W_{\varphi,u}Dh_w\|_{\mathcal{Z}_\beta^\infty} \le C\|W_{\varphi,u}D\|. \tag{38}$$

Inequalities (36), (37) and (38) imply that

$$\frac{(1-|z|^2)^\beta|\varphi(z)|^2}{(1-|\varphi(z)|^2)^{2+\frac{\alpha+2}{p}}}\Big|2u'(z)\varphi'(z)+u(z)\varphi''(z)\Big| \le C\|W_{\varphi,u}D\|, \tag{39}$$

i.e. we have

$$\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\beta|\varphi(z)|^2}{(1-|\varphi(z)|^2)^{2+\frac{\alpha+2}{p}}}\Big|2u'(z)\varphi'(z)+u(z)\varphi''(z)\Big| \le C\|W_{\varphi,u}D\|.$$

Then for $\delta \in (0,1)$,

$$\sup_{\{z:|\varphi(z)|>\delta\}}\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{2+\frac{\alpha+2}{p}}}\Big|2u'(z)\varphi'(z)+u(z)\varphi''(z)\Big| \le C\|W_{\varphi,u}D\|,$$

and by (31),

$$\sup_{\{z:|\varphi(z)|\le\delta\}}\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{1+\frac{\alpha+2}{p}}}\Big|2u'(z)\varphi'(z)+u(z)\varphi''(z)\Big| \le C\|W_{\varphi,u}D\|$$
$$\le C\sup_{\{z:|\varphi(z)|\le\delta\}}(1-|z|^2)^\beta\Big|2u'(z)\varphi'(z)+u(z)\varphi''(z)\Big| \le C\|W_{\varphi,u}D\|,$$

from which it follows that

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{2+\frac{\alpha+2}{p}}}\Big|2u'(z)\varphi'(z)+u(z)\varphi''(z)\Big| \le C\|W_{\varphi,u}D\|. \tag{40}$$

As the proof of inequality (40), by (33) and (37), we have

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{3+\frac{\alpha+2}{p}}}|u(z)| \le C\|W_{\varphi,u}D\|. \tag{41}$$

Similarly, by (29) and (38),

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{1+\frac{\alpha+2}{p}}}|u''(z)| \le C\|W_{\varphi,u}D\|. \tag{42}$$

Therefore, conditions (i)-(iv) hold.

Moreover, from (40), (41) and (42), it follows that

$$\begin{aligned}&\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{1+\frac{\alpha+2}{p}}}|u''(z)| \\ &+\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{2+\frac{\alpha+2}{p}}}\Big|2u'(z)\varphi'(z)+u(z)\varphi''(z)\Big| \\ &+\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{3+\frac{\alpha+2}{p}}}|u(z)| \le C\|W_{\varphi,u}D\|.\end{aligned} \tag{43}$$

By Lemma 2.5, for $f\in A^p_\alpha$ we have

$$\begin{aligned}&\|W_{\varphi,u}Df\|_{\mathcal{Z}^\infty_\beta} = |u(0)f'(\varphi(0))| + |(u\cdot f'\circ\varphi)'(0)| \\ &+\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\Big|u''(z)f'(\varphi(z)) \\ &+2u'(z)f''(\varphi(z))\varphi'(z)+u(z)f'''(\varphi(z))+u(z)f''(\varphi(z))\varphi''(z)\Big| \\ &\le C\|f\|_{A^p_\alpha}\Big(1+\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{1+\frac{\alpha+2}{p}}}|u''(z)| \\ &+\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{3+\frac{\alpha+2}{p}}}|u(z)| \\ &+\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{2+\frac{\alpha+2}{p}}}\Big|2u'(z)\varphi'(z)+u(z)\varphi''(z)\Big|\Big).\end{aligned} \tag{44}$$

From conditions (i)-(iv) and (44), we prove that $W_{\varphi,u}D : A^p_\alpha \to \mathcal{Z}^\infty_\beta$ is bounded. From (43) and (44), we also obtain the asymptotic relation (28).

The following theorem characterizes the compact operator $DW_{\varphi,u} : A^p_\alpha \to \mathcal{Z}^\infty_\beta$.

Theorem 3.3 *Suppose* φ *is an analytic self-map of* $\mathbb{D}$ *and* $u \in H(\mathbb{D})$, *then the bounded operator* $DW_{\varphi,u} : A^p_\alpha \to \mathcal{Z}^\infty_\beta$ *is compact if and only if the following conditions are satisfied:*

$$(i) \qquad \lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}} |u'''(z)| = 0;$$

$$(ii) \qquad \lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{1+\frac{\alpha+2}{p}}} \Big|3u''(z)\varphi'(z) + 3u'(z)\varphi''(z) + u(z)\varphi'''(z)\Big| = 0;$$

$$(iii) \qquad \lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{2+\frac{\alpha+2}{p}}} \Big|u'(z)\varphi'(z)^2 + u(z)\varphi'(z)\varphi''(z)\Big| = 0;$$

$$(iv) \qquad \lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{3+\frac{\alpha+2}{p}}} |u(z)||\varphi'(z)|^3 = 0.$$

Proof. Suppose conditions (i)-(iv) hold. To prove that the operator $DW_{\varphi,u} : A^p_\alpha \to \mathcal{Z}^\infty_\beta$ is compact, by Lemma 2.1, it is enough to prove that if $(f_n)_{n\in\mathbb{N}}$ is a bounded sequence in A^p_α such that $f_n \to 0$ uniformly on every compact subset of $\mathbb{D}$ as $n \to \infty$, then $\lim_{n\to\infty} \|DW_{\varphi,u} f_n\|_{\mathcal{Z}^\infty_\beta} = 0$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in A^p_α with $\sup_{n\in\mathbb{N}} \|f_n\|_{A^p_\alpha} \le M$ and $f_n \to 0$ uniformly on every compact subset of $\mathbb{D}$ as $n \to \infty$.

By the assumptions of the theorem, we have that for any $\varepsilon > 0$, there exits a constant $\delta \in (0,1)$ such that $\delta < |\varphi(z)| < 1$ implies that

$$\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}} |u'''(z)| < \varepsilon/4M, \tag{45}$$

$$\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{1+\frac{\alpha+2}{p}}} \Big|3u''(z)\varphi'(z) + 3u'(z)\varphi''(z) + u(z)\varphi'''(z)\Big| < \varepsilon/4M, \tag{46}$$

$$\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{2+\frac{\alpha+2}{p}}} \Big|u'(z)\varphi'(z)^2 + u(z)\varphi'(z)\varphi''(z)\Big| < \varepsilon/4M, \tag{47}$$

and

$$\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{3+\frac{\alpha+2}{p}}} |u(z)||\varphi'(z)|^3 < \varepsilon/4M. \tag{48}$$

Applying Lemma 2.5, this implies

$$
\begin{aligned}
&\|DW_{\varphi,u}f_n\|_{\mathcal{Z}_\beta^\infty}\\
&= \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|(u\cdot f_n\circ\varphi)'''(z)|+|(u\cdot f_n\circ\varphi)'(0)|+|(u\cdot f_n\circ\varphi)''(0)|\\
&\le \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\big|u'''(z)f_n(\varphi(z))\big|+\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|3u''(z)f_n'(\varphi(z))\varphi'(z)\\
&\quad +3u'(z)f_n'(\varphi(z))\varphi''(z)+u(z)f_n'(\varphi(z))\varphi'''(z)|\\
&\quad +\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|3u'(z)f_n''(\varphi(z))\varphi'(z)^2\\
&\quad +3u(z)f_n''(\varphi(z))\varphi'(z)\varphi''(z)|+\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\big|u(z)f_n'''(\varphi(z))\varphi'(z)^3\big|\\
&\quad +|(u\cdot f_n\circ\varphi)'(0)|+|(u\cdot f_n\circ\varphi)''(0)|\\
&\le \sup_{\{z\in\mathbb{D}:|\varphi(z)|\le\delta\}}(1-|z|^2)^\beta\big|u'''(z)f_n(\varphi(z))\big|\\
&\quad +\sup_{\{z\in\mathbb{D}:\delta\le|\varphi(z)|<1\}}(1-|z|^2)^\beta\big|u'''(z)f_n(\varphi(z))\big|\\
&\quad +\sup_{\{z\in\mathbb{D}:|\varphi(z)|\le\delta\}}(1-|z|^2)^\beta|3u''(z)f_n'(\varphi(z))\varphi'(z)\\
&\quad +3u'(z)f_n'(\varphi(z))\varphi''(z)+u(z)f_n'(\varphi(z))\varphi'''(z)|\\
&\quad +\sup_{\{z\in\mathbb{D}:\delta\le|\varphi(z)|<1\}}(1-|z|^2)^\beta|3u''(z)f_n'(\varphi(z))\varphi'(z)\\
&\quad +3u'(z)f_n'(\varphi(z))\varphi''(z)+u(z)f_n'(\varphi(z))\varphi'''(z)| \qquad (49)\\
&\quad +\sup_{\{z\in\mathbb{D}:|\varphi(z)|\le\delta\}}(1-|z|^2)^\beta|3u'(z)f_n''(\varphi(z))\varphi'(z)^2\\
&\quad +3u(z)f_n''(\varphi(z))\varphi'(z)\varphi''(z)|\\
&\quad +\sup_{\{z\in\mathbb{D}:\delta\le|\varphi(z)|<1\}}(1-|z|^2)^\beta|3u'(z)f_n''(\varphi(z))\varphi'(z)^2\\
&\quad +3u(z)f_n''(\varphi(z))\varphi'(z)\varphi''(z)|\\
&\quad +\sup_{\{z\in\mathbb{D}:|\varphi(z)|\le\delta\}}(1-|z|^2)^\beta\big|u(z)f_n'''(\varphi(z))\varphi'(z)^3\big|\\
&\quad +\sup_{\{z\in\mathbb{D}:\delta\le|\varphi(z)|<1\}}(1-|z|^2)^\beta\big|u(z)f_n'''(\varphi(z))\varphi'(z)^3\big|\\
&\quad +|(u\cdot f_n\circ\varphi)'(0)|+|(u\cdot f_n\circ\varphi)''(0)|\\
&\quad \le \|u'\|_{\mathcal{Z}_\beta^\infty}\sup_{\{z\in\mathbb{D}:|z|\le\delta\}}|f_n(z)|\\
&\quad +C\sup_{\{z\in\mathbb{D}:\delta\le|\varphi(z)|<1\}}\frac{(1-|z|^2)^\beta|u'''(z)|}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\|f_n\|_{A_\alpha^p}\\
&\quad +L_1\sup_{\{z\in\mathbb{D}:|z|\le\delta\}}|f_n'(z)|\\
&\quad +C\sup_{\{z\in\mathbb{D}:\delta\le|\varphi(z)|<1\}}\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{1+\frac{\alpha+2}{p}}}\Big|3u''(z)\varphi'(z)
\end{aligned}
$$

$$
\begin{aligned}
&+3u'(z)\varphi''(z)+u(z)\varphi'''(z)\Big|\|f_n\|_{A^p_\alpha}+L_2\sup_{\{z\in\mathbb{D}:|z|\le\delta\}}|f_n''(z)|\\
&+C\sup_{\{z\in\mathbb{D}:\delta\le|\varphi(z)|<1\}}\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{2+\frac{\alpha+2}{p}}}\Big|3u'(z)\varphi'(z)^2\\
&+3u(z)\varphi'(z)\varphi''(z)\Big|\|f_n\|_{A^p_\alpha}\\
&+L_3\sup_{\{z\in\mathbb{D}:|z|\le\delta\}}\left|f_n'''(z)\right|\\
&+C\sup_{\{z\in\mathbb{D}:\delta\le|\varphi(z)|<1\}}\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{3+\frac{\alpha+2}{p}}}|u(z)||\varphi'(z)|^3\|f_n\|_{A^p_\alpha}\\
\le{}&\|u'\|_{\mathcal{Z}^\infty_\beta}\sup_{\{z\in\mathbb{D}:|z|\le\delta\}}|f_n(z)|+C\varepsilon\\
&+L_1\sup_{\{z\in\mathbb{D}:|z|\le\delta\}}|f_n'(z)|+L_2\sup_{\{z\in\mathbb{D}:|z|\le\delta\}}|f_n''(z)|\\
&+L_3\sup_{\{z\in\mathbb{D}:|z|\le\delta\}}\left|f_n'''(z)\right|+|(u\cdot f_n\circ\varphi)'(0)|+|(u\cdot f_n\circ\varphi)''(0)|,
\end{aligned}
$$

where

$$
\begin{aligned}
L_1&=\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\Big|3u''(z)\varphi'(z)+3u'(z)\varphi''(z)+u(z)\varphi'''(z)\Big|,\\
L_2&=\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\Big|u'(z)\varphi'(z)^2+u(z)\varphi'(z)\varphi''(z)\Big|,
\end{aligned}
$$

and

$$
L_3=\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|u(z)||\varphi'(z)|^3.
$$

Since $\{z\in\mathbb{D}:|z|\le\delta\}$ is compact, it follows that $\sup_{\{z\in\mathbb{D}:|z|\le\delta\}}|f_n(z)|\to 0$. By Cauchy' estimate, if f_n converges to 0 on every compact subset of $\mathbb{D}$ as $n\to\infty$, then f_n', f_n'' and f_n''' converge to 0 on every compact subset of $\mathbb{D}$ as $n\to\infty$. Using these facts, we have $\lim_{n\to\infty}\|DW_{\varphi,u}f_n\|_{\mathcal{Z}^\infty_\beta}=0$. Therefore, $DW_{\varphi,u}:A^p_\alpha\to\mathcal{Z}^\infty_\beta$ is compact.

Now we suppose that $DW_{\varphi,u}:A^p_\alpha\to\mathcal{Z}^\infty_\beta$ is compact. Let $(z_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ with $|\varphi(z_n)|\to 1$ as $n\to\infty$. If such a sequence does not exist, conditions (i)-(iv) are automatically satisfied.

Take

$$
f_n(z)=\frac{(1-|\varphi(z_n)|^2)^{\frac{\alpha+2}{p}}}{(1-z\overline{\varphi(z_n)})^{\frac{2\alpha+4}{p}}}.
$$

Then $\sup_{n\in\mathbb{N}}\|f_n\|_{A^p_\alpha}\le C$ and $f_n\to 0$ uniformly on compacta of $\mathbb{D}$ as $n\to\infty$. Since $DW_{\varphi,u}:A^p_\alpha\to\mathcal{Z}^\infty_\beta$ is compact, $\|DW_{\varphi,u}f_n\|_{\mathcal{Z}^\infty_\beta}\to 0$ as $n\to\infty$. As the proof of Theorem 3.1, we have

$$
\begin{aligned}
&l_1(1-|z_n|^2)^\beta\Big|3\frac{\overline{\varphi(z_n)}u''(z_n)\varphi'(z_n)}{(1-|\varphi(z_n)|^2)^{1+\frac{\alpha+2}{p}}}+3\frac{\overline{\varphi(z_n)}u'(z_n)\varphi''(z_n)}{(1-|\varphi(z_n)|^2)^{1+\frac{\alpha+2}{p}}}\\
&+\frac{\overline{\varphi(z_n)}u(z)\varphi'''(z_n)}{(1-|\varphi(z_n)|^2)^{1+\frac{\alpha+2}{p}}}\Big|\le\|DW_{\varphi,u}f_n\|+\frac{(1-|z_n|^2)^\beta}{(1-|\varphi(z_n)|^2)^{\frac{\alpha+2}{p}}}|u'''(z_n)|\\
&+3l_2\frac{(1-|z_n|^2)^\beta|\varphi(z_n)|^2}{(1-|\varphi(z_n)|^2)^{2+\frac{\alpha+2}{p}}}\Big|u'(z_n)\varphi'(z_n)^2+u(z_n)\varphi'(z)\varphi''(z_n)\Big|\\
&+l_3\frac{(1-|z_n|^2)^\beta|\varphi(z_n)|^3}{(1-|\varphi(z)|^2_n)^{3+\frac{\alpha+2}{p}}}|u(z_n)||\varphi'(z)|^3.
\end{aligned}\tag{50}
$$

Set

$$
\begin{aligned}
g_n(z)=&\frac{2\alpha+4+3p}{\alpha+2}\frac{(1-|\varphi(z_n)|^2)^{\frac{\alpha+2}{p}}}{(1-z\overline{\varphi(z_n)})^{\frac{2\alpha+4}{p}}}+\frac{3(2\alpha+4+3p)}{2\alpha+4+2p}\frac{(1-|\varphi(z_n)|^2)^{2+\frac{\alpha+2}{p}}}{(1-z\overline{\varphi(z_n)})^{2+\frac{2\alpha+4}{p}}}\\
&+\frac{(1-|\varphi(z_n)|^2)^{3+\frac{\alpha+2}{p}}}{(1-z\overline{\varphi(z_n)})^{3+\frac{2\alpha+4}{p}}}.
\end{aligned}
$$

Then

$$
g_n(\varphi(z_n))=\frac{C}{(1-|\varphi(z_n)|)^{\frac{\alpha+2}{p}}},\ g_n'(\varphi(z_n))=0,\ g_n''(\varphi(z_n))=0 \text{ and } g_n'''(\varphi(z_n))=0.
$$

By Lemma 2.3, we know that $g_n\in A^p_\alpha$ with $\sup_{n\in\mathbb{N}}\|g_n\|_{A^p_\alpha}<\infty$ and $g_n\to 0$ uniformly on every compact subset of $\mathbb{D}$ as $n\to\infty$. From this, and since $DW_{\varphi,u}: A^p_\alpha\to\mathcal{Z}^\infty_\beta$ is compact, we obtain

$$
\frac{(1-|z_n|^2)^\beta}{(1-|\varphi(z_n)|^2)^{\frac{\alpha+2}{p}}}|u'''(z_n)|\le\|DW_{\varphi,u}g_n\|_{\mathcal{Z}^\infty_\beta}\to 0,\tag{51}
$$

as $n\to\infty$.

Taking

$$
\begin{aligned}
h_n(z)=&-\frac{(1-|\varphi(z_n)|^2)^{\frac{\alpha+2}{p}}}{(1-z\overline{\varphi(z_n)})^{\frac{2\alpha+4}{p}}}+3\frac{(1-|\varphi(z_n)|^2)^{1+\frac{\alpha+2}{p}}}{(1-z\overline{\varphi(z_n)})^{1+\frac{2\alpha+4}{p}}}-3\frac{(1-|\varphi(z_n)|^2)^{2+\frac{\alpha+2}{p}}}{(1-z\overline{\varphi(z_n)})^{2+\frac{2\alpha+4}{p}}}\\
&+\frac{(1-|\varphi(z_n)|^2)^{3+\frac{\alpha+2}{p}}}{(1-z\overline{\varphi(z_n)})^{3+\frac{2\alpha+4}{p}}},
\end{aligned}
$$

we have

$$
\begin{aligned}
h_n(\varphi(z_n)&=0,\quad h_n'(\varphi(z_n))=0,\quad h_n''(\varphi(z_n))=0,\\
h_n'''(\varphi(z_n))&=C\frac{\overline{\varphi(z_n)}^3}{(1-|\varphi(z_n)|)^{3+\frac{\alpha+2}{p}}},
\end{aligned}
$$

and $h_n \in A^p_\alpha$ with $\sup_{n\in\mathbb{N}} \|h_n\|_{A^p_\alpha} < \infty$ and $h_n \to 0$ uniformly on every compact subset of $\mathbb{D}$ as $n \to \infty$. Since $DW_{\varphi,u} : A^p_\alpha \to \mathcal{Z}^\infty_\beta$ is compact,

$$\frac{(1-|z_n|^2)^\beta|\varphi(z_n)|^3}{(1-|\varphi(z_n)|^2)^{3+\frac{\alpha+2}{p}}}|u(z_n)||\varphi'(z_n)|^3 \le \|DW_{\varphi,u}h_n\|_{\mathcal{Z}^\infty_\beta} \to 0, \tag{52}$$

as $n \to \infty$.

Define

$$\begin{aligned} k_n(z) = &-\frac{2\alpha+4+3p}{2\alpha+4+2p}\frac{(1-|\varphi(z_n)|^2)^{\frac{\alpha+2}{p}}}{(1-z\overline{\varphi(z_n)})^{\frac{2\alpha+4}{p}}} + \frac{3\alpha+6+4p}{\alpha+2+p}\frac{(1-|\varphi(z_n)|^2)^{1+\frac{\alpha+2}{p}}}{(1-z\overline{\varphi(z_n)})^{1+\frac{2\alpha+4}{p}}} \\ &-\frac{6\alpha+12+7p}{2\alpha+4+2p}\frac{(1-|\varphi(z_n)|^2)^{2+\frac{\alpha+2}{p}}}{(1-z\overline{\varphi(z_n)})^{2+\frac{2\alpha+4}{p}}} + \frac{(1-|\varphi(z_n)|^2)^{3+\frac{\alpha+2}{p}}}{(1-z\overline{\varphi(z_n)})^{3+\frac{2\alpha+4}{p}}}. \end{aligned}$$

Then

$$\begin{aligned} k_n(\varphi(z_n)) &= 0, \quad k_n'(\varphi(z_n)) = 0, \quad k_n'''(\varphi(z_n)) = 0, \\ k_n''(\varphi(z_n)) &= C\frac{\overline{\varphi(z_n)}^2}{(1-|\varphi(z_n)|)^{2+\frac{\alpha+2}{p}}}, \end{aligned}$$

and $k_n \in A^p_\alpha$ with $\sup_{n\in\mathbb{N}} \|k_n\|_{A^p_\alpha} < \infty$ and $k_n \to 0$ uniformly on every compact subset of $\mathbb{D}$ as $n \to \infty$. The compactness of $DW_{\varphi,u}$ implies that

$$\begin{aligned} &= \frac{(1-|z_n|^2)^\beta|\varphi(z_n)|^2}{(1-|\varphi(z_n)|^2)^{2+\frac{\alpha+2}{p}}}\Big|u'(z_n)\varphi'(z_n)^2 + u(z_n)\varphi'(z_n)\varphi''(z_n)\Big| \\ &=\le \|DW_{\varphi,u}k_n\|_{\mathcal{Z}^\infty_\beta} \to 0, \end{aligned} \tag{53}$$

as $n \to \infty$.

Since $|\varphi(z_n)| \to 1$ as $n \to \infty$, then by (51), (52) and (53) we prove that conditions (i), (ii) and (iv) hold. Moreover, inequalities (50), (51), (52) and (53) imply that

$$\begin{aligned} &l_1\tfrac{(1-|z_n|^2)^\beta|\varphi(z_n)|}{(1-|\varphi(z_n)|^2)^{1+\frac{\alpha+2}{p}}}\Big|3u''(z_n)\varphi'(z_n) + 3u'(z_n)\varphi''(z_n) + u(z)\varphi'''(z_n)\Big| \\ &\le \|DW_{\varphi,u}f_n\|_{\mathcal{Z}^\infty_\beta} + \|DW_{\varphi,u}g_n\|_{\mathcal{Z}^\infty_\beta} + \|DW_{\varphi,u}h_n\|_{\mathcal{Z}^\infty_\beta} + \|DW_{\varphi,u}k_n\|_{\mathcal{Z}^\infty_\beta} \to 0 \end{aligned}$$

as $n \to \infty$, from which, and since $|\varphi(z_n)| \to 1$ as $n \to \infty$, the condition (iii) is obtained. This completes the proof of the theorem.

Theorem 3.4 *Suppose φ is an analytic self-map of $\mathbb{D}$ and $u \in H(\mathbb{D})$, then the bounded operator $W_{\varphi,u}D : A^p_\alpha \to \mathcal{Z}^\infty_\beta$ is compact if and only if the following conditions are satisfied:*

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{1+\frac{\alpha+2}{p}}}|u''(z)| = 0; \tag{i}$$

$$(ii) \qquad \lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{2+\frac{\alpha+2}{p}}} \big|2u'(z)\varphi'(z)+u(z)\varphi''(z)\big| = 0;$$

$$(iii) \qquad \lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{3+\frac{\alpha+2}{p}}} |u(z)| = 0.$$

Proof. First assume that conditions (i)-(iii) hold. To prove that the operator $W_{\varphi,u}D : A^p_\alpha \to \mathcal{Z}^\infty_\beta$ is compact, by Lemma 2.1, it is enough to certify that if $(f_n)_{n\in\mathbb{N}}$ is a bounded sequence in A^p_α such that $f_n \to 0$ uniformly on every compact subset of $\mathbb{D}$ as $n \to \infty$, then $\lim_{n\to\infty} \|W_{\varphi,u}Df_n\|_{\mathcal{Z}^\infty_\beta} = 0$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in A^p_α with $\sup_{n\in\mathbb{N}} \|f_n\|_{A^p_\alpha} \le M$ and $f_n \to 0$ uniformly on every compact subset of $\mathbb{D}$ as $n \to \infty$.

By the assumptions of the theorem, we have that for any $\varepsilon > 0$, there exits a constant $\delta \in (0,1)$ such that $\delta < |\varphi(z)| < 1$ implies that

$$\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{1+\frac{\alpha+2}{p}}} |u''(z)| < \varepsilon/3M, \tag{54}$$

$$\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{2+\frac{\alpha+2}{p}}} \big|2u'(z)\varphi'(z)+u(z)\varphi''(z)\big| < \varepsilon/3M, \tag{55}$$

and

$$\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{3+\frac{\alpha+2}{p}}} |u(z)| < \varepsilon/3M. \tag{56}$$

Lemma 2.5 implies that

$$\begin{aligned}
\|W_{\varphi,u}Df_n\|_{\mathcal{Z}^\infty_\beta} &= \sup_{z\in\mathbb{D}} (1-|z|^2)^\beta |(u\cdot f_n'\circ\varphi)''(z)| \\
&\quad + |(u\cdot f_n'\circ\varphi)(0)| + |(u\cdot f_n'\circ\varphi)'(0)| \\
&\le \sup_{z\in\mathbb{D}} (1-|z|^2)^\beta \big|u''(z)f_n'(\varphi(z))\big| + \sup_{z\in\mathbb{D}} (1-|z|^2)^\beta \big|2u'(z)f_n''(\varphi(z))\varphi'(z) \\
&\quad + u(z)f_n''(\varphi(z))\varphi''(z)\big| + \sup_{z\in\mathbb{D}} (1-|z|^2)^\beta |u(z)f_n'''(\varphi(z))| \\
&\quad + |(u\cdot f_n'\circ\varphi)(0)| + |(u\cdot f_n'\circ\varphi)'(0)| \\
&\le \sup_{\{z\in\mathbb{D}:|\varphi(z)|\le\delta\}} (1-|z|^2)^\beta \big|u''(z)f_n'(\varphi(z))\big| \\
&\quad + \sup_{\{z\in\mathbb{D}:\delta\le|\varphi(z)|<1\}} (1-|z|^2)^\beta \big|u''(z)f_n'(\varphi(z))\big| \\
&\quad + \sup_{\{z\in\mathbb{D}:|\varphi(z)|\le\delta\}} (1-|z|^2)^\beta \big|2u'(z)f_n''(\varphi(z))\varphi'(z) + u(z)f_n''(\varphi(z))\varphi''(z)\big|
\end{aligned} \tag{57}$$

$$
\begin{aligned}
&+\sup_{\{z\in\mathbb{D}:\delta\le|\varphi(z)|<1\}}(1-|z|^2)^\beta\big|2u'(z)f_n''(\varphi(z))\varphi'(z)+u(z)f_n''(\varphi(z))\varphi''(z)\big|\\
&+\sup_{\{z\in\mathbb{D}:|\varphi(z)|\le\delta\}}(1-|z|^2)^\beta|u(z)f_n'''(\varphi(z))|\\
&+\sup_{\{z\in\mathbb{D}:\delta\le|\varphi(z)|<1\}}(1-|z|^2)^\beta|u(z)f_n'''(\varphi(z))|\\
&+|(u\cdot f_n'\circ\varphi)(0)|+|(u\cdot f_n'\circ\varphi)'(0)|\\
\le\ &\|u\|_{\mathcal{Z}_\beta^\infty}\sup_{\{z\in\mathbb{D}:|z|\le\delta\}}|f_n'(z)|\\
&+C\sup_{\{z\in\mathbb{D}:\delta\le|\varphi(z)|<1\}}\frac{(1-|z|^2)^\beta|u''(z)|}{(1-|\varphi(z)|^2)^{1+\frac{\alpha+2}{p}}}\|f_n\|_{A^p_\alpha}\\
&+L_1\sup_{\{z\in\mathbb{D}:|z|\le\delta\}}|f_n''(z)|\\
&+C\sup_{\{z\in\mathbb{D}:\delta\le|\varphi(z)|<1\}}\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{2+\frac{\alpha+2}{p}}}\big|2u'(z)\varphi'(z)+u(z)\varphi''(z)\big|\|f_n\|_{A^p_\alpha}\\
&+L_2\sup_{\{z\in\mathbb{D}:|z|\le\delta\}}|f_n'''(z)|+C\sup_{\{z\in\mathbb{D}:\delta\le|\varphi(z)|<1\}}\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{3+\frac{\alpha+2}{p}}}\big|u(z)\big|\|f_n\|_{A^p_\alpha}\\
\le\ &\|u\|_{\mathcal{Z}_\beta^\infty}\sup_{\{z\in\mathbb{D}:|z|\le\delta\}}|f_n'(z)|+C\varepsilon+L_1\sup_{\{z\in\mathbb{D}:|z|\le\delta\}}|f_n''(z)|+L_2\sup_{\{z\in\mathbb{D}:|z|\le\delta\}}|f_n'''(z)|\\
&+|(u\cdot f_n'\circ\varphi)(0)|+|(u\cdot f_n'\circ\varphi)'(0)|,
\end{aligned}
$$

where $L_1=\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\big|2u'(z)f_n''(\varphi(z))\varphi'(z)+u(z)f_n''(\varphi(z))\varphi''(z)\big|$ and $L_2=\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|u(z)|$. Since $\{z\in\mathbb{D}:|z|\le\delta\}$ is compact, it follows that $\sup_{\{z\in\mathbb{D}:|z|\le\delta\}}|f_n(z)|\to 0$. By Cauchy' estimate, if f_n converges to 0 on every compact subset of $\mathbb{D}$ as $n\to\infty$, then f_n', f_n'' and f_n''' converge to 0 on every compact subset of $\mathbb{D}$ as $n\to\infty$. Using these facts, we have $\lim_{n\to\infty}\|W_{\varphi,u}Df_n\|_{\mathcal{Z}_\beta^\infty}=0$, i.e. $W_{\varphi,u}D:A^p_\alpha\to\mathcal{Z}_\beta^\infty$ is compact.

Now we assume that $W_{\varphi,u}D:A^p_\alpha\to\mathcal{Z}_\beta^\infty$ is compact. Let $(z_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ with $|\varphi(z_n)|\to 1$ as $n\to\infty$. If such a sequence does not exist, conditions (i)-(iii) are automatically satisfied. Using the sequence $(z_n)_{n\in\mathbb{N}}$ we define

$$
f_n(z)=\frac{(1-|\varphi(z_n)|^2)^{\frac{\alpha+2}{p}}}{(1-z\overline{\varphi(z_n)})^{\frac{2\alpha+4}{p}}}.
$$

Then $\sup_{n\in\mathbb{N}}\|f_n\|_{A^p_\alpha}\le C$ and $f_n\to 0$ uniformly on compacta of $\mathbb{D}$ as $n\to\infty$. Since $W_{\varphi,u}D:A^p_\alpha\to\mathcal{Z}_\beta^\infty$ is compact, then $\|W_{\varphi,u}Df_n\|_{\mathcal{Z}_\beta^\infty}\to 0$ as $n\to\infty$. As the proof of Theorem 3.2, we have

$$
\begin{aligned}
&l_2\frac{(1-|z_n|^2)^\beta|\varphi(z_n)|^2}{(1-|\varphi(z_n)|^2)^{2+\frac{\alpha+2}{p}}}\Big|2u'(z_n)\varphi'(z)+u(z_n)\varphi''(z_n)\Big| \le \|W_{\varphi,u}Df_n\|\\
&+\frac{(1-|z_n|^2)^\beta|\varphi(z_n)|}{(1-|\varphi(z_n)|^2)^{1+\frac{\alpha+2}{p}}}|u''(z_n)|+l_3\frac{(1-|z_n|^2)^\beta|\varphi(z_n)|^3}{(1-|\varphi(z_n)|^2)^{3+\frac{\alpha+2}{p}}}|u(z_n)|.
\end{aligned}
\tag{58}
$$

Taking

$$
\begin{aligned}
&g_n(z)\\
&=-\frac{(1-|\varphi(z_n)|^2)^{\frac{\alpha+2}{p}}}{(1-z\overline{\varphi(z_n)})^{\frac{2\alpha+4}{p}}}+3\frac{(1-|\varphi(z_n)|^2)^{1+\frac{\alpha+2}{p}}}{(1-z\overline{\varphi(z_n)})^{1+\frac{2\alpha+4}{p}}}-3\frac{(1-|\varphi(z_n)|^2)^{2+\frac{\alpha+2}{p}}}{(1-z\overline{\varphi(z_n)})^{2+\frac{2\alpha+4}{p}}}\\
&\quad+\frac{(1-|\varphi(z_n)|^2)^{3+\frac{\alpha+2}{p}}}{(1-z\overline{\varphi(z_n)})^{3+\frac{2\alpha+4}{p}}},
\end{aligned}
\tag{59}
$$

then

$$
g_n(\varphi(z_n))=0,\quad g_n'(\varphi(z_n))=0,\quad g_n''(\varphi(z_n))=0,
$$

$$
g_n'''(\varphi(z_n))=C\frac{\overline{\varphi(z_n)}^3}{(1-|\varphi(z_n)|)^{3+\frac{\alpha+2}{p}}},
$$

and $g_n\in A^p_\alpha$ with $\sup_{n\in\mathbb{N}}\|g_w\|_{A^p_\alpha}<\infty$ and $g_n\to 0$ uniformly on compacta of $\mathbb{D}$ as $n\to\infty$. Since $W_{\varphi,u}D:A^p_\alpha\to\mathcal{Z}^\infty_\beta$ is compact, then

$$
\frac{(1-|z_n|^2)^\beta|\varphi(z_n)|^3}{(1-|\varphi(z_n)|^2)^{3+\frac{\alpha+2}{p}}}|u(z_n)|\le\|W_{\varphi,u}Dg_n\|_{\mathcal{Z}^\infty_\beta}\to 0
\tag{60}
$$

as $n\to\infty$.

Setting

$$
\begin{aligned}
&h_n(z)\\
&=\frac{2\alpha+4+3p}{\alpha+2}\frac{(1-|\varphi(z_n)|^2)^{\frac{\alpha+2}{p}}}{(1-z\overline{\varphi(z_n)})^{\frac{2\alpha+4}{p}}}+\frac{3(2\alpha+4+3p)}{2\alpha+4+2p}\frac{(1-|\varphi(z_n)|^2)^{2+\frac{\alpha+2}{p}}}{(1-z\overline{\varphi(z_n)})^{2+\frac{2\alpha+4}{p}}}\\
&\quad+\frac{(1-|\varphi(z_n)|^2)^{3+\frac{\alpha+2}{p}}}{(1-z\overline{\varphi(z_n)})^{3+\frac{2\alpha+4}{p}}},
\end{aligned}
\tag{61}
$$

we also have

$$
h_n(\varphi(z_n))=0,\ h_n'(\varphi(z_n))=\frac{C\overline{\varphi(z_n)}}{(1-|\varphi(w)|)^{1+\frac{\alpha+2}{p}}},\ h_n''(\varphi(z_n))=0 \text{ and } h_n'''(\varphi(z_n))=0.
$$

By Lemma 2.3, we know that $h_n\in A^p_\alpha$ and $\sup_{n\in\mathbb{N}}\|h_w\|_{A^p_\alpha}<\infty$. From this, and since $W_{\varphi,u}D:A^p_\alpha\to\mathcal{Z}^\infty_\beta$ is compact, we obtain

$$
\frac{(1-|z_n|^2)^\beta}{(1-|\varphi(z_n)|^2)^{1+\frac{\alpha+2}{p}}}|u''(z_n)||\varphi(z_n)|\le\|W_{\varphi,u}Dh_n\|_{\mathcal{Z}^\infty_\beta}\to 0
\tag{62}
$$

as $n \to \infty$.

From (60), (62) and since $|\varphi(z_n)| \to 1$ as $n \to \infty$, then conditions (i) and (iii) hold. Moreover, (58) (60), (62) and $|\varphi(z_n)| \to 1$ as $n \to \infty$ show

$$\begin{aligned} &l_2 \frac{(1-|z_n|^2)^\beta}{(1-|\varphi(z_n)|^2)^{2+\frac{\alpha+2}{p}}} \Big| 2u'(z_n)\varphi'(z) + u(z_n)\varphi''(z_n) \Big| \\ &\le \|W_{\varphi,u} Df_n\|_{\mathcal{Z}_\beta^\infty} + \|W_{\varphi,u} Dg_n\|_{\mathcal{Z}_\beta^\infty} + \|W_{\varphi,u} Dh_n\|_{\mathcal{Z}_\beta^\infty} \to 0 \end{aligned} \tag{63}$$

as $n \to \infty$. Therefore, the condition (ii) is obtained.

From Theorems 3.1-3.4, the following corollaries hold.

Corollary 3.5 *Suppose φ is an analytic self-map of $\mathbb{D}$, then the operator $DC_\varphi : A_\alpha^p \to \mathcal{Z}_\beta^\infty$ is bounded if and only if the following conditions are satisfied:*

$$(i) \qquad \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{1+\frac{\alpha+2}{p}}} |\varphi'''(z)| < \infty;$$

$$(ii) \qquad \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{2+\frac{\alpha+2}{p}}} |\varphi'(z)||\varphi''(z)| < \infty,$$

$$(iii) \qquad \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{3+\frac{\alpha+2}{p}}} |\varphi'(z)|^3 < \infty.$$

Moreover, if the operator $DC_\varphi : A_\alpha^p \to \mathcal{Z}_{\beta,0}^\infty$ is bounded, then

$$\begin{aligned} \|DC_\varphi\|_{A_\alpha^p \to \mathcal{Z}_{\beta,0}^\infty} \asymp &\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{1+\frac{\alpha+2}{p}}} |\varphi'''(z)| \\ &+ \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{2+\frac{\alpha+2}{p}}} |\varphi'(z)||\varphi''(z)| \\ &+ \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{3+\frac{\alpha+2}{p}}} |\varphi'(z)|^3. \end{aligned} \tag{64}$$

Corollary 3.6 *Suppose φ is an analytic self-map of $\mathbb{D}$, then the bounded operator $DC_\varphi : A_\alpha^p \to \mathcal{Z}_\beta^\infty$ is compact if and only if the following conditions are satisfied:*

$$(i) \qquad \lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{1+\frac{\alpha+2}{p}}} |\varphi'''(z)| = 0;$$

$$(ii) \qquad \lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{2+\frac{\alpha+2}{p}}} |\varphi'(z)||\varphi''(z)| = 0,$$

$$(iii) \qquad \lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{3+\frac{\alpha+2}{p}}}|\varphi'(z)|^3 = 0.$$

Corollary 3.7 *Suppose φ is an analytic self-map of $\mathbb{D}$, then the operator $C_\varphi D : A^p_\alpha \to \mathcal{Z}^\infty_\beta$ is bounded if and only if the following conditions are satisfied:*

$$(i) \qquad \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{2+\frac{\alpha+2}{p}}}\big|\varphi''(z)\big| < \infty;$$

$$(ii) \qquad \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{3+\frac{\alpha+2}{p}}} < \infty.$$

Moreover, if the operator $C_\varphi D : A^p_\alpha \to \mathcal{Z}^\infty_{\beta,0}$ is bounded, then

$$(65) \quad \|C_\varphi D\|_{A^p_\alpha\to\mathcal{Z}^\infty_{\beta,0}} \asymp \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{2+\frac{\alpha+2}{p}}}\big|\varphi''(z)\big| + \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{3+\frac{\alpha+2}{p}}}.$$

Corollary 3.8 *Suppose φ is an analytic self-map of $\mathbb{D}$, then the bounded operator $C_\varphi D : A^p_\alpha \to \mathcal{Z}^\infty_\beta$ is compact if and only if the following conditions are satisfied:*

$$(i) \qquad \lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{2+\frac{\alpha+2}{p}}}\big|\varphi''(z)\big| = 0;$$

$$(ii) \qquad \lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{3+\frac{\alpha+2}{p}}} = 0.$$

4. An Appendix

In this section, we formulate several results of the operator $DW_{\varphi,u}$. We also can give similar results of the operator $W_{\varphi,u}D$. Here we omit them.

Theorem 4.1 *Suppose φ is an analytic self-map of $\mathbb{D}$ and $u \in H(\mathbb{D})$, then the bounded operator $DW_{\varphi,u} : A^p_\alpha \to \mathcal{B}^\infty_\beta$ is bounded if and only if the following conditions are satisfied:*

$$(i) \qquad \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}|u''(z)| < \infty;$$

$$(ii) \qquad \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{1+\frac{\alpha+2}{p}}}\big|2u'(z)\varphi'(z) + u(z)\varphi''(z)\big| < \infty;$$

$$(iii) \qquad \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{2+\frac{\alpha+2}{p}}}|u(z)||\varphi'(z)|^2 < \infty.$$

Moreover, if the operator $DW_{\varphi,u}: A^p_\alpha \to \mathcal{B}^\infty_{\beta,0}$ *is bounded, then*

$$\begin{aligned} \|DW_{\varphi,u}\|_{A^p_\alpha\to\mathcal{B}^\infty_{\beta,0}} \asymp & \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}|u''(z)| \\ & + \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{2+\frac{\alpha+2}{p}}}|u(z)||\varphi'(z)|^2 \\ & + \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{1+\frac{\alpha+2}{p}}}\big|2u'(z)\varphi'(z)+u(z)\varphi''(z)\big|. \end{aligned} \tag{66}$$

Theorem 4.2 *Suppose* φ *is an analytic self-map of* $\mathbb{D}$ *and* $u \in H(\mathbb{D})$*, then the bounded bounded operator* $DW_{\varphi,u}: A^p_\alpha \to \mathcal{B}^\infty_\beta$ *is compact if and only if the following conditions are satisfied:*

$$(i) \qquad \lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}|u''(z)| = 0;$$

$$(ii) \qquad \lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{1+\frac{\alpha+2}{p}}}\big|2u'(z)\varphi'(z)+u(z)\varphi''(z)\big| = 0;$$

$$(iii) \qquad \lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{2+\frac{\alpha+2}{p}}}|u(z)||\varphi'(z)|^2 = 0.$$

Theorem 4.3 *Suppose* φ *is an analytic self-map of* $\mathbb{D}$ *and* $u \in H(\mathbb{D})$*, then the bounded operator* $DW_{\varphi,u}: A^p_\alpha \to \mathcal{A}^\infty_\beta$ *is bounded if and only if the following conditions are satisfied:*

$$(i) \qquad \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}|u'(z)| < \infty;$$

$$(ii) \qquad \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{1+\frac{\alpha+2}{p}}}\big|u(z)\big|\big|\varphi'(z)\big| < \infty.$$

Moreover,

$$\|DW_{\varphi,u}\|_{A^p_\alpha\to\mathcal{A}^\infty_\beta} \asymp \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta|u'(z)|}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}} + \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta\big|u(z)\big|\big|\varphi'(z)\big|}{(1-|\varphi(z)|^2)^{1+\frac{\alpha+2}{p}}}. \tag{67}$$

Theorem 4.4 *Suppose φ is an analytic self-map of $\mathbb{D}$ and $u \in H(\mathbb{D})$, then the bounded bounded operator $DW_{\varphi,u} : A^p_\alpha \to \mathcal{A}^\infty_\beta$ is compact if and only if the following conditions are satisfied:*

$$(i) \qquad \lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}} |u'(z)| = 0;$$

$$(ii) \qquad \lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{1+\frac{\alpha+2}{p}}} |u(z)||\varphi'(z)| = 0.$$

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