TAIWANESE JOURNAL OF MATHEMATICS
Vol. 15, No. 5, pp. 2169-2193, October 2011
This paper is available online at http://tjm.math.ntu.edu.tw/index.php/TJM

NONLINEAR PROJECTIONS AND GENERALIZED CONDITIONAL EXPECTATIONS IN BANACH SPACES

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Abstract. Let E be a smooth, strictly convex and reflexive Banach space, let C_* be a closed linear subspace of the dual space E^* of E and let Π_{C_*} be the generalized projection of E^* onto C_*. In this paper, we study the mapping R defined by $R = J^{-1}\Pi_{C_*}J$, where J is the normalized duality mapping from E into E^*. We obtain some results which are related to conditional expectations and martingales in the probability theory.

1. INTRODUCTION

Let E be a smooth Banach space and let E^* be the dual space of E. The function $\phi : E \times E \to \mathbf{R}$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$$

for each $x, y \in E$, where J is the normalized duality mapping from E into E^*. Let C be a nonempty closed convex subset of E and let T be a mapping from C into itself. Then, T is called generalized nonexpansive if the set $F(T)$ of fixed points of T is nonempty and

$$\phi(Tx, y) \le \phi(x, y)$$

for all $x \in C$ and $y \in F(T)$; see Ibaraki and Takahashi [16]. Such nonlinear operators are connected with resolvents of maximal monotone operators in Banach spaces. When E is a smooth, strictly convex and reflexive Banach space and C is a nonempty closed convex subset of E, Alber [2] also defined a nonlinear projection Π_C of E onto C called the generalized projection. Motivated by Alber [2] and Ibaraki and Takahashi [16], Kohsaka and Takahashi [27] proved the following result: Let E be a smooth, strictly convex and reflexive Banach space, let C_* be a

Received October 22, 2008, accepted June 3, 2010.
Communicated by Mau-Hsiang Shih.
2010 *Mathematics Subject Classification*: Primary 47H09; Secondary 54C15, 47N30.
Key words and phrases: Banach space, Nonexpansive retraction, Generalized nonexpansive retraction, Metric projection, Conditional expectation.

nonempty closed convex subset of E^* and let Π_{C_*} be the generalized projection of E^* onto C_*. Then the mapping R defined by $R = J^{-1}\Pi_{C_*}J$ is a sunny generalized nonexpansive retraction of E onto $J^{-1}C_*$. Such retractions are related to conditional expectations in the probability theory.

In this paper, motivated by Kohsaka and Takahashi [27], we study such retractions $R = J^{-1}\Pi_{C_*}J$ when C_* is a closed linear subspace of the dual space E^* of E. We first obtain some fundamental properties for such nonlinear retractions R. Next, we study a relation between such a nonlinear retraction R and the metric projection. Finally, we obtain convergence theorems which are related to martingales in the probability theory.

2. Preliminaries

Throughout this paper, we assume that a Banach space E with the dual space E^* is real. We denote by $\mathbf{N}$ and $\mathbf{R}$ the sets of all positive integers and all real numbers, respectively. We also denote by $\langle x, x^*\rangle$ the dual pair of $x \in E$ and $x^* \in E^*$. A Banach space E is said to be strictly convex if $\|x + y\| < 2$ for $x, y \in E$ with $\|x\| \leq 1$, $\|y\| \leq 1$ and $x \neq y$. A Banach space E is said to be uniformly convex if for any sequences $\{x_n\}$ and $\{y_n\}$ in E such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty} \|x_n + y_n\| = 2$, $\lim_{n\to\infty} \|x_n - y_n\| = 0$ holds. A Banach space E is said to be smooth provided

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in E$ with $\|x\| = \|y\| = 1$. Moreover, E is said to have a Fréchet differentiable norm if for each $x \in E$ with $\|x\| = 1$, this limit is attained uniformly for $y \in E$ with $\|y\| = 1$. E is said to have a uniformly Gâteaux differentiable norm if for each $y \in E$ with $\|y\| = 1$, this limit is attained uniformly for $x \in E$ with $\|x\| = 1$. Let E be a Banach space. With each $x \in E$, we associate the set

$$J(x) = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}.$$

The multivalued operator $J : E \to E^*$ is called the normalized duality mapping of E. From the Hahn-Banach theorem, $Jx \neq \emptyset$ for each $x \in E$. We know that E is smooth if and only if J is single-valued. If E is strictly convex, then J is one-to-one, i.e., $x \neq y \Rightarrow J(x) \cap J(y) = \emptyset$. If E is reflexive, then J is a mapping of E onto E^*. So, if E is reflexive, strictly convex and smooth, then J is single-valued, one-to-one and onto. In this case, the normalized duality mapping J_* from E^* into E is the inverse of J, that is, $J_* = J^{-1}$. If E has a Fréchet differentiable norm, then J is norm to norm continuous. If E has a uniformly Gâteaux differentiable norm, then J is norm to weak^* uniformly continuous on each bounded subset of E; see [32] for more details. Let E be a smooth Banach space and let J be the normalized duality mapping of E. We define the function $\phi : E \times E \to \mathbf{R}$ by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$$

for all $x, y \in E$. We also define the function $\phi_* : E^* \times E^* \to \mathbf{R}$ by

$$\phi_*(x^*, y^*) = \|x^*\|^2 - 2\langle x^*, J^{-1}y^*\rangle + \|y^*\|^2$$

for all $x^*, y^* \in E^*$. It is easy to see that $(\|x\| - \|y\|)^2 \leq \phi(x, y)$ for all $x, y \in E$. Thus, in particular, $\phi(x, y) \geq 0$ for all $x, y \in E$. We also know the following:

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy\rangle \tag{2.1}$$

for all $x, y, z \in E$. It is easy to see that

$$\phi(x, y) = \phi_*(Jy, Jx) \tag{2.2}$$

for all $x, y \in E$. If E is additionally assumed to be strictly convex, then

$$\phi(x, y) = 0 \Leftrightarrow x = y. \tag{2.3}$$

Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E. For an arbitrary point x of E, the set

$$\{z \in C : \phi(z, x) = \min_{y \in C} \phi(y, x)\}$$

is always nonempty and a singletone. Let us define the mapping Π_C of E onto C by $z = \Pi_C x$ for every $x \in E$, i.e.,

$$\phi(\Pi_C x, x) = \min_{y \in C} \phi(y, x)$$

for every $x \in E$. Such Π_C is called the generalized projection of E onto C; see Alber [2]. The following lemma is due to Alber [2] and Kamimura and Takahashi [24].

Lemma 2.1. ([2, 24]). *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E and let $(x, z) \in E \times C$. Then, the following hold:*

(a) $z = \Pi_C x$ *if and only if* $\langle y - z, Jx - Jz\rangle \leq 0$ *for all* $y \in C$;

(b) $\phi(z, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(z, x)$.

From this lemma, we can prove the following lemma.

Lemma 2.2. *Let M be a closed linear subspace of a smooth, strictly convex and reflexive Banach space E and let $(x, z) \in E \times M$. Then, $z = \Pi_M x$ if and only if*

$$\langle J(x) - J(z), m\rangle = 0 \text{ for any } m \in M.$$

The following lemmas are due to Kamimura and Takahashi [24] and Aoyama, Kohsaka and Takahashi [1].

Lemma 2.3. ([24]). *Let E be a smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be sequences of E such that $\{x_n\}$ or $\{y_n\}$ is bounded. Then, $\lim_n \phi(x_n, y_n) = 0$ implies that $\lim_n \|x_n - y_n\| = 0$.*

Lemma 2.4. ([1]). *Let E be a smooth and uniformly convex Banach space, let $\{s_n\}$ be a convergent sequence of real numbers, and let $\{x_n\}$ be a sequence of E such that*

$$\phi(x_n, x_m) \leq |s_n - s_m|$$

for all $m, n \in \mathbf{N}$. Then $\{x_n\}$ converges strongly.

Let D be a nonempty closed convex subset of a smooth Banach space E, let T be a mapping from D into itself and let $F(T)$ be the set of fixed points of T. Then, T is said to be generalized nonexpansive [16] if $F(T)$ is nonempty and $\phi(Tx, u) \leq \phi(x, u)$ for all $x \in D$ and $u \in F(T)$. Let C be a nonempty subset of E and let R be a mapping from E onto C. Then R is said to be a retraction if $R^2 = R$. It is known that if R is a retraction from E onto C, then $F(R) = C$. The mapping R is also said to be sunny if $R(Rx + t(x - Rx)) = Rx$ whenever $x \in E$ and $t \geq 0$. A nonempty subset C of a smooth Banach space E is said to be a generalized nonexpansive retract (resp. sunny generalized nonexpansive retract) of E if there exists a generalized nonexpansive retraction (resp. sunny generalized nonexpansive retraction) R from E onto C. The following lemmas were proved by Ibaraki and Takahashi [16].

Lemma 2.5. ([16]). *Let C be a nonempty closed subset of a smooth and strictly convex Banach space E and let R be a retraction from E onto C. Then, the following are equivalent:*

(a) *R is sunny and generalized nonexpansive;*

(b) *$\langle x - Rx, Jy - JRx \rangle \leq 0$ for all $(x, y) \in E \times C$.*

Lemma 2.6. ([16]). *Let C be a nonempty closed sunny and generalized nonexpansive retract of a smooth and strictly convex Banach space E. Then, the sunny generalized nonexpansive retraction from E onto C is uniquely determined.*

Lemma 2.7. ([16]). *Let C be a nonempty closed subset of a smooth and strictly convex Banach space E such that there exists a sunny generalized nonexpansive retraction R from E onto C and let $(x, z) \in E \times C$. Then, the following hold:*

(a) *$z = Rx$ if and only if $\langle x - z, Jy - Jz \rangle \leq 0$ for all $y \in C$;*

(b) *$\phi(Rx, z) + \phi(x, Rx) \leq \phi(x, z)$.*

Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E. For an arbitrary point x of E, the set

$$\{z \in C : \|z - x\| = \min_{y \in C} \|y - x\|\}$$

is always nonempty and a singletone. Let us define the mapping P_C of E onto C by $z = P_C x$ for every $x \in E$, i.e.,

$$\|P_C x - x\| = \min_{y \in C} \|y - x\|$$

for every $x \in E$. Such P_C is called the metric projection of E onto C; see [32]. The following lemma is in [32].

Lemma 2.8. ([32]). *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E and let $(x, z) \in E \times C$. Then, $z = P_C x$ if and only if $\langle y - z, J(x - z)\rangle \leq 0$ for all $y \in C$.*

An operator $A \subset E \times E^*$ with domain $D(A) = \{x \in E : Ax \neq \emptyset\}$ and range $R(A) = \cup\{Ax : x \in D(A)\}$ is said to be monotone if $\langle x - y, x^* - y^*\rangle \geq 0$ for any $(x, x^*), (y, y^*) \in A$. An operator A is said to be strictly monotone if $\langle x - y, x^* - y^*\rangle > 0$ for any $(x, x^*), (y, y^*) \in A$ $(x \neq y)$. A monotone operator A is said to be maximal if its graph $G(A) = \{(x, x^*) : x^* \in Ax\}$ is not properly contained in the graph of any other monotone operator. If A is maximal monotone, then the set $A^{-1}0 = \{u \in E : 0 \in Au\}$ is closed and convex (see [33] for more details). Let J be the normalized duality mapping from E into E^*. Then, J is monotone. If E is strictly convex, then J is one to one and strictly monotone. The following theorem is well-known; for instance, see [32].

Theorem 2.1. *Let E be a reflexive, strictly convex and smooth Banach space and let $A\colon E \to 2^{E^*}$ be a monotone operator. Then A is maximal if and only if $R(J + rA) = E^*$ for all $r > 0$. Further, if $R(J + A) = E^*$, then $R(J + rA) = E^*$ for all $r > 0$.*

3. Generalized Conditional Expectations

In this section, we discuss sunny generalized nonexpansive retractions which are connected with conditional expectations in the probability theory. We start with two theorems proved by Kohsaka and Takahashi [27].

Theorem 3.1. ([27]). *Let E be a smooth, strictly convex and reflexive Banach space, let C_* be a nonempty closed convex subset of E^* and let Π_{C_*} be the generalized projection of E^* onto C_*. Then the mapping R defined by $R = J^{-1}\Pi_{C_*}J$ is a sunny generalized nonexpansive retraction of E onto $J^{-1}C_*$.*

Theorem 3.2. ([27]). *Let E be a smooth, reflexive and strictly convex Banach space and let D be a nonempty subset of E. Then, the following conditions are equivalent.*

(1) *D is a sunny generalized nonexpansive retract of E;*

(2) *D is a generalized nonexpansive retract of E;*

(3) *JD is closed and convex.*

In this case, D is closed.

Motivated by these theorems, we define the following nonlinear operator: Let E be a reflexive, strictly convex and smooth Banach space and let J be the normalized duality mapping from E onto E^*. Let Y^* be a closed linear subspace of the dual space E^* of E. Then, the generalized conditional expectation E_{Y^*} with respect to Y^* is defined as follows:

$$E_{Y^*} := J^{-1}\Pi_{Y^*}J,$$

where Π_{Y^*} is the generalized projection from E^* onto Y^*. Such generalized conditional expectations are deeply connected with conditional expectations in the probability theory; see [14].

Lemma 3.1. *Let E be a reflexive, strictly convex and smooth Banach space. Let Y be a closed linear subspace of E and let Π_Y be the generalized projection of E onto Y. Then, the following hold.*

(1) *$\|\Pi_Y x\| \leq \|x\|$ for all $x \in E$;*

(2) *$\Pi_Y \alpha x = \alpha \Pi_Y x$ for any $x \in E$ and $\alpha \in \mathbf{R}$.*

Further, if X is a closed linear subspace of E such that $X \subset Y$, then

$$\Pi_X \Pi_Y = \Pi_X.$$

Proof. First, consider the function $\phi : E \times E \to \mathbf{R}$ defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle - \|y\|^2$$

for $x, y \in E$. From Lemma 2.1, we have that for any $x \in E$ and $y \in Y$,

$$\begin{aligned} 0 &\leq \phi(y, x) - \phi(y, \Pi_Y x) \\ &= \|y\|^2 - 2\langle y, Jx\rangle + \|x\|^2 - \|y\|^2 + 2\langle y, J\Pi_Y x\rangle - \|\Pi_Y x\|^2 \\ &= 2\langle y, J\Pi_Y x - Jx\rangle + \|x\|^2 - \|\Pi_Y x\|^2. \end{aligned}$$

Putting $y = 0 \in Y$, we obtain $0 \leq \|x\|^2 - \|\Pi_Y x\|^2$ and hence $\|\Pi_Y x\| \leq \|x\|$. This implies (1). Let us prove (2). It is obvious that $\Pi_Y 0 = 0$. From Lemma 2.2, we have that for any $x \in E$ and $\alpha \in \mathbf{R}$ with $\alpha \neq 0$,

$$\begin{aligned} y = \Pi_Y x &\Leftrightarrow \langle Jx - Jy, m\rangle = 0, \quad \forall m \in Y \\ &\Leftrightarrow \alpha\langle Jx - Jy, m\rangle = 0, \quad \forall m \in Y \\ &\Leftrightarrow \langle \alpha Jx - \alpha Jy, m\rangle = 0, \quad \forall m \in Y \\ &\Leftrightarrow \langle J\alpha x - J\alpha y, m\rangle = 0, \quad \forall m \in Y. \end{aligned}$$

Since Y is a closed linear subspace of E, we have $\alpha y \in Y$. Then we obtain $\alpha y = \Pi_Y \alpha x$ and hence $\alpha \Pi_Y x = \Pi_Y \alpha x$. This implies (2). Let X be a closed linear subspace of E such that $X \subset Y$ and let $x \in E$. From Lemma 2.2, we have $\langle Jx - J\Pi_Y x, y\rangle = 0$ for all $y \in Y$ and $\langle J\Pi_Y x - J\Pi_X \Pi_Y x, z\rangle = 0$ for all $z \in X$. Since $X \subset Y$, we have $\langle Jx - J\Pi_Y x, z\rangle = 0$ for all $z \in X$. Then we have

$$\begin{aligned} 0 &= \langle Jx - J\Pi_Y x + J\Pi_Y x - J\Pi_X \Pi_Y x, z\rangle \\ &= \langle Jx - J\Pi_X \Pi_Y x, z\rangle \end{aligned}$$

for all $z \in X$. From $\Pi_X \Pi_Y x \in X$ and the uniqueness of $\Pi_X x$, we obtain $\Pi_X \Pi_Y x = \Pi_X x$. This implies $\Pi_X \Pi_Y = \Pi_X$. ■

Using Lemma 3.1, we first prove the following theorem.

Theorem 3.3. . *Let E be a reflexive, strictly convex and smooth Banach space. Let Y^* be a closed linear subspace of the dual space E^*. Then, the generalized conditional expectation E_{Y^*} with respect to Y^* has the following properties:*

(1) *For any $x \in E$, $Jx \in Y^* \Leftrightarrow x = E_{Y^*}x$. In particular, $E_{Y^*}0 = 0$;*

(2) $\|E_{Y^*}x\| \le \|x\|$ *for all $x \in E$;*

(3) *For any $x \in E$, $\|E_{Y^*}x\| = \|x\| \Leftrightarrow Jx \in Y^*$;*

(4) $E_{Y^*}\alpha x = \alpha E_{Y^*}x$ *for all $x \in E$ and $\alpha \in \mathbf{R}$;*

(5) *For any $x_1, x_2 \in E$, $E_{Y^*}(x_1 + x_2) = E_{Y^*}(E_{Y^*}x_1 + E_{Y^*}x_2)$.*

Proof. Let J be the normalized duality mapping from E onto E^*. From the definition of E_{Y^*}, we obtain $JE_{Y^*}x = \Pi_{Y^*}Jx$ for all $x \in E$. Let $x \in E$. If $Jx \in Y^*$, then we have $\Pi_{Y^*}Jx = Jx$ and $J^{-1}\Pi_{Y^*}Jx = x$. Then x is a fixed point of E_{Y^*}. Conversely, if $x = E_{Y^*}x$, then $Jx = \Pi_{Y^*}Jx$. Then we have $Jx \in Y^*$. Since $0 = J0 \in Y^*$, we obtain $E_{Y^*}0 = 0$. This implies (1). From (1) in Lemma 3.1, we have $\|\Pi_{Y^*}x^*\| \le \|x^*\|$ for all $x^* \in E^*$. Then, we have

$$\|E_{Y^*}x\| = \|JE_{Y^*}x\| = \|\Pi_{Y^*}Jx\| \le \|Jx\| = \|x\|.$$

This implies (2). From Theorem 3.1, the generalized conditional expectation $E_{Y^*} = J^{-1}\Pi_{Y^*}J$ with respect to $Y^*$is a sunny generalized nonexpansive retraction from E onto $J^{-1}Y^*$. Since E_{Y^*} is sunny, we have that

$$E_{Y^*}\left(E_{Y^*}x + \beta\left(x - E_{Y^*}x\right)\right) = E_{Y^*}x$$

for any $x \in E$ and $\beta \ge 0$. So, we have from (2) that for any $x \in E$,

$$\begin{aligned}\|E_{Y^*}x\| &= \left\|E_{Y^*}\left(E_{Y^*}x + \frac{1}{2}(x - E_{Y^*}x)\right)\right\| \\ &\le \left\|E_{Y^*}x + \frac{1}{2}(x - E_{Y^*}x)\right\| \\ &= \left\|\frac{E_{Y^*}x + x}{2}\right\|.\end{aligned}$$

For $x \in E$, assume $\|E_{Y^*}x\| = \|x\|$. Then, we have

$$\|E_{Y^*}x\| = \|x\| = \left\|\frac{E_{Y^*}x + x}{2}\right\|.$$

Since E is strictly convex, we have $E_{Y^*}x = x$. From (1), we have $Jx \in Y^*$. Conversely, it is obvious that

$$Jx \in Y^* \Rightarrow E_{Y^*}x = x \Rightarrow \|E_{Y^*}x\| = \|x\|.$$

This implies (3). From (2) in Lemma 3.1, we have $\Pi_{Y^*}\alpha x_* = \alpha\Pi_{Y^*}x_*$ for all $x^* \in E^*$ and $\alpha \in \mathbf{R}$. Then we have

$$\begin{aligned}E_{Y^*}\alpha x &= J^{-1}\Pi_{Y^*}J\alpha x \\ &= J^{-1}\Pi_{Y^*}\alpha Jx \\ &= J^{-1}\alpha\Pi_{Y^*}Jx \\ &= \alpha J^{-1}\Pi_{Y^*}Jx = \alpha E_{Y^*}x.\end{aligned}$$

This implies (4). Since J_* is onto and one-to-one, for any $x_1, x_2 \in E$ there exist $x_1^*, x_2^* \in E^*$ such that $x_1 = J_*x_1^*$ and $x_2 = J_*x_2^*$, where J_* is J^{-1}. For $x_1^*, x_2^* \in E^*$, let $y_1^* = \Pi_{Y^*}x_1^*$ and $y_2^* = \Pi_{Y^*}x_2^*$. Further, put

$$z^* = \Pi_{Y^*}J_*^{-1}(J_*y_1^* + J_*y_2^*).$$

Then, from Lemma 2.2, we have that for any $m^* \in Y^*$,

$$\langle J_*x_1^* - J_*y_1^*, m^*\rangle = 0 \quad \textit{and} \quad \langle J_*x_2^* - J_*y_2^*, m^*\rangle = 0$$

and hence

$$\langle J_*x_1^* + J_*x_2^* - (J_*y_1^* + J_*y_2^*), m^*\rangle = 0. \tag{3.1}$$

We also have that for any $m^* \in Y^*$,

$$\langle J_*J_*^{-1}(J_*y_1^* + J_*y_2^*) - J_*z^*, m^*\rangle = 0.$$

This implies

$$\langle J_* y_1^* + J_* y_2^* - J_* z^*, m^* \rangle = 0. \tag{3.2}$$

Then, from (3.1) and (3.2) we have

$$\langle J_* x_1^* + J_* x_2^* - J_* z^*, m^* \rangle = 0.$$

So, we obtain

$$\langle J_* J_*^{-1}(J_* x_1^* + J_* x_2^*) - J_* z^*, m^* \rangle = 0$$

for all $m^* \in Y^*$. Since $z^* \in Y^*$, we obtain $z^* = \Pi_{Y^*} J_*^{-1}(J_* x_1^* + J_* x_2^*)$. Therefore,

$$z^* = \Pi_{Y^*} J_*^{-1}(J_* y_1^* + J_* y_2^*) = \Pi_{Y^*} J_*^{-1}(J_* x_1^* + J_* x_2^*).$$

Setting $y_1 = J_* y_1^*$ and $y_2 = J_* y_2^*$, we have that

$$\begin{aligned} y_1^* = \Pi_{Y^*} x_1^*,\ y_2^* = \Pi_{Y^*} x_2^* &\Leftrightarrow y_1 = J_* \Pi_{Y^*} J_*^{-1} x_1,\ y_2 = J_* \Pi_{Y^*} J_*^{-1} x_2 \\ &\Leftrightarrow y_1 = J^{-1} \Pi_{Y^*} J x_1,\ y_2 = J^{-1} \Pi_{Y^*} J x_2 \\ &\Leftrightarrow y_1 = E_{Y^*} x_1,\ y_2 = E_{Y^*} x_2. \end{aligned}$$

So, we have that

$$\begin{aligned} &\Pi_{Y^*} J_*^{-1}(J_* y_1^* + J_* y_2^*) = \Pi_{Y^*} J_*^{-1}(J_* x_1^* + J_* x_2^*) \\ &\Leftrightarrow \Pi_{Y^*} J(y_1 + y_2) = \Pi_{Y^*} J(x_1 + x_2) \\ &\Leftrightarrow J^{-1} \Pi_{Y^*} J(y_1 + y_2) = J^{-1} \Pi_{Y^*} J(x_1 + x_2) \\ &\Leftrightarrow E_{Y^*}(y_1 + y_2) = E_{Y^*}(x_1 + x_2). \end{aligned}$$

This implies that for any $x_1, x_2 \in E$,

$$E_{Y^*}(x_1 + x_2) = E_{Y^*}(E_{Y^*} x_1 + E_{Y^*} x_2).$$

This completes the proof of (5). ■

Corollary 3.1. *Let $n \in \mathbf{N}$ with $n \geq 2$ and let $x_1, x_2, \ldots, x_n \in E$. Then the following holds:*

$$E_{Y^*}\left(\sum_{i=1}^{n} x_i\right) = E_{Y^*}\left(\sum_{i=1}^{n} E_{Y^*} x_i\right). \tag{3.3}$$

Proof. For $n = 2$, it is obvious from Theorem 3.3. We suppose that for some $k \geq 2$,

$$E_{Y^*}\left(\sum_{i=1}^{k} x_i\right) = E_{Y^*}\left(\sum_{i=1}^{k} E_{Y^*} x_i\right).$$

Then we have

$$\begin{aligned} E_{Y^*}\left(\sum_{i=1}^{k+1} x_i\right) &= E_{Y^*}\left(\sum_{i=1}^{k} x_i + x_{k+1}\right) \\ &= E_{Y^*}\left(E_{Y^*}\sum_{i=1}^{k} x_i + E_{Y^*}x_{k+1}\right) \\ &= E_{Y^*}\left(E_{Y^*}\sum_{i=1}^{k} E_{Y^*}x_i + E_{Y^*}E_{Y^*}x_{k+1}\right) \\ &= E_{Y^*}E_{Y^*}\left(\sum_{i=1}^{k+1} E_{Y^*}x_i\right) = E_{Y^*}\left(\sum_{i=1}^{k+1} E_{Y^*}x_i\right). \end{aligned}$$

By mathematical induction, (3.3) holds. ■

Corollary 3.2. *Let E be a reflexive, strictly convex and smooth Banach space. Let Y^* be a closed linear subspace of the dual space E^*. Then the generalized conditional expectation E_{Y^*} with respect to Y^* is linear when*

$$y_1, y_2 \in J^{-1}Y^* \Rightarrow y_1 + y_2 \in J^{-1}Y^*.$$

Proof. For $x_1, x_2 \in E$, we know that $E_{Y^*}x_1, E_{Y^*}x_2 \in J^{-1}Y^*$. Since $E_{Y^*}x_1 + E_{Y^*}x_2 \in J^{-1}Y^*$, we have

$$E_{Y^*}(x_1 + x_2) = E_{Y^*}(E_{Y^*}x_1 + E_{Y^*}x_2) = E_{Y^*}x_1 + E_{Y^*}x_2.$$

From Theorem 3.3, we also have that $E_{Y^*}\alpha x = \alpha E_{Y^*}x$ for all $x \in E$ and $\alpha \in \mathbf{R}$. So, E_{Y^*} is linear. ■

Theorem 3.4. *Let E be a reflexive, strictly convex and smooth Banach space. Let Y_1^* and Y_2^* be closed linear subspaces of the dual space E^* such that $Y_1^* \subset Y_2^*$. Then, the following hold:*

(1) $(E_{Y_2^*})^{-1}0 \subset (E_{Y_1^*})^{-1}0$;
(2) $E_{Y_2^*}E_{Y_1^*} = E_{Y_1^*}$;
(3) $E_{Y_1^*}E_{Y_2^*} = E_{Y_1^*}$.

Proof. From $J0 = 0$, we have that for any $x \in E$,

$$x \in (E_{Y_2^*})^{-1}0 \Leftrightarrow E_{Y_2^*}x = 0 \Leftrightarrow J^{-1}\Pi_{Y_2^*}Jx = 0 \Leftrightarrow \Pi_{Y_2^*}Jx = 0.$$

Since $Y_1^* \subset Y_2^*$, from Lemma 3.1 we have $\Pi_{Y_1^*}Jx = \Pi_{Y_1^*}\Pi_{Y_2^*}Jx = \Pi_{Y_1^*}0 = 0$. So, we have

$$\Pi_{Y_1^*}Jx = 0 \Leftrightarrow J^{-1}\Pi_{Y_1^*}Jx = 0 \Leftrightarrow E_{Y_1^*}x = 0 \Leftrightarrow x \in (E_{Y_1^*})^{-1}0.$$

This implies $(E_{Y_2^*})^{-1}0 \subset (E_{Y_1^*})^{-1}0$. So, (1) holds. For $x \in E$, we have

$$E_{Y_1^*}x \in J^{-1}Y_1^* \subset J^{-1}Y_2^*.$$

So, we have

$$E_{Y_2^*}E_{Y_1^*}x = E_{Y_1^*}x.$$

This implies (2). From Lemma 3.1, we have that for any $x^* \in E^*$,

$$\Pi_{Y_1^*}\Pi_{Y_2^*}x^* = \Pi_{Y_1^*}x^*.$$

Then we have that for any $x \in E$,

$$\begin{aligned}\Pi_{Y_1^*}\Pi_{Y_2^*}Jx = \Pi_{Y_1^*}Jx &\Rightarrow J^{-1}\Pi_{Y_1^*}JJ^{-1}\Pi_{Y_2^*}Jx \\ &= J^{-1}\Pi_{Y_1^*}Jx \Rightarrow E_{Y_1^*}E_{Y_2^*}x \\ &= E_{Y_1^*}x.\end{aligned}$$

This implies (3). ■

4. Orthogonal Properties of E_{Y^*}

Let E be a normed linear space and let $x, y \in E$. We say that x is orthogonal to y in the sense of Birkhoff-James (or simply, x is BJ-orthogonal to y), denoted by $x \perp y$, if

$$\|x\| \le \|x + \lambda y\|$$

for all $\lambda \in \mathbf{R}$; see [5, 21, 22, 23]. We know that for $x, y \in E$, $x \perp y$ if and only if there exists $f \in J(x)$ with $\langle y, f\rangle = 0$; see [32]. In general, $x \perp y$ does not imply $y \perp x$. An operator T of E into itself is called left-orthogonal (resp. right-orthogonal) if for each $x \in E$, $Tx \perp (x - Tx)$ (resp. $(x - Tx) \perp Tx$); see [25].

Lemma 4.1. *Let E be a normed linear space and let T be an operator of E into itself such that*

$$T(Tx + \beta(x - Tx)) = Tx \tag{4.1}$$

for any $x \in E$ and $\beta \in \mathbf{R}$. Then, the following conditions are equivalent:

(1) $\|Tx\| \le \|x\|$ *for all $x \in E$;*

(2) T *is left-orthogonal.*

Proof. We prove $(1) \Rightarrow (2)$. Since $T(Tx + \beta(x - Tx)) = Tx$ for all $x \in E$ and $\beta \in \mathbf{R}$, we have

$$\begin{aligned} \|Tx\| &= \|T(Tx + \beta(x - Tx))\| \\ &\leq \|Tx + \beta(x - Tx)\| \end{aligned}$$

for any $x \in E$ and $\beta \in \mathbf{R}$. This implies that for each $x \in E$, $Tx \perp (x - Tx)$. Next, we prove $(2) \Rightarrow (1)$. Since T is left-orthogonal, we have

$$\|Tx\| \leq \|Tx + \lambda(x - Tx)\|$$

for any $x \in E$ and $\lambda \in \mathbf{R}$. When $\lambda = 1$, we obtain $\|Tx\| \leq \|x\|$. This completes the proof. ∎

Using Lemma 4.1, we can prove the following theorem.

Theorem 4.1. *Let E be a reflexive, strictly convex and smooth Banach space. Let Y^* be a closed linear subspace of the dual space E^*. Then, the generalized conditional expectation E_{Y^*} with respect to Y^* is left-orthogonal, i.e., for any $x \in E$,*

$$E_{Y^*}x \perp (x - E_{Y^*}x).$$

Proof. We show that E_{Y^*} satisfies (4.1). From the definition of Π_{Y^*}, we have that for any $x \in E$,

$$\langle J_*Jx - J_*\Pi_{Y^*}Jx, m^* \rangle = 0, \quad \forall m^* \in Y^*,$$

where J_* is the normalized duality mapping of E^* into E. This implies that

$$\langle x - E_{Y^*}x, m^* \rangle = 0, \quad \forall m^* \in Y^*. \tag{4.2}$$

Let $x_t = E_{Y^*}x + t(x - E_{Y^*}x)$ for $t \in \mathbf{R}$. Since $JE_{Y^*}x, JE_{Y^*}x_t \in Y^*$ and hence $JE_{Y^*}x - JE_{Y^*}x_t \in Y^*$, from (4.2) we have

$$\langle x_t - E_{Y^*}x_t, JE_{Y^*}x - JE_{Y^*}x_t \rangle = 0 \tag{4.3}$$

and

$$\langle x - E_{Y^*}x, JE_{Y^*}x - JE_{Y^*}x_t \rangle = 0.$$

Since $t(x - E_{Y^*}x) = x_t - E_{Y^*}x$, we have

$$\begin{aligned} 0 &= t\langle x - E_{Y^*}x, JE_{Y^*}x - JE_{Y^*}x_t \rangle \\ &= \langle t(x - E_{Y^*}x), JE_{Y^*}x - JE_{Y^*}x_t \rangle \\ &= \langle x_t - E_{Y^*}x, JE_{Y^*}x - JE_{Y^*}x_t \rangle. \end{aligned}$$

From this and (4.3) we obtain

$$\langle E_{Y^*}x - E_{Y^*}x_t, JE_{Y^*}x - JE_{Y^*}x_t\rangle = 0.$$

Since E is strictly convex, we have $E_{Y^*}x = E_{Y^*}x_t$, that is, E_{Y^*} satisfies (4.1). From Theorem 3.3 and Lemma 4.1, E_{Y^*} is left-orthogonal. ∎

Let Y be a nonempty subset of a Banach space E and let Y^* be a nonempty subset of the dual space E^*. Then, we define the annihilator $Y^*_\perp$ of Y^* and the annihilator $Y^\perp$ of Y as follows:

$$Y^*_\perp = \{x \in E : f(x) = 0 \text{ for all } f \in Y^*\}$$

and

$$Y^\perp = \{f \in E^* : f(x) = 0 \text{ for all } x \in Y\}.$$

The following theorem is related to Alber's result [3].

Theorem 4.2. *Let E be a reflexive, strictly convex and smooth Banach space and let I be the identity operator of E into itself. Let Y^* be a closed linear subspace of the dual space E^* and let E_{Y^*} be the generalized conditional expectation with respect to Y^*. Then, the mapping $I - E_{Y^*}$ is the metric projection of E onto $Y^*_\perp$. Conversely, let Y be a closed linear subspace of E and let P_Y be the metric projection of E onto Y. Then, the mapping $I - P_Y$ is the generalized conditional expectation $E_{Y^\perp}$ with respect to $Y^\perp$, i.e., $I - P_Y = E_{Y^\perp}$.*

Proof. Let $P = I - E_{Y^*}$. From the definition of Π_{Y^*}, we have that for any $x \in E$,

$$\begin{aligned} &\langle J_*Jx - J_*\Pi_{Y^*}Jx, m^*\rangle = 0, \quad \forall m^* \in Y^* \\ &\Leftrightarrow \langle x - E_{Y^*}x, m^*\rangle = 0, \quad \forall m^* \in Y^* \\ &\Leftrightarrow \langle Px, m^*\rangle = 0, \quad \forall m^* \in Y^*. \end{aligned}$$

Then, for any $x \in E$ we have $Px \in Y^*_\perp$. Since $JE_{Y^*}x \in Y^*$, from the definition of $Y^*_\perp$ we have that for any $x \in E$,

$$\langle JE_{Y^*}x, m\rangle = 0, \quad \forall m \in Y^*_\perp.$$

This implies that

$$\langle J(x - Px), m\rangle = 0, \quad \forall m \in Y^*_\perp. \tag{4.4}$$

We know that $Y^*_\perp$ is a closed linear subspace of E. Since $Px \in Y^*_\perp$, (4.4) means that the mapping P is the metric projection of E onto $Y^*_\perp$. Next, let $T = I - P_Y$. From the definition of P_Y we have that for any $x \in E$,

$$\begin{aligned} &\langle J(x - P_Yx), m\rangle = 0, \quad \forall m \in Y \\ &\Leftrightarrow \langle J(Tx), m\rangle = 0, \quad \forall m \in Y. \end{aligned}$$

Then, for any $x \in E$, we have $JTx \in Y^{\perp}$. Since $P_Y x \in Y$, from the definition of $Y^{\perp}$ we have that for any $x \in E$,

$$\langle P_Y x, m^* \rangle = 0, \quad \forall m^* \in Y^{\perp}.$$

This implies that

$$\langle x - Tx, m^* \rangle = 0, \quad \forall m^* \in Y^{\perp}. \tag{4.5}$$

Since the annihilator $Y^{\perp}$ is a closed linear subspace in E^*, from the definition of $E_{Y^{\perp}}$ we have that for $x \in E$,

$$y = E_{Y^{\perp}} x \Leftrightarrow Jy \in Y^{\perp} \text{ and } \langle x - y, m^* \rangle = 0, \quad \forall m^* \in Y^{\perp}.$$

In fact, if $y = E_{Y^{\perp}} x$, we have $Jy \in Y^{\perp}$ and

$$\langle x - y, m^* \rangle = \langle J^{-1}Jx - J^{-1}\Pi_{Y^{\perp}} Jx, m^* \rangle = 0, \quad \forall m^* \in Y^{\perp}.$$

Conversely, if $Jy \in Y^{\perp}$ and $\langle x - y, m^* \rangle = 0$ for all $m^* \in Y^{\perp}$, then we have that $Jy \in Y^{\perp}$ and $\langle J^{-1}Jx - J^{-1}Jy, m^* \rangle = 0$ for all $m^* \in Y^{\perp}$. This implies that $Jy = \Pi_{Y^{\perp}} Jx$ and hence $y = J^{-1}\Pi_{Y^{\perp}} Jx = E_{Y^{\perp}} x$. Since $JTx \in Y^{\perp}$ and $\langle x - Tx, m^* \rangle = 0$ for all $m^* \in Y^{\perp}$, we have $Tx = E_{Y^{\perp}} x$. So, we obtain $T = E_{Y^{\perp}}$ and hence $I - P_Y = E_{Y^{\perp}}$. ■

Let E be a normed linear space and let $Y_1, Y_2 \subset E$ be closed linear subspaces. If $Y_1 \cap Y_2 = \{0\}$ and for any $x \in E$ there exists a unique pair $y_1 \in Y_1, y_2 \in Y_2$ such that

$$x = y_1 + y_2,$$

and any element of Y_1 is BJ-orthogonal to any element of Y_2, i.e., $y_1 \perp y_2$ for any $y_1 \in Y_1, y_2 \in Y_2$, then we represent the space E as

$$E = Y_1 \oplus Y_2 \text{ and } Y_1 \perp Y_2.$$

For an operator T of E into itself, the kernel of T is denoted by $\ker(T)$, i.e.,

$$\ker(T) = \{x \in E : Tx = 0\}.$$

Using Theorem 4.2, we have the following theorem; see also [3, 4, 8, 25].

Theorem 4.3. *Let E be a strictly convex, reflexive and smooth Banach space and let Y^* be a closed linear subspace of the dual space E^* of E such that for any $y_1, y_2 \in J^{-1}Y^*$, $y_1 + y_2 \in J^{-1}Y^*$. Then, $J^{-1}Y^*$ is a closed linear subspace of E and the generalized conditional expectation E_{Y^*} with respect to Y^* is a norm one linear projection from E to $J^{-1}Y^*$. Further, the following hold:*

(1) $E = J^{-1}Y^* \oplus \ker(E_{Y^*})$ *and* $J^{-1}Y^* \perp \ker(E_{Y^*})$;

(2) $I - E_{Y^*}$ *is the metric projection of* E *onto* $\ker(E_{Y^*})$.

Proof. By the assumption, for any $y_1, y_2 \in J^{-1}Y^*$, $y_1+y_2 \in J^{-1}Y^*$. Further, for $y \in J^{-1}Y^*$ and $\alpha \in \mathbf{R}$, we have from $J\alpha y = \alpha Jy \in Y^*$ that $\alpha y \in J^{-1}Y^*$. So, $J^{-1}Y^*$ is a linear subspace of E. From Theorem 3.1, $J^{-1}Y^*$ is closed. Therefore, $J^{-1}Y^*$ is a closed linear subspace of E. For any $x, y \in E$, we have $E_{Y^*}x, E_{Y^*}y \in J^{-1}Y^*$. Since $J^{-1}Y^*$ is a linear subspace of E, we have $E_{Y^*}x + E_{Y^*}y \in J^{-1}Y^*$. From Theorem 3.3, we have that for any $x, y \in E$,

$$\begin{aligned} E_{Y^*}(x+y) &= E_{Y^*}(E_{Y^*}x + E_{Y^*}y) \\ &= E_{Y^*}x + E_{Y^*}y. \end{aligned}$$

So, E_{Y^*} is linear. Since $\|E_{Y^*}x\| \le \|x\|$ for all $x \in E$ and $\|E_{Y^*}y\| = \|y\|$ for all $y \in J^{-1}Y^*$, we have $\|E_{Y^*}\| = 1$. Then, E_{Y^*} is a norm one linear projection of E onto $J^{-1}Y^*$. Let us show (1) and (2). We first show that $E = J^{-1}Y^* \oplus Y^*_\perp$ and $J^{-1}Y^* \perp Y^*_\perp$. Note that $J^{-1}Y^* \cap Y^*_\perp = \{0\}$. In fact, let $u \in J^{-1}Y^* \cap Y^*_\perp$. Then, we have $Ju \in Y^*$ and $u \in Y^*_\perp$. So, we have $\|u\|^2 = \langle Ju, u\rangle = 0$. Then, $J^{-1}Y^* \cap Y^*_\perp = \{0\}$. Further, we have $J^{-1}Y^* \perp Y^*_\perp$. In fact, let $u \in J^{-1}Y^*$ and $v \in Y^*_\perp$. Then, $Ju \in Y^*$ and $v \in Y^*_\perp$. So, $\langle Ju, v\rangle = 0$. This implies $J^{-1}Y^* \perp Y^*_\perp$. Let us show $E = J^{-1}Y^* \oplus Y^*_\perp$. From the definition of Π_{Y^*}, we have that for any $x \in E$ and $m \in Y^*$,

$$\langle x - E_{Y^*}x, m\rangle = \langle J^{-1}Jx - J^{-1}\Pi_{Y^*}Jx, m\rangle = 0.$$

So, for any $x \in E$ we have $x - E_{Y^*}x \in Y^*_\perp$. We also have that for any $x \in E$,

$$x = E_{Y^*}x + x - E_{Y^*}x.$$

Further, since $E_{Y^*}x \in J^{-1}Y^*$, $x - E_{Y^*}x \in Y^*_\perp$ and $J^{-1}Y^* \cap Y^*_\perp = \{0\}$, we have

$$E = J^{-1}Y^* \oplus Y^*_\perp.$$

Finally, setting $Y = \{x - E_{Y^*}x : x \in E\}$, we shall show $Y = \ker(E_{Y^*}) = Y^*_\perp$. For any $x \in E$, we have $E_{Y^*}(x - E_{Y^*}x) = E_{Y^*}x - E_{Y^*}E_{Y^*}x = E_{Y^*}x - E_{Y^*}x = 0$. So, we have $Y \subset \ker(E_{Y^*})$. Conversely, for any $y \in \ker(E_{Y^*})$ we have $E_{Y^*}y = 0$ and hence $y = y - E_{Y^*}y \in Y$. So, we have $\ker(E_{Y^*}) \subset Y$. Then $\ker(E_{Y^*}) = Y$. Next, we show $\ker(E_{Y^*}) = Y^*_\perp$. Since for any $x \in E$, $x - E_{Y^*}x \in Y^*_\perp$, we have $\ker(E_{Y^*}) = Y \subset Y^*_\perp$. Conversely, for any $y \in Y^*_\perp$, we have $y = E_{Y^*}y + y - E_{Y^*}y = 0 + y$. Since $E_{Y^*}y, 0 \in J^{-1}Y^*$, $y - E_{Y^*}y, y \in Y^*_\perp$ and $E = J^{-1}Y^* \oplus Y^*_\perp$, we have $E_{Y^*}y = 0$. This implies $Y^*_\perp \subset \ker(E_{Y^*})$. So, we have $Y^*_\perp = \ker(E_{Y^*})$. This completes the proof of (1). From Theorem 4.2 , we know that $I - E_{Y^*}$ is the metric projection of E onto $Y^*_\perp$. Since $Y^*_\perp = \ker(E_{Y^*})$, we have (2). This completes the proof. ∎

5. Orthogonal Maximal Monotone Operators

Let E be a Banach space and let $B \subset E^* \times E$ be a maximal monotone operator with domain $D(B)$ and range $R(B)$. Then, B is said to be orthogonal if

$$\langle x^*, x\rangle = 0 \text{ for any } x^* \in D(B) \text{ and } x \in Bx^*.$$

Theorem 5.1. *Let E be a reflexive, strictly convex and smooth Banach space. Let Y^* be a closed linear subspace of the dual space E^* and let E_{Y^*} be the generalized conditional expectation with respect to Y^*. Then, the operator $B \subset E^* \times E$ defined by*

$$B = \{(JE_{Y^*}x, x - E_{Y^*}x) : x \in E\}$$

is orthogonal maximal monotone.

Proof. We first show that B is monotone. We have that for any $x, y \in E$,

$$\begin{aligned}
&\langle J(E_{Y^*}x) - J(E_{Y^*}y), x - E_{Y^*}x - (y - E_{Y^*}y)\rangle \\
&\qquad = \langle JJ^{-1}\Pi_{Y^*}Jx - JJ^{-1}\Pi_{Y^*}Jy, x - J^{-1}\Pi_{Y^*}Jx - (y - J^{-1}\Pi_{Y^*}Jy)\rangle \\
&\qquad = \langle \Pi_{Y^*}Jx - \Pi_{Y^*}Jy, x - J^{-1}\Pi_{Y^*}Jx - (y - J^{-1}\Pi_{Y^*}Jy)\rangle \\
&\qquad = \langle \Pi_{Y^*}Jx - \Pi_{Y^*}Jy, J^{-1}Jx - J^{-1}\Pi_{Y^*}Jx - (J^{-1}Jy - J^{-1}\Pi_{Y^*}Jy)\rangle \\
&\qquad = \langle \Pi_{Y^*}Jx - \Pi_{Y^*}Jy, J^{-1}Jx - J^{-1}\Pi_{Y^*}Jx\rangle \\
&\qquad + \langle \Pi_{Y^*}Jx - \Pi_{Y^*}Jy, -(J^{-1}Jy - J^{-1}\Pi_{Y^*}Jy)\rangle \\
&\qquad \geq 0.
\end{aligned}$$

So, B is monotone. Next, we show the maximality of B. Since

$$\begin{aligned}
\{J^{-1}JE_{Y^*}x + BJE_{Y^*}x : x \in E\} &= \{E_{Y^*}x + x - E_{Y^*}x) : x \in E\} \\
&= \{x : x \in E\} = E,
\end{aligned}$$

we have $R(J^{-1} + B) = E$. Then, from Theorem 2.1 the operator B is maximal. For any $x \in E$, we know that $JE_{Y^*}x \in Y^*$. Further, we have that for any $x \in E$ and $m \in Y^*$,

$$\langle x - E_{Y^*}x, m\rangle = \langle J^{-1}Jx - J^{-1}\Pi_{Y^*}Jx, m\rangle = 0.$$

So, for any $x \in E$ we have $x - E_{Y^*}x \in Y^*_\perp$. Then, we have that for any $x \in E$, $\langle JE_{Y^*}x, x - E_{Y^*}x\rangle = 0$. Therefore, the operator B is orthogonal. ■

If $B \subset E^* \times E$ is a maximal monotone operator with domain $D(B)$ and range $R(B)$, then for $\lambda > 0$ and $x \in E^*$, we can define the resolvent $J_\lambda x$ of B as follows:

$$J_\lambda x = \{y \in E : x \in y + \lambda BJy\}.$$

We know from Ibaraki and Takahashi [16] that $J_\lambda : E \to E$ is a single valued mapping. So, we call J_λ the generalized resolvent of B for $\lambda > 0$. We also denote the resolvent J_λ by

$$J_\lambda = (I + \lambda BJ)^{-1}.$$

We know that $D(J_\lambda) = R(I + \lambda BJ)$ and $R(J_\lambda) = D(BJ)$.

Theorem 5.2. *Let E be a reflexive, strictly convex and smooth Banach space and let $B \subset E^* \times E$ be an orthogonal maximal monotone operator. For $\lambda > 0$, let J_λ be the generalized resolvent of B for $\lambda > 0$. Then, the following properties hold:*

(1) $\|J_\lambda x\| \le \|x\|$ *for any* $x \in E$,

(2) $0 \in (BJ)^{-1}0$,

(3) J_λ *is left-orthogonal, i.e.,* $J_\lambda x \perp (x - J_\lambda x)$ *for all* $x \in E$.

Proof. Let $\lambda > 0$. Since B is maximal, then $R(J^{-1} + \lambda B) = E$. So, for any $y \in E$, there exists $x^* \in E^*$ such that $y \in J^{-1}x^* + \lambda Bx^*$. From assumptions of E, for such $x^* \in E^*$ there exists an exactly one element $x \in E$ such that $x^* = Jx$. We also know that

$$\begin{aligned} y \in J^{-1}x^* + \lambda Bx^* &\Leftrightarrow y \in J^{-1}Jx + \lambda BJx \\ &\Leftrightarrow y \in x + \lambda BJx. \end{aligned}$$

Then, for any $y \in E$, there exists $x \in E$ such that $y \in x + \lambda BJx$. We obtain $D(J_\lambda) = E$. So, we have that for any $x \in E$,

$$\begin{aligned} x \in (BJ)^{-1}0 &\Leftrightarrow 0 \in BJx \\ &\Leftrightarrow 0 \in \lambda BJx \\ &\Leftrightarrow x \in x + \lambda BJx \\ &\Leftrightarrow x = J_\lambda x. \end{aligned}$$

For any $x \in E$ and $\lambda > 0$, we have from definition of $J_\lambda x$ that $x \in J_\lambda x + \lambda BJJ_\lambda x$. So, we have that for $z \in BJJ_\lambda x$ with $x = J_\lambda x + \lambda z$,

$$\begin{aligned} \|J_\lambda x\|\|x\| &\ge \langle JJ_\lambda x, x\rangle \\ &= \langle JJ_\lambda x, J_\lambda x + \lambda z\rangle \\ &= \langle JJ_\lambda x, J_\lambda x\rangle + \lambda\langle JJ_\lambda x, z\rangle \\ &= \langle JJ_\lambda x, J_\lambda x\rangle = \|J_\lambda x\|^2. \end{aligned}$$

Then, we have $\|J_\lambda x\| \le \|x\|$. This implies (1). Since $0 \in D(J_\lambda)$, we have $\|J_\lambda 0\| = 0$. So, we obtain $J_\lambda 0 = 0$ and $0 \in (BJ)^{-1}0$. This implies (2). Let us show (3). We have

$$\begin{aligned}\langle JJ_\lambda x, x - J_\lambda x\rangle &= \langle JJ_\lambda x, x\rangle - \langle J_\lambda x, JJ_\lambda x\rangle \\ &= \|J_\lambda x\|^2 - \|J_\lambda x\|^2 = 0.\end{aligned}$$

So, we obtain that for $\beta \in \mathbf{R}$,

$$\begin{aligned}\|J_\lambda x\|\|J_\lambda x + \beta(x - J_\lambda x)\| &\geq \langle JJ_\lambda x, J_\lambda x + \beta(x - J_\lambda x)\rangle \\ &= \|J_\lambda x\|^2 + \beta\langle JJ_\lambda x, x - J_\lambda x\rangle \\ &= \|J_\lambda x\|^2.\end{aligned}$$

Then, we obtain $\|J_\lambda x + \beta(x - J_\lambda x)\| \geq \|J_\lambda x\|$ for all $\beta \in \mathbf{R}$. From this, we obtain that $J_\lambda x \perp (x - J_\lambda x)$. This implies (3). ∎

6. Generalized Martingales

Let E be a reflexive, strictly convex and smooth Banach space and let J be the normalized duality mapping from E onto E^*. Let $\{Y_n^*\}$ be a sequence of closed linear subspaces of the dual space E^* such that

$$Y_1^* \subseteq Y_2^* \subseteq \cdots \subseteq E^*$$

and let $\{x_n\}$ be a sequence of elements of E. Then, a sequence $\{x_n, Y_n^*\}$ is said to be adapted if $Jx_n \in Y_n^*$ for all $n \geq 1$. A sequence $\{a_n, Y_n^*\}$ is predictable if $a_1 = 0$ and $Ja_n \in Y_{n-1}^*$ for $n \geq 2$. A sequence $\{x_n, Y_n^*\}$ is a generalized martingale if it satisfies the following:

1. $Jx_n \in Y_n^*$ for all $n \geq 1$;
2. $E_{Y_n^*} x_{n+1} = x_n$ for all $n \geq 1$.

Theorem 6.1. *Let E be a reflexive, strictly convex and smooth Banach space. Let $\{Y_n^*\}$ be a sequence of closed linear subspaces of the dual space E^* such that*

$$Y_1^* \subseteq Y_2^* \subseteq \cdots \subseteq E^*$$

and for any $n \geq 2$ and for any $\tilde{y} \in J^{-1}Y_{n-1}^*$,

$$J^{-1}Y_n^* - \tilde{y} = J^{-1}Y_n^* \tag{6.1}$$

and let $\{x_n, Y_n^*\}$ be an adapted sequence in E with $x_1 = 0$. Then, there exist a generalized martingale $\{m_n, Y_n^*\}$ with $m_1 = 0$ and a predictable sequence $\{a_n, Y_n^*\}$ such that for every $n \geq 1$, Doob's decomposition [13, 34]

$$x_n = m_n + a_n \tag{6.2}$$

holds. Further, this decomposition is unique.

Proof. When $n = 1$, the equation (6.2) holds. To use an inductive method, suppose that for some $k \in \mathbf{N}$,

$$x_k = m_k + a_k, \tag{6.3}$$

where $\{m_n, Y_n^*\}_{n\le k}$ with $m_1 = 0$ is a generalized martingale and $\{a_n, Y_n^*\}_{n\le k}$ is a predictable sequence. From assumptions of $\{Y_n^*\}$, for any $y^* \in Y_k^*$ and $a_k \in J^{-1}Y_{k-1}^*$ we can find an element $y \in J^{-1}Y_k^*$ such that

$$y^* = \Pi_{Y_k^*} J(y - a_k). \tag{6.4}$$

In fact, from $y^* \in Y_k^*$ we have $J^{-1}y^* \in J^{-1}Y_k^*$. From (6.1), there exists an element $y \in J^{-1}Y_k^*$ such that $y - a_k = J^{-1}y^*$. So, we have $J(y - a_k) = y^*$ and hence $y^* = \Pi_{Y_k^*} y^* = \Pi_{Y_k^*} J(y - a_k)$. We know that $E_{Y_k^*}(x_{k+1} - x_k) \in J^{-1}Y_k^*$. So, taking $y^* \in Y_k^*$ such that $J^{-1}y^* = E_{Y_k^*}(x_{k+1} - x_k)$, from (6.4) we can find $y = a_{k+1} \in J^{-1}Y_k^*$ such that

$$E_{Y_k^*}(x_{k+1} - x_k) = J^{-1}y^* = J^{-1}\Pi_{Y_k^*} J(a_{k+1} - a_k) = E_{Y_k^*}(a_{k+1} - a_k).$$

Then, we can get a_{k+1} in $J^{-1}Y_k^*$ such that

$$E_{Y_k^*}(x_{k+1} - x_k) = E_{Y_k^*}(a_{k+1} - a_k).$$

Putting $m_{k+1} := x_{k+1} - a_{k+1}$, we have

$$\begin{aligned}
E_{Y_k^*}(m_{k+1}) &= E_{Y_k^*}\left(x_{k+1} - a_{k+1}\right)\\
&= E_{Y_k^*}\left(x_{k+1} - a_{k+1} + a_k - a_k\right)\\
&= E_{Y_k^*}\Big(E_{Y_k^*} x_{k+1} - E_{Y_k^*}(a_{k+1} - a_k) - E_{Y_k^*} a_k\Big)\\
&= E_{Y_k^*}\Big(E_{Y_k^*} x_{k+1} - E_{Y_k^*}(x_{k+1} - x_k) - E_{Y_k^*} a_k\Big)\\
&= E_{Y_k^*}\left(x_{k+1} - x_{k+1} + x_k - a_k\right)\\
&= E_{Y_k^*}\left(x_k - a_k\right)\\
&= E_{Y_k^*}\left(m_k\right) = m_k.
\end{aligned}$$

From assumptions of $\{Y_n^*\}$, we have $m_{k+1} \in J^{-1}Y_{k+1}^*$. Then we have

$$x_{k+1} = m_{k+1} + a_{k+1}, \tag{6.5}$$

where $\{m_n, Y_n^*\}_{n\le k+1}$ with $m_1 = 0$ is a generalized martingale and $\{a_n\}_{n\le k+1}$ is a predictable sequence.

To show the uniqueness, we use the mathematical induction. When $n = 1$, the decomposition is unique. We suppose that for a fixed $k \in \mathbf{N}$, the decomposition is unique for $n \leq k$. If there exists another decomposition for $n = k + 1$

$$x_{k+1} = \tilde{m}_{k+1} + \tilde{a}_{k+1}, \tag{6.6}$$

where $\{\tilde{m}_n, Y_n^*\}_{n \leq k+1}$ is a generalized martingale and $\{\tilde{a}_n\}_{n \leq k+1}$ is a predictable sequence such that for $n \leq k$, $\tilde{m}_n = m_n$ and $\tilde{a}_n = a_n$. We have

$$\begin{aligned}
\tilde{a}_{k+1} &= E_{Y_k^*}(\tilde{a}_{k+1}) \\
&= E_{Y_k^*}\left(x_{k+1} - \tilde{m}_{k+1}\right) \\
&= E_{Y_k^*}\left(E_{Y_k^*} x_{k+1} - E_{Y_k^*} \tilde{m}_{k+1}\right) \\
&= E_{Y_k^*}\left(E_{Y_k^*} x_{k+1} - m_k\right) \\
&= E_{Y_k^*}\left(E_{Y_k^*} x_{k+1} - E_{Y_k^*} m_{k+1}\right) \\
&= E_{Y_k^*}\left(x_{k+1} - m_{k+1}\right) \\
&= E_{Y_k^*}\left(a_{k+1}\right) = a_{k+1}.
\end{aligned}$$

Then, there exists a unique decomposition for $n = k + 1$. This completes the proof. ∎

Lemma 6.1. *Let E be a reflexive and strictly convex Banach space with a Fréchet differentiable norm. Let Y be a closed linear subspaces of E. Then the generalized projection Π_Y of E onto Y is norm to weak continuous. Moreover, let the dual E^* have a Fréchet differentiable norm. Then, Π_Y is norm to norm continuous.*

Proof. Let $\{x_n\}$ be a sequence in E such that $x_n \to x \in E$. Since $\{x_n\}$ is bounded, from Lemma 2.1 the sequence $\{\Pi_Y x_n\} \subset Y$ is bounded. Then, there exists $y_0 \in Y$ such that $\Pi_Y x_{n_i} \rightharpoonup y_0$ as $i \to \infty$. From the assumption, we have $Jx_{n_i} \to Jx$ as $i \to \infty$. So, we have

$$\begin{aligned}
|\langle \Pi_Y x_{n_i}, x_{n_i}\rangle - \langle y_0, Jx\rangle| &= |\langle \Pi_Y x_{n_i} - y_0 + y_0, x_{n_i}\rangle - \langle y_0, Jx\rangle| \\
&= \langle \Pi_Y x_{n_i} - y_0, Jx_{n_i}\rangle - \langle y_0, Jx_{n_i} - Jx\rangle| \\
&\to 0
\end{aligned}$$

as $i \to \infty$. Then, we have $\langle \Pi_Y x_{n_i}, Jx_{n_i}\rangle \to \langle y_0, Jx\rangle$ as $i \to \infty$. From the lower semicontinuity of the norm, we have

$$\begin{aligned}
&\liminf_{i\to\infty} \phi(\Pi_Y x_{n_i}, x_{n_i}) \\
&= \liminf_{i\to\infty} \left(\|\Pi_Y x_{n_i}\|^2 - 2\langle \Pi_Y x_{n_i}, Jx_{n_i}\rangle + \|x_{n_i}\|^2\right) \\
&\geq \|y_0\|^2 - 2\langle y_0, Jx\rangle + \|x\|^2 \\
&= \phi(y_0, x).
\end{aligned} \tag{6.7}$$

On the other hand, from definition of Π_Y we have that for any $y \in Y$, $\phi(\Pi_Y x_{n_i}, x_{n_i}) \leq \phi(y, x_{n_i})$. Then, we have

$$\begin{aligned}
\liminf_{i\to\infty} \phi(\Pi_Y x_{n_i}, x_{n_i}) &\leq \liminf_{i\to\infty} \phi(y, x_{n_i}) \\
&= \liminf_{i\to\infty} \left(\|y\|^2 - 2\langle y, Jx_{n_i}\rangle + \|x_{n_i}\|^2\right) \\
&= \|y\|^2 - 2\langle y, x\rangle + \|x\|^2 \\
&= \phi(y, x)
\end{aligned} \tag{6.8}$$

for any $y \in Y$. From (6.7) and (6.8),

$$\phi(y_0, x) = \min_{y\in Y} \phi(y, x). \tag{6.9}$$

Then we have $\Pi_Y x = y_0$. Every weakly convergent subsequence of $\{\Pi_Y x_n\}$ converges to the unique point $\Pi_Y x$ weakly. Then, $\{\Pi_Y x_n\}$ converges to $\Pi_Y x$ weakly.

Moreover, if E^* has a Fréchet differentiable norm, then E has the Kadec-Klee property. To show $\Pi_Y x_n \to \Pi_Y x$, it is sufficient to prove $\|\Pi_Y x_n\| \to \|y_0\|$ as $n \to \infty$. Since $\Pi_Y x_n \rightharpoonup y_0$ and hence $y_0 \in Y$, from Lemma 2.1 we have $\phi(\Pi_Y x_n, x_n) \leq \phi(y_0, x_n)$ for every $n \in \mathbf{N}$. Since $x_n \to x$ and $\Pi_Y x_n \rightharpoonup y_0$, as in (6.7) we have

$$\begin{aligned}
\phi(y_0, x) &\leq \liminf_{n\to\infty} \phi(\Pi_Y x_n, x_n) \\
&\leq \limsup_{n\to\infty} \phi(\Pi_Y x_n, x_n) \\
&\leq \lim_{n\to\infty} \phi(y_0, x_n) = \phi(y_0, x).
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
\|y_0\|^2 - 2\langle y_0, Jx\rangle + \|x\|^2 &= \phi(y_0, x) \\
&= \lim_{n\to\infty} \phi(\Pi_Y x_n, x_n) \\
&= \lim_{n\to\infty} \left(\|\Pi_Y x_n\|^2 - 2\langle \Pi_Y x_n, Jx_n\rangle + \|x_n\|^2\right) \\
&= \lim_{n\to\infty} \|\Pi_Y x_n\|^2 - 2\langle y_0, Jx\rangle + \|x\|^2.
\end{aligned}$$

Then $\|\Pi_Y x_n\| \to \|y_0\|$ holds. So, $\Pi_Y x_n \to y_0 = \Pi_Y x$ and hence Π_Y is norm-to-norm continuous. ∎

Theorem 6.2. *Let E be a uniformly convex and uniformly smooth Banach space and let $\{Y_n^*\}$ be a family of closed linear subspaces Y_n^*, $n \geq 1$, of the dual space E^* such that*

$$Y_1^* \subseteq Y_2^* \subseteq \cdots \subseteq E^*.$$

Then, a generalized martingale $\{x_n, Y_n^*\}$ in E converges strongly if and only if there exits an element $x \in E$ such that

$$x_n = E_{Y_n^*}x.$$

Proof. We suppose that a generalized martingale $\{x_n, Y_n^*\}$ in E converges to $x \in E$ in norm. The mappings J, J^{-1} and $\Pi_{Y_n^*}$ are norm to norm continuous. From Theorem 3.4, we have that for $n \in \mathbf{N}$,

$$\begin{aligned} E_{Y_n^*}x_{n+2} &= E_{Y_n^*}E_{Y_{n+1}^*}x_{n+2} \\ &= E_{Y_n^*}x_{n+1} = x_n. \end{aligned}$$

Further, we have

$$\begin{aligned} E_{Y_n^*}x_{n+3} &= E_{Y_n^*}E_{Y_{n+2}^*}x_{n+3} \\ &= E_{Y_n^*}x_{n+2} = x_n. \end{aligned}$$

Similarly, we can show that

$$E_{Y_n^*}x_m = x_n$$

for all $m, n \in \mathbf{N}$ with $m \geq n$. Since $x_m \to x$ as $m \to \infty$, we have that for any $n \in \mathbf{N}$,

$$E_{Y_n^*}x_m = J^{-1}\Pi_{Y_n^*}Jx_m \to J^{-1}\Pi_{Y_n^*}Jx = E_{Y_n^*}x$$

as $m \to \infty$. Then, for a fixed $n \in \mathbf{N}$,

$$E_{Y_n^*}x = \lim_{m\to\infty} E_{Y_n^*}x_m = x_n.$$

Conversely, suppose that there exists an element $x \in E$ such that

$$x_n = E_{Y_n^*}x = J^{-1}\Pi_{Y_n^*}Jx.$$

Then, we have $Jx_n = \Pi_{Y_n^*}Jx \in Y_n^*$. Further, we have

$$E_{Y_n^*}x_{n+1} = J^{-1}\Pi_{Y_n^*}Jx_{n+1} = J^{-1}\Pi_{Y_n^*}\Pi_{Y_{n+1}^*}Jx = x_n.$$

So, $\{x_n, Y_n^*\}$ is a generalized martingale.

Since $\{Y_n^*\}$ is a sequence of sets in E^* such that

$$Y_1^* \subseteq Y_2^* \subseteq \cdots \subseteq E^*,$$

the sequence $\{\Pi_{Y_n^*} Jx\}$ converges strongly. In fact, we have that for any $n \in \mathbf{N}$ and $x \in E$,

$$\phi_*(\Pi_{Y_{n+1}^*} Jx, Jx) \leq \phi_*(\Pi_{Y_n^*} Jx, Jx).$$

So, $\lim_{n\to\infty} \phi_*(\Pi_{Y_n^*} Jx, Jx)$ exists. Since for any $m, n \in \mathbf{N}$ with $m \geq n$,

$$\phi_*(\Pi_{Y_m^*} Jx, Jx) + \phi_*(\Pi_{Y_n^*} Jx, \Pi_{Y_m^*} Jx) \leq \phi_*(\Pi_{Y_n^*} Jx, Jx),$$

we have

$$\phi_*(\Pi_{Y_n^*} Jx, \Pi_{Y_m^*} Jx) \leq \phi_*(\Pi_{Y_n^*} Jx, Jx) - \phi_*(\Pi_{Y_m^*} Jx, Jx).$$

From Lemma 2.4, $\{\Pi_{Y_n^*} Jx\}$ is a Cauchy sequence in E^*. So, $\{\Pi_{Y_n^*} Jx\}$ converges strongly to some $y^* \in E^*$. Since J^{-1} is continuous, we have

$$x_n = J^{-1}\Pi_{Y_n^*} Jx \to J^{-1} y^*.$$

This completes the proof. ∎

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