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VALUE DISTRIBUTION OF PRODUCTS OF MEROMORPHIC FUNCTIONS AND THEIR DIFFERENCES

Zong-Xuan Chen

Abstract. In this paper, we study zeros of difference product $f(z)^n\Delta f(z)$ ($n \geq 2$), and the value distribution of difference product $f(z)\Delta f(z)$, where $f(z)$ is a transcendental entire function of finite order, $\Delta f(z) = f(z+c) - f(z)$, where $c\ (\neq 0)$ is a constant such that $f(z+c) \not\equiv f(z)$.

1. Introduction and Results

In this paper, we use the basic notions of Nevanlinna's theory (see [10, 17]). In addition, we use the notation $\sigma(f)$ to denote the order of growth of the meromorphic function $f(z)$, $\lambda(f)$ to denote the exponent of convergence of zeros of $f(z)$.

Hayman proved the following theorem in [11].

Theorem A. If $f(z)$ is a transcendental integral function and $n \geq 2$ is an integer, then $f(z)^n f'(z)$ assumes all values except possibly zero infinitely often.

Clunie [7] proved that if $n = 1$, then Theorem A remains valid.

Recently, many papers (see [1-6, 8, 9, 12-16]) focus on complex difference. They obtain many new results on difference utilizing the value distribution theory of meromorphic functions.

Laine and Yang [15] proved the following theorem.

Theorem B. Let $f(z)$ be a transcendental entire function of finite order, and c be a non-zero complex constant. Then for $n \geq 2$, $f(z)^n f(z+c)$ assumes every non-zero value $a \in \mathbf{C}$ infinitely often.

Liu and Yang [16] proved the following theorems.

Theorem C. Let $f(z)$ be a transcendental entire function of finite order, and c be a non-zero complex constant. Then for $n \geq 2$, $f(z)^n f(z+c) - p(z)$ has infinitely many zeros, where $p(z) \not\equiv 0$ is a polynomial in z.

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Theorem D. Let $f(z)$ be a transcendental entire function of finite order, and c be a non-zero complex constant, $\Delta f(z) = f(z+c) - f(z) \not\equiv 0$. Then for $n \geq 2$, $f(z)^n \Delta f(z) - p(z)$ has infinitely many zeros, where $p(z) \not\equiv 0$ is a polynomial in z.

In Theorems B, C, D, authors proved that when $n \geq 2$, $f(z)^n f(z+c)$ (or $f(z)^n \Delta f(z)$) assume every value $a \in \mathbf{C}\backslash\{0\}$ infinitely often.

The following Example 1 shows that $f(z)^n \Delta f(z)$ may have only finitely many zeros, may also have infinitely many zeros.

Example 1. Suppose that $c = 1$ and

$$f_1(z) = e^z,\ f_2(z) = e^{z^2},\ f_3(z) = \sin z.$$

Thus,

$$H_2^{(1)} = f_1(z)^2 \Delta f(z) = e^{3z}(e-1);$$

$$H_2^{(2)} = f_2(z)^2 \Delta f(z) = e^{3z^2}(e^{2z+1}-1);$$

$$H_2^{(3)} = f_3(z)^2 \Delta f(z) = \sin^2 z(\sin(z+1) - \sin z).$$

From $H_2^{(j)}$ $(j = 1, 2, 3)$, we see that $H_2^{(2)}$ and $H_2^{(3)}$ have infinitely many zeros, but $H_2^{(1)}$ has only finitely many zeros.

Thus, it is natural to ask what condition will guarantee $f(z)^n \Delta f(z)$ $(n \geq 2)$ has infinitely many zeros?

In this paper, we answer this problem, and prove the following Theorem 1.

Theorem 1. Let f be a transcendental entire function of finite order and let $c \in \mathbf{C}\backslash\{0\}$ be a constant satisfying $f(z+c) \not\equiv f(z)$. Set $H_n(z) = f(z)^n \Delta f(z)$ where $\Delta f(z) = f(z+c) - f(z)$, $n \geq 2$ is an integer. Then the following statements hold.

(i) If $f(z)$ satisfies $\sigma(f) \neq 1$, or has infinitely many zeros, then $H_n(z)$ has infinitely many zeros.

(ii) If $f(z)$ has only finitely many zeros and $\sigma(f) = 1$, then $H_n(z)$ has only finitely many zeros.

Remark 1. From Theorem 1(i), we see that $f(z)^n \Delta f(z)$ is differ from $f(z)^n$ $f(z+c)$ $(n \geq 2)$. For example, the function $f(z) = e^{z^2}$ has no zero, and $f(z)^2$ $f(z+c) = e^{3z^2+2cz+c^2}$ (where $c \in \mathbf{C}\backslash\{0\}$ is a constant satisfying $f(z+c) \not\equiv f(z)$) has no zero either. But $f(z)^2 \Delta f(z) = e^{3z^2}(e^{2cz+c^2} - 1)$ has infinitely many zeros.

By Theorem 1 and Theorem D, we easily obtain the following corollary.

Corollary 1. Let f be a transcendental entire function of finite order and let $c \in \mathbf{C}\backslash\{0\}$ be a constant satisfying $f(z+c) \not\equiv f(z)$. Set $H_n(z) = f(z)^n \Delta f(z)$ where $\Delta f(z) = f(z+c) - f(z)$, $n \geq 2$ is an integer.

If $\sigma(f) \neq 1$, or has infinitely many zeros, then $H_n(z)$ takes every value $a \in \mathbf{C}$ (including $a = 0$) infinitely often.

The other aim of this paper is to study the value distribution of difference product $f(z)\Delta f(z)$, i.e. the case $n = 1$. We prove the following Theorems 2-5.

Theorem 2. Let f be a finite order transcendental entire function with a finite Borel exceptional value d, and let $c \in \mathbf{C}\backslash\{0\}$ be a constant satisfying $f(z+c) \not\equiv f(z)$. Set $H(z) = f(z)\Delta f(z)$ where $\Delta f(z) = f(z+c) - f(z)$. Then the following statements hold.

(i) $H(z)$ takes every non-zero value $a \in \mathbf{C}$ infinitely often and satisfies $\lambda(H - a) = \sigma(f)$.

(ii) If $d \neq 0$, then $H(z)$ has no any finite Borel exceptional value.

(iii) If $d = 0$, then 0 is also the Borel exceptional value of $H(z)$. So that $H(z)$ has no non-zero finite Borel exceptional value.

Remark 2. From Theorem 2, we see that $f(z)\Delta f(z)$ is differ from $f(z)f(z+c)$. For example, the function $f(z) = e^z + 1$ has the Borel exceptional value 1, and

$$f(z)f(z + \pi i) = 1 - e^{2z}$$

has the Borel exceptional value 1 either. But by Theorem 2, we see that $f(z)\Delta f(z)$ (with $c = \pi i$) has no finite Borel exceptional value.

Theorem 3. Let f be a transcendental entire function of finite order and let $c \in \mathbf{C}\backslash\{0\}$ be a constant satisfying $f(z+c) \not\equiv f(z)$. Set $H(z) = f(z)\Delta f(z)$ where $\Delta f(z) = f(z+c) - f(z)$.

If $f(z)$ has infinitely many multi-order zeros, then $H(z)$ takes every value $a \in \mathbf{C}$ (including $a = 0$) infinitely often.

Theorem 4. Let f be a transcendental entire function of finite order and let $c \in \mathbf{C}\backslash\{0\}$ be a constant satisfying $f(z+c) \not\equiv f(z)$. Set $H(z) = f(z)\Delta f(z)$ where $\Delta f(z) = f(z+c) - f(z)$.

If there exists an infinite sequence $\{z_n\}$ satisfying $f(z_n) = f(z_n + c) = 0$, then $H(z)$ takes every value $a \in \mathbf{C}$ (including $a = 0$) infinitely often.

Theorem 5. Let f be a transcendental entire function of finite order and let $c \in \mathbf{C}\backslash\{0\}$ be a constant satisfying $f(z+c) \not\equiv f(z)$. Set $H(z) = f(z)\Delta f(z)$ where $\Delta f(z) = f(z+c) - f(z)$.

(i) If $f(z)$ has only finitely many zeros and $\sigma(f) \neq 1$, or has infinitely many zeros, then $H(z)$ has infinitely many zeros.

(ii) If $f(z)$ has only finitely many zeros and $\sigma(f) = 1$, then $H(z)$ has only finitely many zeros.

Example 2. An entire function $f(z) = e^{z^2}$ satisfies Theorem 2(iii), it has the Borel exceptional value 0, and

$$H(z) = e^{2z^2}\left[e^{2cz+c^2} - 1\right]$$

has also the Borel exceptional value 0 since $\lambda(H) = 1 < \sigma(H) = 2.$

Simultaneity, $f(z) = e^{z^2}$ also satisfies Theorem 5(i), although $f(z)$ has no zero, $H(z)$ has infinitely many zeros since $\sigma(f) \neq 1.$

Example 3. An entire function $f(z) = e^z + 1$ satisfies Theorem 2(ii), although it has the Borel exceptional value 1 $(\neq 0)$,

$$H(z) = e^z(e^z + 1)(e^c - 1)\ (c \neq 2k\pi i\ (k \text{ is an integer}))$$

has no finite Borel exceptional value.

2. The Proofs of Theorems 1

We need the following lemmas for the proof of Theorem 1.

Lemma 2.1. ([18, p.79-80]). Let $f_j(z)$ $(j = 1, \cdots, n)$ $(n \geq 2)$ be meromorphic functions, $g_j(z)$ $(j = 1, \cdots, n)$ be entire functions, and satisfy

(i) $\sum_{j=1}^{n} f_j(z)e^{g_j(z)} \equiv 0$;

(ii) when $1 \leq j < k \leq n$, $g_j(z) - g_k(z)$ is not a constant;

(iii) when $1 \leq j \leq n,\ 1 \leq h < k \leq n,$

$$T(r, f_j) = o\{T(r, e^{g_h - g_k})\}\ (r \to \infty, r \notin E),$$

where $E \subset (1, \infty)$ is of finite linear measure or finite logarithmic measure.

Then $f_j(z) \equiv 0$ $(j = 1, \cdots, n).$

Lemma 2.2. (see [8]). Let f be a non-constant finite-order meromorphic solution of

$$f^n P(z, f) = Q(z, f),$$

where $P(z, f),\ Q(z, f)$ are difference polynomials in f, and let $\delta < 1.$ If the degree of $Q(r, f)$ as a polynomial in f and its shifts is at most n, then

$$m(r, P(z, f)) = o\left(\frac{T(r + |c|, f)}{r^{\delta}} + o(T(r, f)\right).$$

for all r outside of a possible exceptional set with finite logarithmic measure.

Lemma 2.3. Let f be a transcendental entire function of finite order and let $c \in \mathbf{C}\backslash\{0\}$ be a constant satisfying $f(z+c) \not\equiv f(z)$. Then $H_n(z) = f(z)^n \Delta f(z)$ $(n \geq 1)$ is transcendental.

Proof. If $H_n(z) \equiv 0$, then $\Delta f(z) \equiv 0$ which contradicts our condition $f(z+c) \not\equiv f(z)$.

Now we suppose that

$$H_n(z) = f(z)^n \Delta f(z) = P(z) \tag{2.1}$$

where $P(z)$ $(\not\equiv 0)$ is a polynomial. Applying Lemma 2.2 to (2.1), we obtain that

$$T(r, \Delta f) = m(r, \Delta f) = S(r, f)$$

for all r outside of a possible exceptional set with finite logarithmic measure. Thus,

$$T\left(r, \frac{1}{\Delta f}\right) = S(r, f) \tag{2.2}$$

for all r outside of a possible exceptional set with finite logarithmic measure. By (2.1) and (2.2), we obtain that

$$T(r, f^n) = T\left(r, \frac{P(z)}{\Delta f(z)}\right) \leq T(r, P) + T\left(r, \frac{1}{\Delta f(z)}\right) = S(r, f).$$

This is a contradiction. Hence $H_n(z)$ is a transcendental entire function.

The Proof of Theorem 1.

(i) If $f(z)$ has infinitely many zeros, then $H_n(z)$ has infinitely many zeros since $\Delta f(z)$ is an entire function and $\Delta f(z) \not\equiv 0$.

Now we suppose that $f(z)$ has only finitely many zeros and $\sigma(f) \neq 1$. Thus since f is transcendental, $f(z)$ can be written as the form

$$f(z) = g(z) e^{h(z)}$$

where $g(z)$ $(\not\equiv 0)$, $h(z)$ are polynomials, $\deg h(z) \geq 2$. Thus

$$f(z+c) = g(z+c) e^{h(z+c)}.$$

Now we suppose that $H_n(z)$ has only finitely many zeros. By Lemma 2.3, we see that $H_n(z)$ is transcendental. So, $H_n(z)$ can be written as

$$H_n(z) = g(z)^n g(z+c) e^{nh(z)+h(z+c)} - g(z)^{n+1} e^{(n+1)h(z)} = g_1(z) e^{h_1(z)}, \tag{2.3}$$

where $g_1(z)$ $(\not\equiv 0)$, $h_1(z)$ are polynomials, $\deg h_1(z) \geq 1$. Set

$$h(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_0, \ a_m \neq 0,$$

where $a_m, \cdots, a_0$ are constants. By $\sigma(f) \neq 1$, we see that $m \geq 2$. Thus,

$$h(z+c) = a_m z^m + (a_m mc + a_{m-1})z^{m-1} + a'_{m-2}z^{m-2} \cdots + a'_0,$$

where $a'_{m-2}, \cdots, a'_0$ are constants. Since $m \geq 2$ and

$$(n+1)a_{m-1} \neq a_m mc + (n+1)a_{m-1},$$

we see that $(n+1)h(z) - (nh(z) + h(z+c))$ is not a constant.

If $nh(z) + h(z+c) - h_1(z)$ and $(n+1)h(z) - h_1(z)$ are not constants, then by (2.3) and Lemma 2.1, we see that

$$g(z)^n g(z+c) \equiv 0,\ g(z)^{n+1} \equiv 0,\ g_1(z) \equiv 0$$

which is a contradiction.

If $nh(z) + h(z+c) - h_1(z) = \delta$ where δ is a constant, then by (2.3), we have

$$[g(z)^n g(z+c) - e^{-\delta} g_1(z)]e^{nh(z)+h(z+c)} - g(z)^{n+1}e^{(n+1)h(z)} = 0. \tag{2.4}$$

By (2.4) and Lemma 2.1, we obtain that

$$g(z)^n g(z+c) - e^{-\delta} g_1(z) \equiv 0,\ g(z)^{n+1} \equiv 0$$

which is also a contradiction.

If $(n+1)h(z) - h_1(z)$ is a constant, then using the same method, we also obtain a contradiction.

Hence, $H_n(z)$ has infinitely many zeros.

(ii) Suppose that $f(z)$ has only finitely many zeros and $\sigma(f) = 1$. Then $f(z)$ can be written as the form

$$f(z) = p^*(z)e^{bz+d}$$

where $p^*(z)$ ($\not\equiv 0$) is a polynomial, b ($\neq 0$) and d are constants. Thus

$$f(z+c) = p^*(z+c)e^{bc}e^{bz+d}$$

and

$$H_n(z) = \{(p^*(z))^n (p^*(z+c)e^{bc} - p^*(z))\}e^{(n+1)(bz+d)}.$$

By the condition $f(z+c) \not\equiv f(z)$ of the theorem, we see that $p^*(z+c)e^{bc} - p^*(z) \not\equiv 0$. Hence $H_n(z)$ has only finitely many zeros.

3. The Proofs of Theorems 2

First, we prove (ii) and (iii)

(ii) Suppose that d $(\neq 0)$ is the Borel exceptional value of $f(z)$. Then $f(z)$ can be written as the form

$$f(z) = d + p(z)e^{\alpha z^k}$$

where k is a positive integer, α $(\neq 0)$ ia a constant, $p(z)$ $(\not\equiv 0)$ is an entire function satisfying

$$\sigma(p) < \sigma(f) = k.$$

Thus

$$f(z+c) = d + p(z+c)p_1(z)e^{\alpha z^k}$$

$p_1(z)$ $(\not\equiv 0)$ is an entire function satisfying $\sigma(p_1) = k-1$. So that,

$$H(z) = p(z)[p(z+c)p_1(z) - p(z)]e^{2\alpha z^k} + d[p(z+c)p_1(z) - p(z)]e^{\alpha z^k}. \tag{3.1}$$

Since $f(z) \not\equiv f(z+c)$, we see that

$$p(z+c)p_1(z) - p(z) \not\equiv 0. \tag{3.2}$$

By (3.1) and (3.2), we see that

$$\sigma(H) = \sigma(f) = k. \tag{3.3}$$

If $H(z)$ has the Borel exceptional value d^*, then

$$H(z) = d^* + p^*(z)e^{\beta z^k}, \tag{3.4}$$

where β $(\neq 0)$ ia a constant, $p^*(z)$ $(\not\equiv 0)$ is an entire function satisfying

$$\sigma(p^*) < \sigma(H) = k.$$

By (3.1) and (3.4), we have

$$p(z)[p(z+c)p_1(z) - p(z)]e^{2\alpha z^k}$$

$$+d[p(z+c)p_1(z) - p(z)]e^{\alpha z^k} - p^*(z)e^{\beta z^k} - d^* = 0. \tag{3.5}$$

If $\beta \neq \alpha$ and $\beta \neq 2\alpha$, then by Lemma 2.1 and (3.5), we can obtain that

$$p(z+c)p_1(z) - p(z) \equiv 0$$

This contradicts (3.2).

If $\beta = 2\alpha$ or $\beta = \alpha$, then using the same method as above, we also obtain a contradiction.

Hence $H(z)$ has no the Borel exceptional value.

(iii) Now suppose that $d = 0$ is the Borel exceptional value of $f(z)$. Using the same method as above, we obtain (3.1) with $d = 0$, i.e.

$$H(z) = p(z)[p(z+c)p_1(z) - p(z)]e^{2\alpha z^k}. \tag{3.6}$$

Since $p(z)[p(z+c)p_1(z) - p(z)] \not\equiv 0$ and

$$\sigma(p(z)[p(z+c)p_1(z) - p(z)]) < k, \tag{3.7}$$

by (3.6) and (3.7), we see that $H(z)$ has the finite Borel exceptional value 0. So that $H(z)$ has no non-zero finite Borel exceptional value.

Finally, we prove (i).

By assert of (ii) and (iii), we see that if $f(z)$ has the finite Borel exceptional value, then any non-zero finite value a must not be the Borel exceptional value of $H(z)$. Hence $H(z)$ takes the value a infinitely often. By (3.3), we obtain $\lambda(H - a) = \sigma(H) = \sigma(f)$.

4. The Proofs of Theorems 3 and 4

The Proof of Theorem 3. Clearly, if $a = 0$, then $H(z)$ has infinitely many zeros since $\Delta f(z)$ is an entire function and $f(z)$ has infinitely many zeros.

Now we suppose that $a \neq 0$. Suppose that $H(z) - a$ has only finitely many zeros. Then $H(z) - a$ can be written as the form

$$H(z) = f(z)f(z+c) - f(z)^2 - a = p(z)e^{q(z)} \tag{4.1}$$

where $p(z),\ q(z)$ are polynomials. By Lemma 2.3, we see that $p(z) \not\equiv 0$, $\deg q(z) \geq 1$. Differentiating (4.1) and eliminating $e^{q(z)}$, we obtain that

$$\frac{[f(z)f(z+c)]'}{f(z)f(z+c)} - \frac{[2f(z)]'}{f(z+c)}$$

$$= \frac{p'(z) + p(z)q'(z)}{p}\left\{1 - \frac{f(z)}{f(z+c)} - \frac{a}{f(z)f(z+c)}\right\} \tag{4.2}$$

Since $p(z) \not\equiv 0,\ q(z)$ are polynomials and $\deg q(z) \geq 1$, we can see that $p'(z) + p(z)q'(z)) \not\equiv 0$. Since $f(z)$ has infinitely many multi-order zeros, we see that there is a sufficiently large point z_0 such that $f(z)$ has zero at the point z_0 of multiplicity $k \geq 2$, and $p'(z_0) + p(z_0)q'(z_0) \neq 0,\ p(z_0) \neq 0$ at the same time.

If $f(z+c)$ has zero at z_0 of multiplicity $k_c \geq 1$, then $\frac{[f(z)f(z+c)]'}{f(z)f(z+c)}$ has a simple pole at z_0; $-\frac{[2f(z)]'}{f(z+c)}$ has pole at z_0 of multiplicity $k_c - k + 1$; $\frac{f(z)}{f(z+c)}$ has pole at

z_0 of multiplicity $k_c - k$; but $\frac{a}{f(z)f(z+c)}$ has pole at z_0 of multiplicity $k_c + k$. This shows (4.2) is a contradiction.

If $f(z_0 + c) \neq 0$, then $\frac{[f(z)f(z+c)]'}{f(z)f(z+c)}$ has a simple pole at z_0; $-\frac{[2f(z_0)]'}{f(z+c)} = 0$; $\frac{f(z_0)}{f(z_0+c)} = 0$. But $\frac{a}{f(z)f(z+c)}$ has pole at z_0 of multiplicity $k \geq 2$. This shows (4.2) is also a contradiction.

Hence $H(z)$ takes every value a infinitely often.

The Proof of Theorem 4. Using the same method as in the proof of Theorem 3, we can prove Theorem 4.

5. The Proof OF Theorem 5

(i) If $f(z)$ has infinitely many zeros, then $H(z)$ has infinitely many zeros since $\Delta f(z)$ is an entire function and $\Delta f(z) \not\equiv 0$.

Now we suppose that $f(z)$ has only finitely many zeros and $\sigma(f) \neq 1$. Thus $f(z)$ can be written as the form

$$f(z) = p(z)e^{h(z)} \tag{5.1}$$

where $p(z)$ $(\not\equiv 0)$, $h(z)$ are polynomials, $\deg h(z) \geq 2$. Thus

$$f(z+c) = p(z+c)e^{h(z+c)}.$$

By Lemma 2.3, we see that $H(z)$ is transcendental. If $H(z)$ has only finitely many zeros, then $H(z)$ can be written as the form

$$H(z) = p(z)p(z+c)e^{h(z)+h(z+c)} - p(z)^2 e^{2h(z)} = p^* e^{h^*(z)} \tag{5.2}$$

where $p^*(z)$ $(\not\equiv 0)$, $h^*(z)$ are polynomials and $\deg h^*(z) \geq 1$. Since $\deg h(z) \geq 2$, we see that $[h(z) + h(z+c)] - 2h(z)$ is not constant.

If $h^*(z) - [h(z) + h(z+c)]$ and $h^*(z) - 2h(z)$ are not constants, then by Lemma 2.1 and (5.2), we obtain that

$$p(z)^2 \equiv 0,\ p(z)p(z+c) \equiv 0$$

which is a contradiction.

If either $h^*(z) - [h(z) + h(z+c)]$ or $h^*(z) - 2h(z)$ is constant, then using the same method, we get that

$$p(z)^2 \equiv 0 \text{ or } p(z)p(z+c) \equiv 0.$$

Both are contradictions. Hence $H(z)$ has infinitely many zeros.

(ii) Suppose that $f(z)$ has only finitely many zeros and $\sigma(f) = 1$. Using the same method as in the proof of Theorem 1(ii), we can finish the proof of Theorem 5.

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Zong-Xuan Chen
School of Mathematical Sciences
South China Normal University
Guangzhou, 510631
P. R. China
E-mail: chzx@vip.sina.com