TAIWANESE JOURNAL OF MATHEMATICS
Vol. 15, No. 4, pp. 1575-1580, August 2011
This paper is available online at http://www.tjm.nsysu.edu.tw/

CYCLE ADJACENCY OF PLANAR GRAPHS AND 3-COLOURABILITY

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Abstract. Suppose G is a planar graph. Let H_G be the graph with vertex set $V(H_G) = \{C : C$ is a cycle of G with $|C| \in \{4, 6, 7\}\}$ and $E(H_G) = \{C_iC_j : C_i$ and C_j are adjacent in $G\}$. We prove that if any 3-cycles and 5-cycles are not adjacent to i-cycles for $3 \le i \le 7$, and H_G is a forest, then G is 3-colourable.

1. INTRODUCTION

As every planar graph is 4-colourable, a natural question is which planar graphs are 3-colourable. It is known [10] that to decide whether a planar graph is 3-colourable is NP-complete. So attention is concentrated in finding sufficient conditions for planar graphs to be 3-colourable. By Grötzsch Theorem, triangle-free planar graphs are 3-colourable. In 1976, Steinberg conjectured that every planar graph without 4- and 5-cycles is 3-colourable (see [11]). This conjecture has received a lot of attention and there are many partial results and related open problems. Erdös (see [13]) suggested the following relaxation of Steinberg's conjecture: Determine the minimum integer k, if it exists, such that every planar graph without cycles of length l for $4 \le l \le k$ is 3-colourable. Abbott and Zhou [1] proved that such a k exists and $k \le 11$. This result was improved to $k \le 10$ in [2], then to $k \le 9$ in [3, 12], and to $k \le 7$ in [7].

The following theorems were proved by Borodin et al. in [7].

Theorem 1.1. *Every planar graph without cycles of length from 4 to 7 is 3-colourable.*

For the purpose of using induction, instead of proving Theorem 1.1 directly, they proved the following stronger statement.

Received November 19, 2008, accepted March 23, 2010.
Communicated by Hung-Lin Fu.
2000 *Mathematics Subject Classification*: 05C15.
Key words and phrases: Planar graph, Number, Cycle adjacency.
*Partially supported by NSC 95-2115-M-110-013-MY3.

Theorem 1.2. *Suppose G is a planar graph without cycles of length 4 to 7 and f_0 is a face of G of length $8 \leq i \leq 11$. Then every proper 3-colouring of the vertices of f_0 can be extended to a proper 3-colouring of G.*

The distance between two cycles C, C' of a graph G is the shortest distance between vertices of C and C'. Two cycles are *adjacent* if they have at least one edge in common. Havel asked in 1969 the question whether there is a constant C such that every planar graph with minimum distance between triangles at least C is 3-colourable. This question also remains open. However, it was proved in [9] that if a planar graph G has no 5-cycles and every two triangles have distance at least 4, then G is 3-colourable. This distance requirement between triangles is reduced to 3 in [4, 14] and then to 2 in [5]. These results motivated the following two conjectures:

Conjecture 1.3. ([9]). *Every planar graph without 5-cycles and without adjacent triangles is 3-colourable.*

Conjecture 1.4. ([6]). *Every planar graph without triangles adjacent to cycles of length 3 or 5 is 3-colourable.*

Conjecture 1.4 is stronger than Conjecture 1.3, and Conjecture 1.3 is stronger than Steinberg's conjecture. These conjectures remain unsettled and stimulate the study of 3-colourability of planar graphs which satisfy specific adjacency relations among short cycles. In [8], it was proved that if G is a planar graph in which no i-cycle is adjacent to a j-cycle whenever $3 \leq i \leq j \leq 7$, then G is 3-colourable.

In this paper, we consider planar graphs in which cycles of lengths $4, 6, 7$ may be adjacent to each other, but the adjacency is rather limited. For a planar graph G, let H_G be the graph with vertex set $V(H_G) = \{C : C \text{ is a cycle of } G \text{ with } |C| \in \{4, 6, 7\}\}$ and $E(H_G) = \{C_iC_j : C_i \text{ and } C_j \text{ are adjacent in } G\}$. We prove the following result:

Theorem 1.5. *For a planar graph G, if any 3-cycles and 5-cycles are not adjacent to i-cycles whenever $3 \leq i \leq 7$, and H_G is a forest, then G is 3-colourable.*

2. Proof of Theorem 1.5

For a face f, denote by $b(f)$ the set of edges on the boundary of f. A k-vertex is a vertex of degree k. A k-face is a face f with $|b(f)| = k$. For a vertex v, $N(v)$ denotes the set of neighbors of v. For a cycle C of G, $int(C)$ and $ext(C)$ denote the sets of vertices lie in the interior and exterior of C, respectively. A cycle C is called a separating cycle if $int(C) \neq \emptyset$ and $ext(C) \neq \emptyset$. Let $c_i(G)$ be the number of cycles of length i in G. If u, v are two vertcies on C, we use $C[u, v]$ to denote the path of C clockwisely from u to v, and let $C(u, v) = C[u, v] \setminus \{u, v\}$,

$C[u, v) = C[u, v] \setminus \{v\}$, $C(u, v] = C[u, v] \setminus \{u\}$. For each path P and cycle C, we denote by $|P|$ and $|C|$ the number of vertices of P and C. Let Ω be the set of connected planar graphs satisfying the assumption of Theorem 1.5.

Theorem 1.5 follows from the following lemma:

Lemma 2.1. *Suppose $G \in \Omega$ and f_0 is an i-face of G with $3 \leq i \leq 11$. Then every proper 3-colouring of the vertices of f_0 can be extended to the whole G.*

If Lemma 2.1 is true, then for any $G \in \Omega$, either G has no triangles, and hence by Grötzsch theorem, G is 3-colourable, or G has a triangle C, and it follows from Lemma 2.1 that any proper 3-colouring of C can be extended to a proper 3-colouring of the interior as well as of the exterior of C. So it remains to prove Lemma 2.1. Assume the lemma is not true and G is a counterexample with

(1) $c(G) = c_4(G) + c_5(G) + c_6(G) + c_7(G)$ is minimum.
(2) subject to (1), $|V(G)| + |E(G)|$ is minimum.

Assume the unbounded face f^* is an i-face with $3 \leq i \leq 11$ and ϕ is a proper 3-colouring of the vertices of f^* which cannot be extended to G. Let C^* be the boundary cycle of f^*.

By the minimality of G, G is 2-connected, and hence each face is a cycle. Moreover, each vertex $v \in int(C^*)$ has degree at least 3, for otherwise, one can first extend the colouring of C^* to $G - v$, and then extend it to v. Also G has no separating cycles of length 3 to 11, because if C is such a cycle, then we can first extend ϕ to $G \setminus int(C)$. Then extend this colouring to $G \setminus ext(C)$. Therefore, G has a proper 3-colouring.

Observe that C^* has no chord, because if $e = uv$ is a chord of C^*, then $G - e$ is a smaller counterexample. Moreover, any cycle of G of length $4 \leq i \leq 7$ has no chord, for otherwise, we either have a 3-cycle or a 5-cycle adjacent to an i-cycle for some $3 \leq i \leq 7$, or we have two 4-cycles and a 6-cycle that are pairwise adjacent (so these three cycles form a cycle in H_G, contrary to our assumption).

If $4 \leq |C^*| \leq 7$, then let G' be the graph obtained from G by adding $11 - |C^*|$ vertices on one edge of C^*. Then $c(G') < c(G)$ and $G' \in \Omega$. The colouring of C^* can be easily extended to the added degree 2 vertices. By the minimality of G, the colouring of the outer cycle of G' can be extended to a 3-colouring of G'. Hence, G is 3-colourable, contrary to our assumption. Thus we may assume that $|C^*| \neq 4, 5, 6, 7$.

Claim 1. *For each internal face f, there exists another internal face f' such that f and f' have exactly one edge in common. Moreover, any two internal k-faces with $4 \leq k \leq 7$ have at most one edge in common.*

Proof. Let f be an internal face of G and let C be the boundary cycle of f. Certainly there is another internal face adjacent to f. Assume for each internal face

f' adjacent to f, $b(f) \cap b(f')$ contains at least two edges. Then either $b(f) \cap b(f')$ contains two edges e_1, e_2 that have a vertex in common or $C - b(f) \cap b(f')$ contains at least two segments. If $b(f) \cap b(f')$ contains two edges e_1, e_2 and $e_1 \cap e_2 \neq \emptyset$, then $e_1 \cap e_2$ is a cut-vertex or an internal 2-vertex, which is a contradiction. Thus we assume that for each internal face f' adjacent to f, $C - b(f) \cap b(f')$ has at least two segments. Note that at most one the segments of $C - b(f) \cap b(f')$ intersects C^*. Let $\beta(f')$ be the minimum length of those segments of $C - b(f) \cap b(f')$ that do not intersect C^*. Choose f' so that $\beta(f')$ is minimum. Let P be a segment of $C - b(f) \cap b(f')$ of length $\beta(f')$ and $P \cap C^* = \emptyset$. Let $f'' \neq f$ be a face with $b(f'') \cap P \neq \emptyset$. Then $b(f'') \cap b(f)$ is contained in P. This implies that $\beta(f'') < \beta(f')$, in contrary to the choice of f'.

Suppose $4 \leq i, j \leq 7$ and there exist an internal i-face f and an internal j-face f' such that $e_1, e_2 \in b(f) \cap b(f')$. If $e_1 \cap e_2 \neq \emptyset$, then $e_1 \cap e_2$ is a cut-vertex or an internal 2-vertex. If $e_1 \cap e_2 = \emptyset$, then there are three cycles of length between 3 to 7 adjacent to each other, again contrary to our assumption. ■

Claim 2. *Suppose f is an internal k-face with $4 \leq k \leq 7$ and $C = b(f)$. If $|V(f) \cap C^*| \geq 2$ and $u, v \in V(f) \cap C^*$, then either $C[u, v]$ or $C[v, u]$ is a segment of C^*.*

Proof. Suppose none of $C[u, v]$ and $C[v, u]$ is a segment of C^*. Then $C[u, v] \cup C^*[v, u]$ and $C[v, u] \cup C^*[u, v]$ are separating cycles. Let $q = |C(u, v)|$, $p = |C(v, u)|$. Since any separating cycle has length at least 12, it follows that $|C^*| \geq (12 - p) + (12 - q) - 2 = 22 - (p + q) > 11$, contrary to our assumption. ■

Claim 3. *G contains no internal k-faces with $4 \leq k \leq 7$.*

Proof. Suppose G contains an internal k-face for some $k \in \{4, 5, 6, 7\}$. Since H_G is acyclic, there is an internal k_1-face f_1 with $k_1 \in \{4, 5, 6, 7\}$ such that f_1 is adjacent to at most one face of length 4 to 7.

If f_1 is adjacent to a face of length 4 to 7, then let f_2 to be the unique face adjacent to f_1 of length $k_2 \in \{4, 5, 6, 7\}$. Otherwise let f_2 to be a face which has exactly one edge in common with f_1. Let C_1, C_2 be the boundary cycles of f_1, f_2, respectively.

By Claim 1, $C_1 \cap C_2$ contains exactly one edge xy. For $i = 1, 2$, let u_i be the other neighbour of x in C_i, and let v_i be the other neighbour of y in C_i.

Since C^* has no chord, at most one of x, y belong to C^*. First we consider the case that one of x, y, say x, lies on C^*. If $u_1 \notin C^*$ or $N(y) \cap C^* = \{x\}$, then let G' be the graph obtained from G by identifying u_1 and y into a vertex u^*. It is easy to see that $G' \in \Omega$, and $c(G') \leq c(G)$ and $|V(G')| + |E(G')| < |V(G)| + |E(G)|$. By the minimality of G, the colouring of C^* can be extended to a proper 3-colouring ϕ of G'. By assigning the colour of u^* to u_1 and y, we obtain a proper 3-colouring of G that is an extension of the colouring of C^*. This is in contrary to our assumption. So we have $u_1 \in C^*$ and $N(y) \cap C^* - \{x\} \neq \emptyset$.

If $v_1 \in C^*$, then by Claim 2, $C_1[v_1, u_1] = C^*[v_1, u_1]$. If $C_2(x, y) \not\subset C^*$, then $C' = C^*[x, v_1] \cup v_1yx$ is a separating cycle. But $|C'| \leq |C^*| \leq 11$, which is a contradiction. If $C_2(x, y) \subset C^*$, then $v_2 \in C^*$. Since f_1 is adjacent to at most one face of length 4 to 7, so $|C^*(v_2, v_1)| \geq 5$. If each of f_1, f_2 has length at least 6, then $|C^*[v_1, v_2]| \geq 9$. If f_1 has length 4, then f_2 has length at least 6; If f_1 has length 5, then f_2 has length at least 8; If f_1 has length 6, then f_2 has length at least 4, for otherwise we would have two 4-cycles and a 6-cycle that are pairwise adjacent, in contrary to our assumption. This implies that $|C^*[v_1, v_2]| \geq 7$. In any case, this is a contradiction as $|C^*| \leq 11$. Thus we assume that $v_1 \notin C^*$.

Let $t \in N(y) \cap C^* \setminus \{x\}$. Since $v_1 \notin C^*$, $C^*[t, x] \cup xyt$ is a separating cycle. This implies that $|C^*[t, x]| \geq 11$. Since f_1 is not adjacent to a 3-cycle, $|C^*[x, t]| \geq 3$, contrary to the assumption that $|C^*| \leq 11$.

Suppose $C^* \cap \{x, y\} = \emptyset$. If $u_1 \notin C^*$, then identify u_1 and y. If $v_1 \notin C^*$, then identify v_1 and x. By the minimality of G, the resulting graph G' has a proper 3-colouring which is an extension of the colouring of C^*. This induces a proper 3-colouring of G which is an extension of the colouring of C^*. Thus we assume $u_1, v_1 \in C^*$.

If there exists $t \in C^* \cap N(x) \setminus \{u_1\}$, then $|C^*[u_1, t]| \geq 7$ and $|C^*[t, v_1]| \geq 6$, otherwise f_1 is adjacent to another cycle of length at most 7. Similarly, if there exists $t \in C^* \cap N(y) \setminus \{v_1\}$, then $|C^*[u_1, t]| \geq 6$ and $|C^*[t, v_1]| \geq 7$. In both cases we have $|C^*| \geq 12$, which is a contradiction. So we assume $C^* \cap N(x) = \{u_1\}$ and $C^* \cap N(y) = \{v_1\}$. In particular, $u_2 \notin C^*$ and $v_2 \notin C^*$. If $|f_1| \geq 6$, then $C^*[u_1, v_1] \cup v_1yxu_1$ is a separating cycle. This implies that $|C^*[u_1, v_1]| \geq 10$ and $|C^*| \geq 12$, which is a contradiction. If $|f_1| = 4$, then we identify u_1 and y. Hence G has a proper 3-colouring by minimality. If $|f_1| = 5$, let $C_1 \setminus \{u_1, v_1, x, y\} = \{t\}$, then we identify t and x. Hence G has a proper 3-colouring by minimality, this is a contradiction. This complete the proof of Claim 3. ∎

Since $|C^*| \neq 4, 5, 6, 7$, and G has no separating cycles of length 3 to 11. Claim 3 implies that G has no cycles of length 4 to 7. If $8 \leq |C^*| \leq 11$, then by applying Theorem 1.2, we can extend the 3-colouring of C^* to the whole G. If $|C^*| = 3$, then by applying Theorem 1.1, G is 3-colourable, and we can extend the 3-colouring of C^* to the whole G by permuting the colours. Hence this means that there is no counterexample. This complete the proof of Lemma 2.1.

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